Int. J. Nonlinear Anal. Appl. 11 (2020) No. 1, 247-258
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2020.4274

Carathéodory's Inequality on the Right half plane

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(Communicated by Madjid Eshaghi Gordji)

Abstract

In this paper, a boundary version of Carathéodory's inequality on the right half plane is investigated. $Z(s)$ is an analytic function defined in the right half of the s-plane.We derive inequalities for the modulus of $Z(s)$ function, $|Z'(0)|$, by assuming the $Z(s)$ function is also analytic at the boundary point $s=0$ on the imaginary axis and finally, the sharpness of these inequalities is proved.

Keywords: Analytic function, Schwarz lemma, Carathéodory's inequality.
2010 MSC: 30C80, 32A10.

1. Introduction

Let f be an analytic function in the unit disc $D=\{z:|z|<1\}$, $f(0)=0$ and $|f(z)|<1$ for $|z|<1$. In accordance with the classical Schwarz lemma, for any point z in the disc D, we have $|f(z)|\leq|z|$ and $|f'(0)|\leq 1$. Equality in these inequalities (in the first one, for $z\neq 0$) occurs only if $f(z)=\lambda z$, $|\lambda|=1$ ([6], p.329). The Schwarz lemma is one of the most important results in the classical complex analysis,which has become a crucial theme in many branches of mathematical research for over a hundred years. On the other hand, in the book [7], Sharp Real-Parts Theorem's (in particular Carathéodory's inequalities), which are frequently used in the theory of entire functions and analytic function theory, have been studied. Also, a boundary version of the Carathéodory's inequality is considered in unit disc and novel results are obtained in [10, 11]. At first, as in Schwarz lemma, Carathéodory's inequality at right half plane will be presented.

Let $Z(s)=1+c_1(s-1)+c_2(s-1)^2+...$ be an analytic in the right half plane with $\Re Z(s)\leq A$ $(A>1)$ for $\Re s\geq 0$.

Consider the function

$$f(z)=\frac{Z(s)-1}{Z(s)+1-2A},\ z=\frac{s-1}{s+1}.$$

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Received: April 2019 *Revised:* February 2020

Here, the function $f(z)$ is an analytic function in D, $f(0)=0$ and $|f(z)|<1$ for $z\in D$. Now, let us show that $|f(z)|<1$ for $|z|<1$. Since

$$\begin{aligned}\left|Z\left(\frac{1+z}{1-z}\right)-1\right|^2 &= \left(Z\left(\frac{1+z}{1-z}\right)-1\right)\left(\overline{Z\left(\frac{1+z}{1-z}\right)}-1\right)\\ &= \left|Z\left(\frac{1+z}{1-z}\right)\right|^2-Z\left(\frac{1+z}{1-z}\right)-\overline{Z\left(\frac{1+z}{1-z}\right)}+1\end{aligned}$$

and

$$\begin{aligned}\left|Z\left(\frac{1+z}{1-z}\right)+1-2A\right|^2 &= \left(Z\left(\frac{1+z}{1-z}\right)+1-2A\right)\left(\overline{Z\left(\frac{1+z}{1-z}\right)}+1-2A\right)\\ &= \left|Z\left(\frac{1+z}{1-z}\right)\right|^2+(1-2A)\,Z\left(\frac{1+z}{1-z}\right)\\ &\quad +(1-2A)\,\overline{Z\left(\frac{1+z}{1-z}\right)}+(1-2A)^2,\end{aligned}$$

we obtain

$$\begin{aligned}&\left|Z\left(\frac{1+z}{1-z}\right)-1\right|^2-\left|Z\left(\frac{1+z}{1-z}\right)+1-2A\right|^2\\ =\ & -2\,(1-A)\left(Z\left(\frac{1+z}{1-z}\right)+\overline{Z\left(\frac{1+z}{1-z}\right)}\right)+4A-4A^2\\ =\ & -4\,(1-A)\,\Re Z\left(\frac{1+z}{1-z}\right)+4A-4A^2\\ \leq\ & -4\,(1-A)\,A+4A-4A^2=0.\end{aligned}$$

Therefore, we have $|f(z)|<1$ for $|z|<1$.

From the Schwarz lemma, we take

$$|f'(0)|\leq 1.$$

Since

$$f'(z)=\frac{\frac{4}{(1-z)^2}Z'\left(\frac{1+z}{1-z}\right)(1-A)}{\left(Z\left(\frac{1+z}{1-z}\right)+1-2A\right)^2},$$

we obtain

$$|f'(0)|=\left|\frac{4Z'\,(1)\,(1-A)}{(Z\,(1)+1-2A)^2}\right|\leq 1$$

and

$$|Z'\,(1)|\leq A-1.$$

This result is sharp with the function

$$Z(s)=(1-A)\,s+A.$$

We thus obtain the following lemma.

