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Common fixed point theorems on Branciari metric spaces via simulation functions

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Abstract

In this article, we secure couple of exciting common fixed point theorems via simulation functions in Branciari metric spaces context. These results improve, complement and generalize the recent fixed point theorems of Aydi et al. [Results Math., 71(2017), no. 1-2, 73-92] and few others also. Our findings are aptly endorsed by some interesting non-trivial examples which also illustrate the usefulness of these generalizations. Finally, we discuss an application of our conceived results to a certain type of integral equations.

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1. Introduction and preliminaries

The concept of a simulation function η and the notion of $\mathcal{F}$-contractions with respect to η were defined by Khojasteh et al. [19]. They generalized the Banach contraction principle [6] and established several fixed point theorems via such auxiliary functions in complete metric spaces. Firstly we note down the definition of a simulation function.

Definition 1.1. *[19] A map $\eta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is said to be a simulation function if the following properties hold:*

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($\eta 1$) $\eta(0,0)=0$;
($\eta 2$) $\eta(t,s)<t-s$ *for each* $s,t\in[0,\infty)$;
($\eta 3$) *for any two sequences* (s_n) *and* (t_n) *in* $[0,\infty)$ *such that* $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n > 0$, *we have* $\limsup_{n\to\infty} \ \eta(t_n,s_n)<0$.

The family of simulation functions is denoted by $\mathcal{F}$. The following is the definition of a $\mathcal{Z}$-contraction with respect to a simulation function η.

Definition 1.2. *[19] Suppose $f:X\to X$ be any self-mapping and $\eta\in\mathcal{Z}$ be a simulation function. Then f is said to be a $\mathcal{Z}$-contraction with respect to η, if for all $x,y\in X$,*

$$\eta(d(fx,fy),d(x,y))\geq 0$$

holds.

For some examples, notions and interesting results on simulation functions, the readers are referred to [3, 8, 9, 18, 19, 21, 23, 25, 26]. Now we recall the idea of (c)-comparison functions. Let us consider the set of functions $\psi:[0,\infty)\to[0,\infty)$ such that

(Ψ_1) ψ is non-decreasing;

(Ψ_2) $\sum_{n=1}^{+\infty}\psi^n(t)<\infty$ for all $t>0$, where ψ^n is the n^{th}-iterate of ψ.

These functions are known in the literature as (c)-comparison functions. The family of such functions are denoted by Ψ. Also, it can be easily proved that if ψ is a (c)-comparison function, then $\psi(t)<t$ for any $t>0$. Now we recollect the notion of α-admissible mappings.

Definition 1.3. *[5] Given that $f,g:X\to X$ are two self-maps and $\alpha:X\times X\to[0,\infty)$. Then the pair (f,g) is said to be α-admissible if*

$$x,y\in X,\ \ \alpha(x,y)\geq 1 \textit{ implies } \min\{\alpha(fx,fy),\alpha(gx,fy),\alpha(fx,gy),\alpha(gx,gy)\}\geq 1. \tag{1.1}$$

If $f=g$, then f is called α-admissible [27].

On the other hand, in 2000, Branciari [7] initiated the concept of Branciari (or rectangular) metric spaces, where the triangular inequality is replaced by a rectangular one. Such spaces are explored thoroughly and as a result, many fixed point results have come forth in this setting (see for example [1, 2, 4, 11, 12, 13, 14, 15, 16, 17, 22, 29]). Here we note the following proposition by Kirk and Shahzad [20] in Branciari metric spaces which is useful in the sequel.

Proposition 1.4. *[20] Suppose that (x_n) is a Cauchy sequence in a Branciari metric space such that*

$$\lim_{n\to\infty} d(x_n,u)=\lim_{n\to\infty} d(x_n,z)=0$$

where $u,z\in X$. Then $u=z$.

Recently, Aydi et al. [5] coined the notion of (α,ψ)-Meir-Keeler-contractions in the setting of Branciari metric spaces and obtained some common fixed point theorems involving these contractions. Further, the authors also introduced the concept of a generalized (α-ψ)-contractive pair of mappings as follows:

Definition 1.5. *Let (X,d) be a Branciari metric space and $f,g : X \to X$ be two given mappings. We say that (f,g) is a generalized $(\alpha$-$\psi)$-contractive pair of mappings if there are two functions $\alpha : X \times X \to [0,\infty)$ and $\psi \in \Psi$ such that*

$$\alpha(x,y)d(fx,gy) \leq \psi(M_{f,g}(x,y)) \text{ and } \alpha(x,y)d(gx,fy) \leq \psi(M_{g,f}(x,y)) \tag{1.2}$$

for all $x,y \in X$, where

$$M_{h,k}(x,y) = \max\{d(x,y), d(x,hx), d(y,ky)\},$$

for $h,k : X \to X$.

