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On the Fine Spectra of the Zweier Matrix as an Operator Over the Weighted Sequence Space $\ell_p(w)$

R. Lashkaripour, J. Fathi*

Department of Mathematic, Faculty of Mathematics, University of Sistan and Baluchestan, Zahedan, Iran.

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Abstract

In the present paper, the fine spectrum of the Zweier matrix as an operator over the weighted sequence space $\ell_p(w)$, have been examined.

Keywords: Spectrum of an Operator, Matrix Mapping, Zweier Matrix, Weighted Sequence Space.
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1. Introduction

In functional analysis, the spectrum of an operator generalizes the notion of eigenvalues for matrices. The spectrum of an operator over a Banach space is partitioned into three parts, which are the point spectrum, the continuous spectrum and the residual spectrum. The calculation of three parts of the spectrum of an operator is called calculating the fine spectrum of the operator.

By ω, we denote the space of all real or complex valued sequences. Any vector subspace of ω is called a sequence space. For any Banach space X, let X^* denote the space of all continuous linear functionals f on X with the norm $||f|| = \sup\{|f(x)| : ||x|| \leq 1\}$. If X is any subset of ω then the set $X^\beta = \{a \in \omega : \sum_{k=0}^{\infty} a_k x_k \;\; conveges \;\; for \;\; all \;\; x \in X\}$ is called the $\beta-$dual of X. For $1 \leq p < \infty$ we write l_p for the space of all p-absolutely convergent series, with

$$\|x\|_{\ell_p} := (\sum_{k=0}^{\infty} |x_k|^p)^{\frac{1}{p}}.$$

*Corresponding author
Email addresses: lashkari@hamoon.usb.ac.ir (R. Lashkaripour), fathi756@gmail.com (J. Fathi)

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Let (w_n) be a decreasing, positive sequence. By $\ell_p(w)$ we mean the space of all sequences $x = (x_n)$ having $\sum\limits_{k=0}^{\infty} w_k|x_k|^p$ convergent, with norm $\|x\|_{\ell_p(w)} = (\sum\limits_{k=0}^{\infty} w_k|x_k|^p)^{\frac{1}{p}}$.
When $w_n = 1$ for all n, this coincides with ℓ_p in the usual sense. This space is a BK with AK with respect to

$$\|x\|_{\ell_p(w)} = \left(\sum_{k=0}^{\infty} w_k|x_k|^p\right)^{\frac{1}{p}},$$

by ([17] , Theorem 4.3.6 and 4.3.12). And we define

$$\ell_\infty(w) = \{x = (x_k) \in v : \|x\|_{\ell_\infty(w)} := \sup_n w_n|x_n| < \infty\}$$

Let μ and ν be two sequence spaces and $A = (a_{n,k})$ be an infinite matrix operator of real or complex numbers $a_{n,k}$, where $n, k \in \{0, 1, 2, \ldots\}$. We say that A defines a matrix mapping from μ into ν and denote it by $A : \mu \longrightarrow \nu$, if for every sequence $x = (x_k) \in \mu$ the sequence $Ax = ((Ax)_n)$, the A-transform of x, is in ν, where $(Ax)_n = \sum\limits_{k=0}^{\infty} a_{n,k}x_k$.
The Zweier matrix Z_s represented by the following band matrix,

$$Z_s = \begin{bmatrix} s & 0 & 0 & 0 & 0 & \cdots \\ 1-s & s & 0 & 0 & 0 & \cdots \\ 0 & 1-s & s & 0 & 0 & \cdots \\ 0 & 0 & 1-s & s & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where s is a real number such that $s \neq 0, 1$.

The fine spectrum of the Cesaro operator on the sequence space ℓ_p for $(1 < p < \infty)$ has been studied by Gonzalez [10]. Also, Wenger [16] examined the fine spectrum of the integer power of the Cesaro operator over c, and Rhoades [15] generalized this result to the weighted mean methods. Reade [14] worked the spectrum of the Cesaro operator over the sequence space c_0. Okutoyi [13] computed the spectrum of the Cesaro operator over the sequence space bv. The fine spectrum of the Rhally operators on the sequence spaces c_0 and c is studied by Yildirim [18]. The fine spectra of the Cesaro operator over the sequences space c_0 and bv_p have determined by Akhmedov and Basar [1, 4]. Akhmedov and Basar [2, 3] have studied the fine spectrum of the difference operator Δ over the sequence spaces ℓ_p, and bv_p, where $(1 \leq p < \infty)$. The fine spectrum of the Zweier matrix as an operator over the sequence spaces ℓ_1 and bv_1 have been examined by Altay and Karakus [7]. Altay and Basar [5, 6] determined the fine spectrum of the difference operator Δ over the sequence spaces c_0, c and ℓ_p, where $(0 < p < 1)$. The fine spectrum of the difference operator Δ over the sequence spaces ℓ_1 and bv is investigated by Kayaduman and Furkan [11].