Lemma 1.1. *Let $Z(s) = 1 + c_1(s-1) + c_2(s-1)^2 + ...$ be an analytic in the right half plane with $\Re Z(s) \leq A$ $(A > 1)$ for $\Re s \geq 0$. Then we have the inequality*

$$|Z'(1)| \leq A - 1. \tag{1.1}$$

This result is sharp and the extremal function is

$$Z(s) = (1 - A)s + A.$$

It is an elementary consequence of Schwarz lemma that if f extends continuously to some boundary point b with $|b| = 1$, and if $|f(b)| = 1$ and $f'(b)$ exists, then $|f'(b)| \geq 1$, which is known as the Schwarz lemma on the boundary. In [9], R. Osserman proposed the boundary refinement of the classical Schwarz lemma as follows:

Let $f : D \to D$ be an analytic function with $f(0) = 0$. Assume that there is a $b \in \partial D$ so that f extends continuously to b, $|f(b)| = 1$ and $f'(b)$ exists. Then

$$|f'(b)| \geq \frac{2}{1 + |f'(0)|}. \tag{1.2}$$

Thus, by the classical Schwarz lemma, it follows that

$$|f'(b)| \geq 1. \tag{1.3}$$

Inequality (1.3) is sharp, with equality possible for each value of $|f'(0)|$. In addition, for $b = 1$ in the inequality (1.2), equality occurs for the function $f(z) = z\frac{z+m}{1+mz}$, $m \in [0, 1]$. Also, $|f'(b)| > 1$ unless $f(z) = ze^{i\theta}$, θ real. Inequality (1.3) and its generalizations have important applications in geometric theory of functions and they are still hot topics in the mathematics literature [1, 2, 3, 4, 5, 8, 9].

In this paper, we studied "a boundary Carathéodory's inequalities" on the right half plane as analog the Schwarz lemma at the boundary [9]. We present an analytic to understand the behaviour of the derivative of $Z(s)$ function at the zero on the right half plane. In the resulting theorems of the analysis, assuming that $\Re Z(0) = A$, a lower boundary for modulus of the derivative of the $Z(s)$ function at the zero on right half plane, $|Z'(0)|$, are obtained.

Also, we target to find the answer of the question: "What can be said about $Z'(s)$ when it is considered at the boundary? The answer of the question relies on the boundary analysis of the Caratheodory inequality, that is, analysis of $Z(s)$ function at $s = 0$. As a result, in our study, we give a bounded version of Caratheodory inequality on the right half-plane. Moreover, by assuming $Z(s)$ is also analytic at the boundary point $s = 0$ on the imaginer axis, we shall give an estimate for $|Z'(0)|$ from below using Taylor expansion coefficients. The sharpness of this inequality is also proved.

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2. Main Results

In this section, a boundary version of Carathéodory's inequality on the right half plane is investigated. $Z(s)$ is an analytic function defined in the right half of the s-plane. We derive inequalities for the modulus of $Z(s)$ function, $|Z'(0)|$, by assuming the $Z(s)$ function is also analytic at the boundary point $s = 0$ on the imaginary axis and finally, the sharpness of these inequalities is proved. We have following results, which can be offered as the boundary refinement of Carathéodory's inequality on the right half plane.

Theorem 2.1. *Let $Z(s) = 1 + c_1(s-1) + c_2(s-1)^2 + \dots$ be an analytic in the right half plane with $\Re Z(s) \leq A$ for $\Re s \geq 0$. Suppose that $Z(s)$ is analytic at the point $s = 0$ of the imaginary axis with $\Re Z(0) = A$. Then*

$$|Z'(0)| \geq A - 1. \tag{2.1}$$

The inequality (2.1) *is sharp with extremal function*

$$Z(s) = s(1-A) + A.$$

Proof . Consider the function

$$f(z) = \frac{Z\left(\frac{1+z}{1-z}\right) - 1}{Z\left(\frac{1+z}{1-z}\right) + 1 - 2A}, \ z = \frac{s-1}{s+1}.$$

Here, the function $f(z)$ is analytic in the unit disc D, $|f(z)| < 1$ for $z \in D$ and $|f(b)| = 1$ for $-1 = c \in \partial D$. If we take the derivative of the function $f(z)$, then we get

$$f'(z) = \frac{\frac{4}{(1-z)^2} Z'\left(\frac{1+z}{1-z}\right)(1-A)}{\left(Z\left(\frac{1+z}{1-z}\right) + 1 - 2A\right)^2}.$$