In this paper , we study the fine spectra of the Zweier matrix Z_s as an operator over the weighted sequence space $\ell_p(w)$.

2. Preliminaries, Background and Notations

Let X and Y be Banach spaces and $T : X \longrightarrow Y$, also be a bounded linear operator. By $R(T)$, we denote the range of T, i.e.,

$$R(T) = \{y \in Y : y = Tx, x \in X\}.$$

By $B(X)$, we denote the set of all bounded linear operator on X into itself. If X is any Banach space and $T \in B(X)$ then the *adjoint* T^* of T is a bounded linear operator on the dual X^* of X defined by $(T^*\psi)(x) = \psi(Tx)$ for all $\psi \in X^*$ and $x \in X$ with $\|T\| = \|T^*\|$.
Let $X \neq \phi$ be a complex normed space and $T : \mathcal{D}(T) \longrightarrow X$, also be a bounded linear operator with domain $\mathcal{D} \subseteq X$. With T, we associate the operator $T_\lambda = T - \lambda I$, where λ is a complex number and I is the identity operator on $\mathcal{D}(T)$, if T_λ has an inverse, which is linear, we denote it by T_λ^{-1}, that is $T_\lambda^{-1} = (T - \lambda I)^{-1}$ and call it the *resolvent* operator of T.

The name resolvent is appropriate, since T_λ^{-1} helps to solve the equation $T_\lambda x = y$. Thus, $x = T_\lambda^{-1} y$ provided T_λ^{-1} exists. More important, the investigation of properties of T_λ^{-1} will be basic for an understanding of the operator T itself. Naturally, many properties of T_λ and T_λ^{-1} depend on λ, and spectral theory is concerned with those properties. For instance, we shall be interested in the set of all λ in the complex plane such that T_λ^{-1} exists. Boundedness of T_λ^{-1} is another property that will be essential. We shall also ask for what λ the domain of T_λ^{-1} is dense in X, to name just a few aspects. For our investigation of T, T_λ and T_λ^{-1}, we shall need some basic concepts in spectral theory which are given as follows(see [11], pp. 370-371]):

Definition 2.1. *Let $X \neq \phi$ be a complex normed space and $T : \mathcal{D}(T) \longrightarrow X$, be a linear operator with domain $\mathcal{D} \subseteq X$. A regular value of T is a complex number λ such that*
$(R1)$ T_λ^{-1} *exists,*
$(R2)$ T_λ^{-1} *is bounded,*
$(R3)$ T_λ^{-1} *is defined on a set which is dense in X.*

The *resolvent* set $\rho(T, X)$ of T is the set of all *regular* value λ of T. Its complement $\sigma(T, X) = \mathbb{C} - \rho(T, X)$ in the complex plane $\mathbb{C}$ is called the *spectrum* of T. Furthermore, the spectrum $\sigma(T, X)$ is partitioned into three disjoint sets as follows:
The *point spectrum* $\sigma_p(T, X)$ is the set of all $\lambda \in \mathbb{C}$ such that T_λ^{-1} dose not exist. The element of $\sigma_p(T, X)$ is called *eigenvalue* of T.
The *continuous spectrum* $\sigma_c(T, X)$ is the set of all $\lambda \in \mathbb{C}$ such that T_λ^{-1} exists and satisfies $(R3)$ but not $(R2)$, that is, T_λ^{-1} is unbounded.
The *residual spectrum* $\sigma_r(T, X)$ is the set of all $\lambda \in \mathbb{C}$ such that T_λ^{-1} exists but do not satisfy $(R3)$, that is, the domain of T_λ^{-1} is not dense in X. The condition $(R2)$ may or may not holds good.