Therefore, from (1.3), we obtain

$$1 \leq |f'(-1)| = \left| \frac{Z'(0)(1-A)}{(Z(0) + 1 - 2A)^2} \right|.$$

Since

$$\begin{aligned} |Z(0) + 1 - 2A|^2 &\geq (\Re(Z(0) + 1 - 2A))^2 \\ &= (\Re Z(0) + 1 - 2A)^2 = (A + 1 - 2A)^2 = (1-A)^2, \end{aligned}$$

we obtain

$$1 \leq \frac{|Z'(0)||1-A|}{|1-A|^2} = \frac{|Z'(0)|}{|1-A|}$$

and

$$|Z'(0)| \geq A - 1.$$

Now, we shall show that the inequality (2.1) is sharp. Let

$$Z(s) = s(1-A) + A.$$

Then

$$Z'(s) = 1 - A$$

and

$$|Z'(0)| = A - 1.$$

□

Theorem 2.2. *Under the same assumptions as in Theorem 2.1, we have the inequality*

$$|Z'(0)| \geq \frac{2(A-1)^2}{A-1+|Z'(1)|}. \tag{2.2}$$

The result is sharp with the function

$$Z(s) = \frac{1-\alpha\frac{s-1}{s+1}\left[\left(\frac{s-1}{s+1}\right)^2 - \frac{s-1}{s+1}\alpha\right](1-2A)}{1-2\alpha\frac{s-1}{s+1}+\left(\frac{s-1}{s+1}\right)^2},$$

where $\alpha = \frac{|Z'(1)|}{A-1}$ *is an arbitrary number from* $[0,1]$ *(see* (1.1)).

Proof . Let $f(z)$ be the same as in the proof of Theorem 2.1. Therefore, from (1.2), we take

$$\frac{2}{1+|f'(0)|} \leq |f'(-1)| \leq \frac{|Z'(0)|}{|1-A|}.$$

Also, since

$$f'(0) = \frac{4Z'(1)(1-A)}{(Z(1)+1-2A)^2}$$

and

$$|f'(0)| = \frac{|Z'(1)|}{A-1},$$

we obtain

$$\frac{2}{1+\frac{|Z'(1)|}{A-1}} \leq \frac{|Z'(0)|}{|1-A|}$$

and

$$|Z'(0)| \geq \frac{2(A-1)^2}{A-1+|Z'(1)|}.$$

Now, we shall show that the inequality (2.2) is sharp. Let

$$Z\left(\frac{1+z}{1-z}\right) = \frac{1-\alpha z-(z^2-\alpha z)(1-2A)}{1-2\alpha z+z^2}.$$

Then

$$\begin{aligned}\frac{2}{(1-z)^2}Z'\left(\frac{1+z}{1-z}\right) &= \frac{(-\alpha-(2z-\alpha)(1-2A))(1-2\alpha z+z^2)}{(1-2\alpha z+z^2)^2}\\ &\quad -\frac{(-2\alpha+2z)(1-\alpha z-(z^2-\alpha z)(1-2A))}{(1-2\alpha z+z^2)^2}\end{aligned}$$

and for $z=-1\in\partial D$

$$\begin{aligned}\frac{1}{2}Z'(0) &= \frac{(-\alpha-(-2-\alpha)(1-2A))(2+2\alpha)}{4(1+\alpha)^2}\\ &\quad -\frac{(-2\alpha-2)(1+\alpha-(1+\alpha)(1-2A))}{4(1+\alpha)^2}.\end{aligned}$$

Therefore, we have

$$|Z'(0)| = 2\frac{A-1}{1+\alpha}.$$

Since $\alpha = \frac{|Z'(1)|}{A-1}$, we take

$$|Z'(0)| = 2\frac{A-1}{1+\alpha} = |Z'(0)| = 2\frac{A-1}{1+\frac{|Z'(1)|}{A-1}} = \frac{2(A-1)^2}{A-1+|Z'(1)|}.$$

□

Theorem 2.3. *Let $Z(s) = 1 + c_1(s-1) + c_2(s-1)^2 + ...$ be an analytic in the right half plane with $\Re Z(s) \leq A$ for $\Re s \geq 0$. Suppose that $Z(s)$ is analytic at the point $s = 0$ of the imaginary axis with $\Re Z(0) = A$. Then we have the inequality*

$$|Z'(0)| \geq (A-1)\left(1 + \frac{2\left(A-1-|c_1|^2\right)}{(A-1)^2 - |c_1|^2 + |(c_1+2c_2)(1-A) - c_1^2|}\right). \tag{2.3}$$

The equality in (2.3) *occurs for the function*

$$Z(s) = \frac{As^2 + 2s(1-A) + A}{s^2+1}.$$

Proof . Let $B(z) = z$, $z \in D$ and $f(z)$ be as in Theorem 2.1. $B(z)$ is analytic in D. The maximum principle implies that for each $z \in D$, we have $|f(z)| \leq |B(z)|$. Therefore,

$$p(z) = \frac{f(z)}{B(z)}$$

is analytic function in D and $|p(z)| \leq 1$ for $z \in D$. In particular, we take

$$\begin{aligned} p(z) &= \frac{Z\left(\frac{1+z}{1-z}\right) - 1}{\left(Z\left(\frac{1+z}{1-z}\right) + 1 - 2A\right)z} = \frac{c_1\left(\frac{2z}{1-z}\right) + c_2\left(\frac{2z}{1-z}\right)^2 + c_3\left(\frac{2z}{1-z}\right)^3 + ...}{\left(2 - 2A + c_1\left(\frac{2z}{1-z}\right) + c_2\left(\frac{2z}{1-z}\right)^2 + c_3\left(\frac{2z}{1-z}\right)^3 + ...\right)z} \\ &= \frac{c_1\left(\frac{2}{1-z}\right) + c_2\left(\frac{2}{1-z}\right)^2 z + c_3\left(\frac{2}{1-z}\right)^3 z^2 + ...}{2(1-A) + c_1\left(\frac{2z}{1-z}\right) + c_2\left(\frac{2z}{1-z}\right)^2 + c_3\left(\frac{2z}{1-z}\right)^3 + ...}, \end{aligned}$$

$$|p(0)| = \frac{|c_1|}{A-1} \leq 1$$

and

$$|p'(0)| = \frac{|(c_1+2c_2)(1-A) - c_1^2|}{(1-A)^2}.$$

If we take $|p(0)| = 1$, then by the maximum principle, we have $\frac{f(z)}{B(z)} = e^{i\theta}$, $\theta \in \mathbb{R}$, $f(z) = B(z)e^{i\theta}$ and

$$Z\left(\frac{1+z}{1-z}\right) = \frac{1 + (1-2A)ze^{i\theta}}{1 - ze^{i\theta}}.$$

Therefore, we may assume that

$$Z\left(\frac{1+z}{1-z}\right) \not\equiv \frac{1 + (1-2A)ze^{i\theta}}{1 - ze^{i\theta}},$$

and so $|p(0)| < 1$. Moreover, since the expression $\frac{bf'(b)}{f(b)}$ is a real number greater than or equal to 1 ([2]) and $\Re Z(0) = A$ yields $|f(b)| = 1$ for $-1 = b \in \partial D$, we take

$$\frac{bf'(b)}{f(b)} = \left|\frac{bf'(b)}{f(b)}\right| = |f'(b)|.$$

Also, since $|f(z)| \leq |B(z)|$, we get

$$\frac{1-|f(z)|}{1-|z|} \geq \frac{1-|B(z)|}{1-|z|}.$$

Passing to limit in the last inequality yields

$$|f'(b)| \geq |B'(b)|.$$

Thus, we obtain

$$\frac{bf'(b)}{f(b)} = |f'(b)| \geq |B'(b)| = \frac{bB'(b)}{B(b)}.$$

The auxliary function

$$g(z) = \frac{p(z)-p(0)}{1-\overline{p(0)}p(z)}$$

satisfies the hypothesis of the Schwarz lemma at the boundary. That is, $g(z)$ is analytic function, $g(0) = 0$, $|g(z)| < 1$ for $z \in D$ and $|g(b)| = 1$ for $-1 = b \in \partial D$. Thus, we obtain

$$\begin{aligned}
\frac{2}{1+|g'(0)|} &\leq |g'(-1)| = \frac{1-|p(0)|^2}{\left|1-\overline{p(0)}p(-1)\right|^2}|p'(-1)| \\
&\leq \frac{1+|p(0)|}{1-|p(0)|}\left|\frac{f'(-1)}{B(-1)} - \frac{f(-1)B'(-1)}{B^2(-1)}\right| \\
&= \frac{1+|p(0)|}{1-|p(0)|}\left(|f'(-1)| - |B'(-1)|\right) \\
&\leq \frac{1+\frac{|c_1|}{A-1}}{1-\frac{|c_1|}{A-1}}\left(\frac{|Z'(0)|}{A-1} - 1\right).
\end{aligned}$$