Goldberg's classification of operator $T_\lambda = (T - \lambda I)$ (see [9], PP. 58-71): Let X be a Banach space and $T_\lambda = (T - \lambda I) \in B(X)$, where λ is a complex number. Again let $R(T_\lambda)$ and T_λ^{-1} be

denote the range and inverse of the operator T_λ, respectively. Then following possibilities may occur:
(A) $R(T_\lambda) = X$,
(B) $R(T_\lambda) \neq \overline{R(T_\lambda)} = X$,
(C) $\overline{R(T_\lambda)} \neq X$,
and
(1) T_λ is injective and T_λ^{-1} is continuous,
(2) T_λ is injective and T_λ^{-1} is discontinuous,
(3) T_λ is not injective.
If these possibilities are combined in all possible ways, nine different states are created. These are labelled by: $A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2$ *and* C_3. If λ is a complex number such that $T_\lambda \in A_1$ *or* $T_\lambda \in B_1$, then λ is in the resolvent set $\rho(T, X)$ of T on X. The other classifications give rise to the fine spectrum of T. We use $\lambda \in B_2\sigma(T, X)$ means the operator $T_\lambda \in B_2$, i.e. $R(T_\lambda) \neq \overline{R(T_\lambda)} = X$ and T_λ is injective but T_λ^{-1} is discontinuous. Similarly others.

The spectra radius $r_\sigma(T)$ of an operator $T \in B(X)$ on a complex Banach space X is the radius of the smallest closed disk centered at the origin of the complex $\lambda-$plane , i.e.,

$$r_\sigma(T) = \sup_{\lambda \in \sigma(T,X)} |\lambda|$$

and containing $\sigma(T, X)$. It is obvious that the inequality

$$r_\sigma(T) \leq \|T\|$$

holds for the spectra radius of a bounded linear operator T on a complex Banach space.

Lemma 2.2. *([9], p.59). A linear operator T has a dense range if and only if the adjoint T^* is one to one.*

Lemma 2.3. *([9], p.60). The adjoint operator T^* is onto if and and only if T has a bounded inverse.*

Lemma 2.4. *Let $\sum_{n=0}^{\infty} b_n x^n$ be a power series and r be its radius of convergence. Also, let $\{a_n\}$ be a positive sequence such that $\lim_{n\to\infty} \sqrt[n]{a_n} = 1$. Then the radius of convergence of power series $\sum_{n=0}^{\infty} a_n b_n x^n$ is r.*

3. The Fine Spectra of the Operator Z_s Over the Weighted Sequence Space $\ell_p(w)$, $(1 \leq p < \infty)$

In this section, the dual of the space $\ell_p(w)$ is determined. Moreover the point spectrum, the continuous spectrum and the residual spectrum of the operator Z_s over the weighted sequence space $\ell_p(w), (1 \leq p < \infty)$ have been examined.

Theorem 3.1. *If* $1 < p < \infty$, *and* $q = \frac{p}{p-1}$ *then* $\ell_p^*(w) \cong \ell_q\left(\frac{1}{w^{\frac{q}{p}}}\right)$, *where* $\frac{1}{w^{\frac{q}{p}}} = \left(\frac{1}{w_k^{\frac{q}{p}}}\right)_{k=0}^{\infty}$ *and* $\ell_1^*(w) \cong \ell_\infty(\frac{1}{w})$.

Proof . We write $z = w^{\frac{1}{p}}$, that is $z_k = w_k^{\frac{1}{p}}$ for $k = 0, 1, 2, ...,$ and observe that $x \in \ell_p(w)$ if and only if $y = zx = (z_k x_k)_{k=0}^{\infty} \in \ell_p$. Also given any sequence a, we put $b = \frac{a}{z} = (\frac{a_k}{z_k})_{k=0}^{\infty}$ for $k = 0, 1, 2, ...,$ and observe that obviously $a_k x_k = b_k y_k$ for $k = 0, 1, 2, ...,$ and so $\sum_{k=0}^{\infty} a_k x_k$ converges for all $x \in \ell_p(w)$ if and only if $\sum_{k=0}^{\infty} b_k y_k$ converges for all $y \in \ell_p$. Therefore $a \in \ell_p^\beta(w)$ if and only if $b = (\frac{a}{z}) \in \ell_p^\beta = \ell_q \quad (q = \infty \quad for \quad p = 1)$. If $1 < p < \infty$ then $a \in \ell_p^\beta(w)$ if and only if

$$\sum_{k=0}^{\infty} |b_k|^q = \sum_{k=0}^{\infty} \left|\frac{a_k}{z_k}\right|^q = \sum_{k=0}^{\infty} \frac{1}{w^{\frac{q}{p}}}.|a_k|^p < \infty, \quad that \quad is, \quad a \in \ell_q\left(\frac{1}{w^{\frac{q}{p}}}\right);$$

if $p = 1$ then $a \in \ell_p^\beta(w)$ if and only if

$$\sup_k \left|\frac{a_k}{z_k}\right| = \sup_k \left|\frac{a_k}{w_k}\right| < \infty, \quad that \quad is, \quad a \in \ell_\infty\left(\frac{1}{w}\right).$$

Since $\ell_p(w)$ has AK for $1 \le p < \infty$, the spaces $\ell_p^\beta(w)$ and $\ell_p^*(w)$ are isomorphic by ([17], Theorem 7.2.9) and it is clear that the norms on $\ell_p^\beta(w)$ and $\ell_p^*(w)$ are the same, and this completes the proof. □

Theorem 3.2. $Z_s : \ell_p(w) \longrightarrow \ell_p(w)$ *is a bounded linear operator satisfying the inequality*

$$\|Z_s\|_{\ell_p(w)} \le |s| + |1 - s|.$$

Proof . The linearity of the operator Z_s is trivial and so is omitted. Let $x = (x_k)$ be an arbitrary sequence in $\ell_p(w)$. Then, using Minkowski's inequality and taking $x_{-1} = 0$, we have

$$\begin{aligned}
\|Z_s x\|_{\ell_p(w)} &= \left(\sum_{n=0}^{\infty} w_n |Z_s x_n|^p\right)^{\frac{1}{p}} \\
&= \left(\sum_{n=0}^{\infty} w_n |(1-s)x_{n-1} + s x_n|^p\right)^{\frac{1}{p}} && (x_{-1} = 0) \\
&\le \left(\sum_{n=0}^{\infty} w_{n-1} |(1-s)x_{n-1}|^p\right)^{\frac{1}{p}} + \left(\sum_{n=0}^{\infty} w_n |s x_n|^p\right)^{\frac{1}{p}} && (w_{-1} = 0) \\
&= \left(|1-s|^p \sum_{n=0}^{\infty} w_{n-1} |x_{n-1}|^p\right)^{\frac{1}{p}} + \left(|s|^p \sum_{n=0}^{\infty} w_n |x_n|^p\right)^{\frac{1}{p}} \\
&= (|s| + |1-s|)\|x\|_{\ell_p(w)},
\end{aligned}$$

Therefore

$$\|Z_s\|_{\ell_p(w)} \le |s| + |1 - s|. \tag{3.1}$$

□

Theorem 3.3. $\sigma(Z_s, \ell_1(w)) = \{\lambda \in C : |\lambda - s| \leq |1 - s|\}$.

Proof . Suppose $y = (y_k) \in \ell_1(w)$. Solving the equation $(Z_s - \lambda I)x = y$ for x in terms of y, one can calculate that

$$x_k = \sum_{j=0}^{k} \frac{(s-1)^{k-j}}{(s-\lambda)^{k-j+1}}\, y_j \qquad\qquad k = 0, 1, 2, \ldots .$$

Then, we have

$$\begin{aligned}\sum_{k=0}^{\infty} w_k|x_k| &= \sum_{k=0}^{\infty} w_k \left|\sum_{j=0}^{k} \frac{(s-1)^{k-j}}{(s-\lambda)^{k-j+1}}\, y_j\right| \\ &\leq \frac{1}{|s-\lambda|}\left(\sum_{j=0}^{\infty}\left|\frac{s-1}{s-\lambda}\right|^j\right)\left(\sum_{k=0}^{\infty} w_k|y_k|\right).\end{aligned}$$

One can see by the ratio test that the series

$$\sum_{j=0}^{\infty}\left|\frac{s-1}{s-\lambda}\right|^j$$

and

$$\sum_{k=0}^{\infty} w_k|x_k|$$

converge, if $|s-\lambda| > |s-1|$. Hence, the equation $(Z_s - \lambda I)x = y$ has an unique solution $x \in \ell_1(w)$, if $|s-\lambda| > |s-1|$ and $(Z_s - \lambda I)^{-1}$ is a bounded linear operator for such $\lambda' s$, which is what we wished to prove. □

We should remark for the reader that from now on the index p has different meanings in the notation of the spaces $\ell_p(w)$, $\ell_p^*(w)$ and in the spectrums $\sigma_p(Z_s, \ell_p(w))$, $\sigma_p(Z_s^*, \ell_p^*(w))$ which occur the following theorems.