Since

$$g'(z) = \frac{1-|p(0)|^2}{\left(1-\overline{p(0)}p(z)\right)^2}p'(z)$$

and

$$|g'(0)| = \frac{|p'(0)|}{1-|p(0)|^2} = \frac{\left|(c_1+2c_2)(1-A)-c_1^2\right|}{(A-1)^2-|c_1|^2},$$

we obtain

$$\frac{2}{1+\frac{\left|(c_1+2c_2)(1-A)-c_1^2\right|}{(A-1)^2-|c_1|^2}} \leq \frac{A-1+|c_1|}{A-1-|c_1|}\left(\frac{|Z'(0)|}{A-1} - 1\right)$$

and

$$|Z'(0)| \geq (A-1)\left(1+\frac{2\left(A-1-|c_1|^2\right)}{(A-1)^2-|c_1|^2+\left|(c_1+2c_2)(1-A)-c_1^2\right|}\right).$$

To show that the inequality (2.3) is sharp, take the analytic function

$$Z(s) = \frac{As^2 + 2s(1-A) + A}{s^2+1}.$$

Then

$$Z'(s) = \frac{(2As + 2(1-A))(s^2+1) - 2s(As^2 + 2s(1-A) + A)}{(s^2+1)^2}$$

and

$$|Z'(0)| = 2(A-1).$$

On the other hand, we take

$$1 + c_1(s-1) + c_2(s-1)^2 + ... = \frac{As^2 + 2s(1-A) + A}{s^2+1},$$

$$\begin{aligned} c_1(s-1) + c_2(s-1)^2 + ... &= \frac{As^2 + 2s(1-A) + A}{s^2+1} - 1 \\ &= \frac{As^2 + 2s(1-A) + A - s^2 - 1}{s^2+1} \\ &= \frac{(A-1)(s-1)^2}{s^2+1} \end{aligned}$$

and

$$c_1 + c_2(s-1) + ... = \frac{(A-1)(s-1)}{s^2+1}.$$

Passing to limit in the last equality yields $c_1 = 0$. Similarly, using straightforward calculations, we get $c_2 = \frac{A-1}{2}$.

Therefore, we obtain

$$(A-1)\left(1 + \frac{2\left(A - 1 - |c_1|^2\right)}{(A-1)^2 - |c_1|^2 + \left|(c_1 + 2c_2)(1-A) - c_1^2\right|}\right) = 2(A-1).$$

□

Theorem 2.4. *Let $Z(s) = 1 + c_1(s-1) + c_2(s-1)^2 + ...$ be an analytic in the right half plane with $\Re Z(s) \leq A$ for $\Re s \geq 0$ and $Z(s_1) = 1$ for $\Re s_1 > 0$. Suppose that $Z(s)$ is analytic at the point $s = 0$ of the imaginary axis with $\Re Z(0) = A$. Then*

$$\begin{aligned} |Z'(0)| \geq (A-1)\Bigg(1 + \frac{\Re s_1}{|s_1|^2} + \frac{(A-1)|s_1-1| - |Z'(1)||s_1+1|}{(A-1)|s_1-1| + |Z'(1)||s_1+1|} \\ \times \left[1 + \frac{(A-1)^2\left(1 - \frac{4\Re s_1}{|s_1+1|^2}\right) + |Z'(s_1)|\Re s_1|Z'(1)| - (A-1)|Z'(s_1)|\Re s_1 - (A-1)|Z'(1)|}{(A-1)^2\left(1 - \frac{4\Re s_1}{|s_1+1|^2}\right) + |Z'(s_1)|\Re s_1|Z'(1)| + (A-1)|Z'(s_1)|\Re s_1 + (A-1)|Z'(1)|}\frac{\Re s_1}{|s_1|^2}\right]\Bigg). \end{aligned} \tag{2.4}$$

The inequality (2.4) is sharp, with equality for each possible values $|Z'(1)|$ and $|Z'(s_1)|$.

Proof . Let

$$q(z) = \frac{z-a}{1-\overline{a}z}.$$

Also, let $k : D \to D$ be an analytic function and a point $a \in D$ in order to satisfy

$$\left|\frac{k(z)-k(a)}{1-\overline{k(a)}k(z)}\right| \le \left|\frac{z-a}{1-\overline{a}z}\right| = |q(z)|$$

and

$$|k(z)| \le \frac{|k(a)|+|q(z)|}{1+|k(a)|\,|q(z)|} \tag{2.5}$$

by Schwarz-pick lemma [6]. If $v : D \to D$ is an analytic function and $0 < |a| < 1$, letting

$$k(z) = \frac{v(z)-v(0)}{z\left(1-\overline{v(0)}v(z)\right)}$$

in (2.5), we obtain

$$\left|\frac{v(z)-v(0)}{z\left(1-\overline{v(0)}v(z)\right)}\right| \le \frac{\left|\frac{v(a)-v(0)}{a\left(1-\overline{v(0)}v(a)\right)}\right|+|q(z)|}{1+\left|\frac{v(a)-v(0)}{a\left(1-\overline{v(0)}v(a)\right)}\right||q(z)|}$$

and

$$|v(z)| \le \frac{|v(0)|+|z|\,\frac{|C|+|q(z)|}{1+|C||q(z)|}}{1+|v(0)|\,|z|\,\frac{|C|+|q(z)|}{1+|C||q(z)|}}, \tag{2.6}$$

where

$$C = \frac{v(a)-v(0)}{a\left(1-\overline{v(0)}v(a)\right)}.$$

If we take

$$v(z) = \frac{f(z)}{z\frac{z-a}{1-\overline{a}z}},$$

where

$$v(a) = \frac{f'(a)\left(1-|a|^2\right)}{a}, \quad v(0) = \frac{f'(0)}{-a}.$$

Then

$$C = \frac{\frac{f'(a)\left(1-|a|^2\right)}{a}+\frac{f'(0)}{a}}{a\left(1+\frac{f'(a)\left(1-|a|^2\right)}{a}\overline{\frac{f'(0)}{a}}\right)},$$

where $|C| \le 1$. Let $|v(0)| = \beta$ and

$$\mathrm{D} = \frac{\left|\frac{f'(a)\left(1-|a|^2\right)}{a}\right|+\left|\frac{f'(0)}{a}\right|}{|a|\left(1+\left|\frac{f'(a)\left(1-|a|^2\right)}{a}\right|\left|\frac{f'(0)}{a}\right|\right)}.$$

From (2.6), we get

$$|f(z)| \le |z|\,|q(z)|\,\frac{\beta+|z|\,\frac{\mathrm{D}+|q(z)|}{1+\mathrm{D}|q(z)|}}{1+\beta\,|z|\,\frac{\mathrm{D}+|q(z)|}{1+\mathrm{D}|q(z)|}}$$

and

$$\frac{1-|f(z)|}{1-|z|} \geq \frac{1+\beta\,|z|\,\frac{\mathrm{D}+|q(z)|}{1+\mathrm{D}|q(z)|}-\beta\,|z|\,|q(z)|-|q(z)|\,|z|^2\,\frac{\mathrm{D}+|q(z)|}{1+\mathrm{D}|q(z)|}}{(1-|z|)\left(1+\beta\,|z|\,\frac{\mathrm{D}+|q(z)|}{1+\mathrm{D}|q(z)|}\right)} = r(z). \tag{2.7}$$

Let $\kappa(z)=1+\beta\,|z|\,\frac{\mathrm{D}+|q(z)|}{1+\mathrm{D}|q(z)|}$ and $\tau(z)=1+\mathrm{D}\,|q(z)|$. Then

$$r(z)=\frac{1-|z|^2\,|q(z)|^2}{(1-|z|)\,\kappa(z)\tau(z)}+\mathrm{D}\,|q(z)|\,\frac{1-|z|^2}{(1-|z|)\,\kappa(z)\tau(z)}+\mathrm{D}\beta\,|z|\,\frac{1-|q(z)|^2}{(1-|z|)\,\kappa(z)\tau(z)}.$$

Since

$$\lim_{z\to-1}\kappa(z)=\lim_{z\to-1}1+\alpha\,|z|\,\frac{\mathrm{D}+|q(z)|}{1+\mathrm{D}\,|q(z)|}=1+\beta,$$
$$\lim_{z\to-1}\tau(z)=\lim_{z\to-1}1+\mathrm{D}\,|q(z)|=1+\mathrm{D}$$

and

$$1-|q(z)|^2=1-\left|\frac{z-a}{1-\overline{a}z}\right|^2=\frac{\left(1-|a|^2\right)\left(1-|z|^2\right)}{|1-\overline{a}z|^2}, \tag{2.8}$$

passing to the limit in (2.7) and using (2.8) gives

$$\begin{aligned}
|f'(-1)| &\geq \frac{2}{(1+\beta)\,(1+\mathrm{D})}\left(1+\frac{1-|a|^2}{|1+a|^2}+\mathrm{D}+\beta\mathrm{D}\frac{1-|a|^2}{|1+a|^2}\right)\\
&= 1+\frac{1-|a|^2}{|1+a|^2}+\frac{1-\beta}{1+\beta}\left(1+\frac{1-\mathrm{D}}{1+\mathrm{D}}\frac{1-|a|^2}{|1+a|^2}\right).
\end{aligned}$$