Theorem 3.4. $\sigma_p(Z_s, \ell_p(w)) = \phi$.

Proof . Suppose that $Z_s x = \lambda x$ for $x \neq (0, 0, 0, \ldots)$ in $\ell_p(w)$. Then, by solving the system of linear equations

$$\begin{cases} sx_0 = \lambda x_o \\ (1-s)x_0 + sx_1 = \lambda x_1 \\ (1-s)x_1 + sx_2 = \lambda x_2 \\ \quad\vdots \\ (1-s)x_k + sx_{k+-1} = \lambda x_k \\ \quad\vdots \end{cases}$$

we find that, if x_{n_0} is the first non-zero entry of the sequence $x = (x_n)$, then $\lambda = s$ and $(1-s)x_{n_0} + sx_{n_0+1} = sx_{n_0+1}$, we get $(1-s)x_{n_0} = 0$. Since $s \neq 1$ this contradicts the fact that $x_{n_0} \neq 0$, which means that $\sigma_p(Z_s, \ell_p(w)) = \phi$ and this completes the proof. □

In the following theorems, for the case $1 < p < \infty$, we suppose that $\lim_{n\to\infty} \sqrt[n]{w_n} = 1$.

Theorem 3.5. $\sigma_p(Z_s^*, \ell_p^*(w)) = \begin{cases} \{\lambda \in C : |\lambda - s| < |1-s|\}, & (1 < p < \infty) \\ \{\lambda \in C : |\lambda - s| \leq |1-s|\}, & (p = 1). \end{cases}$

Proof . Suppose $Z_s^* g = \lambda g$ for $g \neq (0,0,0,\ldots)$ in $\ell_p^*(w)$. Then, by solving the system of linear equations

$$\begin{cases} sg_0 + (1-s)g_1 &= \lambda g_o \\ sg_1 + (1-s)g_2 &= \lambda g_1 \\ &\vdots \\ sg_k + (1-s)g_{k+1} &= \lambda g_k \\ &\vdots \end{cases}$$

we obtain that

$$g_k = \left(\frac{\lambda - s}{1-s}\right)^k g_0, \qquad (k \in \mathbb{N}).$$

By Lemma 2.4, $g = (g_k) \in \ell_p^*(w)$ if and only if $|\lambda - s| < |1-s|$ and $g = (g_k) \in \ell_1^*(w)$ if and only if $|\lambda - 1| \leq |1-s|$. This establishes the statement. □

Theorem 3.6. $\sigma_r(Z_s, \ell_p(w)) = \begin{cases} \{\lambda \in C : |\lambda - s| < |1-s|\}, & (1 < p < \infty) \\ \{\lambda \in C : |\lambda - s| \leq |1-s|\}, & (p = 1). \end{cases}$

Proof . We show that the operator $Z_s - \lambda I$ has an inverse and $\overline{R(Z_s - \lambda I)} \neq \ell_p(w)$ for λ satisfying $|\lambda - s| < |1-s|$, whenever $1 < p < \infty$. For $\lambda \neq s$ the operator $Z_s - \lambda I$ is triangle and has an inverse. For $\lambda = s$ the operator $Z_s - \lambda I$ is one to one, and so has an inverse. Applying Theorem 3.5, we deduce that $Z_s^* - \lambda I$ is not one to one. Now, Lemma 2.2 yields the fact that the range of the operator $Z_s - \lambda I$ is not dense in $\ell_p(w)$ and this step concludes the proof of the first part of the theorem.
In the same way, we deal with the case $p = 1$. □