Moreover, since

$$\begin{aligned}
\frac{1-\beta}{1+\beta} &= \frac{1-|v(0)|}{1+|v(0)|}=\frac{1-\left|\frac{f'(0)}{a}\right|}{1+\left|\frac{f'(0)}{a}\right|}=\frac{|a|-|f'(0)|}{|a|+|f'(0)|}\\
&= \frac{\left|\frac{s_1-1}{s_1+1}\right|-\frac{|Z'(1)|}{A-1}}{\left|\frac{s_1-1}{s_1+1}\right|+\frac{|Z'(1)|}{A-1}}
\end{aligned}$$

$$\begin{aligned}
\frac{1-\mathrm{D}}{1+\mathrm{D}} &= \frac{1-\dfrac{\left|\frac{f'(a)\left(1-|a|^2\right)}{a}\right|+\left|\frac{f'(0)}{a}\right|}{|a|\left(1+\left|\frac{f'(a)(1-|a|^2)}{a}\right|\left|\frac{f'(0)}{a}\right|\right)}}{1+\dfrac{\left|\frac{f'(a)(1-|a|^2)}{a}\right|+\left|\frac{f'(0)}{a}\right|}{|a|\left(1+\left|\frac{f'(a)(1-|a|^2)}{a}\right|\left|\frac{f'(0)}{a}\right|\right)}}\\
&= \frac{1-\dfrac{\frac{\frac{|s_1+1|^2}{4}\frac{|Z'(s_1)|}{A-1}\left(1-\left|\frac{s_1-1}{s_1+1}\right|^2\right)}{\left|\frac{s_1-1}{s_1+1}\right|}+\frac{\frac{|Z'(1)|}{A-1}}{\left|\frac{s_1-1}{s_1+1}\right|}}{\left|\frac{s_1-1}{s_1+1}\right|\left(1+\frac{\frac{|s_1+1|^2}{4}\frac{|Z'(s_1)|}{A-1}\left(1-\left|\frac{s_1-1}{s_1+1}\right|^2\right)}{\left|\frac{s_1-1}{s_1+1}\right|}\frac{\frac{|Z'(1)|}{A-1}}{\left|\frac{s_1-1}{s_1+1}\right|}\right)}}{1+\dfrac{\frac{\frac{|s_1+1|^2}{4}\frac{|Z'(s_1)|}{A-1}\left(1-\left|\frac{s_1-1}{s_1+1}\right|^2\right)}{\left|\frac{s_1-1}{s_1+1}\right|}+\frac{\frac{|Z'(1)|}{A-1}}{\left|\frac{s_1-1}{s_1+1}\right|}}{\left|\frac{s_1-1}{s_1+1}\right|\left(1+\frac{\frac{|s_1+1|^2}{4}\frac{|Z'(s_1)|}{A-1}\left(1-\left|\frac{s_1-1}{s_1+1}\right|^2\right)}{\left|\frac{s_1-1}{s_1+1}\right|}\frac{\frac{|Z'(1)|}{A-1}}{\left|\frac{s_1-1}{s_1+1}\right|}\right)}},
\end{aligned}$$

$$\frac{1-\mathrm{D}}{1+\mathrm{D}} = \frac{\left|\frac{s_1-1}{s_1+1}\right|\left(1+\frac{\frac{|s_1+1|^2}{4}\frac{|Z'(s_1)|}{A-1}\left(1-\left|\frac{s_1-1}{s_1+1}\right|^2\right)}{\left|\frac{s_1-1}{s_1+1}\right|}\frac{\frac{|Z'(1)|}{A-1}}{\left|\frac{s_1-1}{s_1+1}\right|}\right)-\frac{\frac{|s_1+1|^2}{4}\frac{|Z'(s_1)|}{A-1}\left(1-\left|\frac{s_1-1}{s_1+1}\right|^2\right)}{\left|\frac{s_1-1}{s_1+1}\right|}-\frac{\frac{|Z'(1)|}{A-1}}{\left|\frac{s_1-1}{s_1+1}\right|}}{\left|\frac{s_1-1}{s_1+1}\right|\left(1+\frac{\frac{|s_1+1|^2}{4}\frac{|Z'(s_1)|}{A-1}\left(1-\left|\frac{s_1-1}{s_1+1}\right|^2\right)}{\left|\frac{s_1-1}{s_1+1}\right|}\frac{\frac{|Z'(1)|}{A-1}}{\left|\frac{s_1-1}{s_1+1}\right|}\right)+\frac{\frac{|s_1+1|^2}{4}\frac{|Z'(s_1)|}{A-1}\left(1-\left|\frac{s_1-1}{s_1+1}\right|^2\right)}{\left|\frac{s_1-1}{s_1+1}\right|}+\frac{\frac{|Z'(1)|}{A-1}}{\left|\frac{s_1-1}{s_1+1}\right|}}$$

$$=\frac{\left|\frac{s_1-1}{s_1+1}\right|^2+\frac{|s_1+1|^2}{4}\frac{|Z'(s_1)|}{A-1}\left(1-\left|\frac{s_1-1}{s_1+1}\right|^2\right)\frac{|Z'(1)|}{A-1}-\frac{|s_1+1|^2}{4}\frac{|Z'(s_1)|}{A-1}\left(1-\left|\frac{s_1-1}{s_1+1}\right|^2\right)-\frac{|Z'(1)|}{A-1}}{\left|\frac{s_1-1}{s_1+1}\right|^2+\frac{|s_1+1|^2}{4}\frac{|Z'(s_1)|}{A-1}\left(1-\left|\frac{s_1-1}{s_1+1}\right|^2\right)\frac{|Z'(1)|}{A-1}+\frac{|s_1+1|^2}{4}\frac{|Z'(s_1)|}{A-1}\left(1-\left|\frac{s_1-1}{s_1+1}\right|^2\right)+\frac{|Z'(1)|}{A-1}},$$

and

$$1-\left|\frac{s_1-1}{s_1+1}\right|^2=\frac{4\Re s_1}{|s_1+1|^2}$$

$$\left|1+\frac{s_1-1}{s_1+1}\right|^2=\frac{4\,|s_1|^2}{|s_1+1|^2},$$

we take

$$|f'(-1)| \geq \left(1+\frac{\Re s_1}{|s_1|^2}+\frac{(A-1)|s_1-1|-|Z'(1)||s_1+1|}{(A-1)|s_1-1|+|Z'(1)||s_1+1|}\right.$$
$$\left.\times\left[1+\frac{(A-1)^2\left(1-\frac{4\Re s_1}{|s_1+1|^2}\right)+|Z'(s_1)|\Re s_1|Z'(1)|-(A-1)|Z'(s_1)|\Re s_1-(A-1)|Z'(1)|}{(A-1)^2\left(1-\frac{4\Re s_1}{|s_1+1|^2}\right)+|Z'(s_1)|\Re s_1|Z'(1)|+(A-1)|Z'(s_1)|\Re s_1+(A-1)|Z'(1)|}\frac{\Re s_1}{|s_1|^2}\right]\right).$$

From definition of $f(z)$, we have

$$f'(z)=\frac{\frac{4}{(1-z)^2}Z'\left(\frac{1+z}{1-z}\right)(1-A)}{\left(Z\left(\frac{1+z}{1-z}\right)+1-2A\right)^2}$$

and

$$|f'(-1)| \leq \frac{|Z'(0)|}{A-1}.$$

Thus, we obtain the inequality (2.4).

Now, we shall show that the inequality (2.4) is sharp.

Since $v(z)=\frac{f(z)}{z\frac{z-a}{1-\overline{a}z}}$ is an analytic function in the unit disc D and $|v(z)| \leq 1$ for $|z|<1$, we obtain

$$|f'(0)| \leq |a|$$

and

$$|f'(a)| \leq \frac{|a|}{1-|a|^2}.$$

We take $a \in (-1,0)$ and arbitrary two numbers x and y. Let

$$\mathtt{M}=\frac{\frac{x\left(1-|a|^2\right)}{a}+\frac{x}{a}}{a\left(1+xy\frac{1-|a|^2}{a}\right)}=\frac{1}{a^2}\frac{y\left(1-|a|^2\right)+x}{1+xy\frac{1-|a|^2}{a}}.$$

The auxiliary function

$$f(z)=z\frac{z-a}{1-\overline{a}z}\frac{\frac{-x}{a}+z\frac{\mathtt{M}+\frac{z-a}{1-\overline{a}z}}{1+\mathtt{M}\frac{z-a}{1-\overline{a}z}}}{1-\frac{x}{a}z\frac{\mathtt{M}+\frac{z-a}{1-\overline{a}z}}{1+\mathtt{M}\frac{z-a}{1-\overline{a}z}}} \tag{2.9}$$

is analytic in D and $|f(z)|<1$ for $z \in D$.

From (2.9), with the simple calculations, we obtain

$$f'(-1) = -\left(1 + \frac{1-a^2}{(1+a)^2} + \frac{x+a}{-x+a}\left(1 + \frac{1+\mathtt{M}}{1-\mathtt{M}}\frac{1-a^2}{(1+a)^2}\right)\right)$$

and

$$|f'(-1)| = 1 + \frac{1-a^2}{(1+a)^2} + \frac{x+a}{-x+a}\left(1 + \frac{1+\mathtt{M}}{1-\mathtt{M}}\frac{1-a^2}{(1+a)^2}\right).$$

Thus, since $0 < \Re s_1 < 1$, $\Im s_1 = 0$, $a = \frac{s_1-1}{s_1+1}$ and choosing suitable signs of the numbers x and y, the last equality shows that (2.4) is sharp. □

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