Theorem 3.7. $s \in C_1\sigma(Z_s, \ell_p(w))$

Proof . By Theorem 3.5 and Lemma 2.2,, we have $Z_s - sI \in C$. Additionally, s is not in $\sigma_p(Z_s, \ell_p(w))$ by Theorem 3.4, Hence, $Z_s - sI$ has an inverse. Thus, $Z_s - sI \in (1) \cup (2)$. To establish the fact $Z_s - sI \in (1)$, it is enough to show, by Lemma 2.3,, that $Z_s^* - sI$ is onto. For a given $y = (y_k) \in \ell_q(w)$, we must find that $x = (x_k) \in \ell_q(w)$ such that $(Z_s^* - sI)x = y$. A direct calculation yields that

$$x_n = \frac{1}{1-s} y_{n-1},$$

for all $n \in \mathbb{N}$. This means $Z_s^* - sI$ is onto, as desired. □

Theorem 3.8. $\sigma_c(Z_s, \ell_1(w)) = \phi$.

Proof . Since $\sigma_r(Z_s, \ell_1(w)) = \{\lambda \in C : |\lambda - s| \leq |1-s|\}$, $\sigma_p(Z_s, \ell_1(w)) = \phi$ and $\sigma(Z_s, \ell_1(w))$ is the disjoint union of the parts $\sigma_p(Z_s, \ell_1(w))$, $\sigma_r(Z_s, \ell_1(w))$ and $\sigma_c(Z_s, \ell_1(w))$ we deduce that $\sigma_c(Z_s, \ell_1(w)) = \phi$. □

Theorem 3.9. *For $1 < p < \infty$, $\sigma_c(Z_s, \ell_p(w)) = \{\lambda \in C : |\lambda - s| = |1 - s|\}$.*

Proof . Since $\lambda \neq s$, $Z_s - \lambda I$ is a triangle and has an inverse. Therefore, $Z_s^* - \lambda I$ is one to one from Lemma 2.2, which is what we wished to prove. □

Theorem 3.10. *For $1 < p < \infty$, $\sigma(Z_s, \ell_p(w)) = \{\lambda \in C : |\lambda - s| \leq |1 - s|\}$.*

Proof . Since $\sigma_r(Z_s, \ell_p(w)) = \{\lambda \in C : |\lambda - s| < |1 - s|\}$, $\sigma_p(Z_s, \ell_p(w)) = \phi$, $\sigma_c(Z_s, \ell_p(w)) = \{\lambda \in C : |\lambda - s| = |1 - s|\}$ and $\sigma(Z_s, \ell_p(w))$ is the disjoint union of the parts $\sigma_p(Z_s, \ell_p(w))$, $\sigma_r(Z_s, \ell_p(w))$ and $\sigma_c(Z_s, \ell_p(w))$ we deduce that $\sigma(Z_s, \ell_p(w)) = \{\lambda \in C : |\lambda - s| \leq |1 - s|\}$. □

Theorem 3.11. $\|Z_s\|_{\ell_p(w)} = |s| + |1 - s|$.

Proof . Since $r_\sigma(T) = |s| + |1 - s|$ applying Theorems 3.3 and 3.10, we have

$$|s| + |1 - s| \leq \|Z_s\|_{\ell_p(w)}.$$

Combining the last inequality with (3.1) we obtain the result. □

If we set $w_n = 1$ for all n, then we have the following corollary.

Corollary 3.12. (1) *$Z_s \in B(\ell_p)$ with the norm $\|Z_s\|_{\ell_p} = |s| + |1 - s|$,*

(2) $\sigma(Z_s, \ell_p) = \{\lambda \in C : |\lambda - s| \leq |1 - s|\}$,

(3) $\sigma_p(Z_s, \ell_p) = \phi$,

(4) $$\sigma_p(Z_s^*, \ell_p^*) = \begin{cases} \{\lambda \in C : |\lambda - s| < |1 - s|\}, & (1 < p < \infty) \\ \{\lambda \in C : |\lambda - s| \leq |1 - s|\}, & (p = 1). \end{cases}$$

(5) $$\sigma_r(Z_s, \ell_p) = \begin{cases} \{\lambda \in C : |\lambda - s| < |1 - s|\}, & (1 < p < \infty) \\ \{\lambda \in C : |\lambda - s| \leq |1 - s| & (p = 1). \end{cases}$$

(6) $$\sigma_c(Z_s, \ell_p) = \begin{cases} \{\lambda \in C : |\lambda - s| = |1 - s|\}, & (1 < p < \infty) \\ \phi, & (p = 1). \end{cases}$$

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