In this paper, firstly we introduce the new notion of a generalized Λ-contraction pair of mappings with respect to any simulation function η. Further, we secure a couple of common fixed point theorems in complete Branciari metric spaces concerning such contractions. The obtained results generalize the findings of Aydi et al. [5] and unify many other existing results in the literature. However, our findings are suitably validated by constructive numerical examples. Moreover, we investigate for some existence and uniqueness criteria to guarantee a unique common solution to a pair of integral equations via our obtained findings.

2. Main Results

In this section, we first introduce the notion of a generalized Λ-contraction pair. Then we secure couple of common fixed point results involving the aforementioned contractions. Throughout the article, $\mathbb{N}$ and $\mathbb{N}_0$ stands for the set of natural numbers and whole numbers.

Definition 2.1. *Let (X,d) be a Branciari metric space and $f,g : X \to X$ be two self-mappings. We say that (f,g) is a generalized Λ-contraction pair of mappings with respect to a simulation function η, if there are two functions $\alpha : X \times X \to [0,\infty)$ and $\psi \in \Psi$ such that for all $x,y \in X$*

$$(a) \quad \eta\Big(\alpha(x,y)d(fx,gy), M_{f,g}(x,y)\Big) \geq 0 \quad \text{and} \quad (b) \quad \eta\Big(\alpha(x,y)d(gx,fy), M_{g,f}(x,y)\Big) \geq 0. \tag{2.1}$$

Whenever $f = g$, the mapping f is said to be a generalized Λ-contraction with respect to η. In the case when either (a) or (b) holds, (f,g) is said to be a semi-generalized Λ-contraction pair of mappings with respect to η.

In the following, we state our first main result.

Theorem 2.2. *Let (X,d) be a complete Branciari metric space, $f,g : X \to X$ be two self-maps and $\eta \in \mathcal{F}$. Suppose that*

(i) (f,g) is a generalized Λ-contraction pair of mappings with respect to η;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0,fx_0) \geq 1$, $\alpha(x_0,gx_0) \geq 1$ and $\alpha(x_0,fgx_0) \geq 1$;
(iii) both of f and g are continuous and for any sufficiently large $n \in \mathbb{Z}^+$, $(fg)^n x_0 = (gf)^n x_0$.

Then there exists a common fixed point $u \in X$ of f and g.

Proof . From assumption *(ii)*, there exists $x_0 \in X$ such that $\alpha(x_0,fx_0) \geq 1$, $\alpha(x_0,gx_0) \geq 1$ and $\alpha(x_0,fgx_0) \geq 1$. We construct a sequence (x_n) in X as follows:

$$x_n = \begin{cases} gx_{n-1}, & \text{if } n \text{ is even,} \\ fx_{n-1}, & \text{if } n \text{ is odd} \end{cases} \tag{2.2}$$

for all $n \in \mathbb{N}$. So $x_1 = fx_0$ and $x_2 = gx_1$ for all $n \in \mathbb{N}_0$. Since the pair (f, g) is α-admissible, we have

$$\alpha(x_0, x_1) = \alpha(x_0, fx_0) \geq 1 \Rightarrow \alpha(x_1, x_2) = \alpha(fx_0, gx_1) = \alpha(fx_0, gfx_0) \geq 1.$$

By induction, we obtain

$$\alpha(x_n, x_{n+1}) \geq 1, \text{ for all } n \in \mathbb{N}_0. \tag{2.3}$$

Starting with $\alpha(x_0, x_2) = \alpha(x_0, gfx_0) \geq 1 \Rightarrow \alpha(x_1, x_3) = \alpha(fx_0, fx_2) = \alpha(fx_0, f(gfx_0)) \geq 1$, and so,

$$\alpha(x_n, x_{n+2}) \geq 1, \text{ for all } n \in \mathbb{N}_0. \tag{2.4}$$

Suppose that there exists n_0 such that $x_{2n_0} = x_{2n_0+1}$ for some $n_0 \in \mathbb{N}$. Then $u = x_{2n_0}$ is a common fixed point of f and g. Indeed, $u = x_{2n_0} = x_{2n_0+1} = fx_{2n_0} = fu$. Now, we show that $d(x_{2n_0+1}, x_{2n_0+2}) = 0$. Since

$$\begin{aligned} 0 &\leq \eta\big(\alpha(x_{2n_0}, x_{2n_0+1})d(x_{2n_0+1}, x_{2n_0+2}), M_{f,g}(x_{2n_0}, x_{2n_0+1})\big) \\ &= \eta\big(\alpha(x_{2n_0}, x_{2n_0+1})d(x_{2n_0+1}, x_{2n_0+2}), \max\{d(x_{2n_0}, x_{2n_0+1}), d(x_{2n_0+1}, x_{2n_0+2})\}\big) \\ &= \eta\big(\alpha(x_{2n_0}, x_{2n_0+1})d(x_{2n_0+1}, x_{2n_0+2}), d(x_{2n_0+1}, x_{2n_0+2})\big) \\ &< d(x_{2n_0+1}, x_{2n_0+2}) - \alpha(x_{2n_0}, x_{2n_0+1})d(x_{2n_0+1}, x_{2n_0+2}). \end{aligned}$$

Therefore

$$\alpha(x_{2n_0}, x_{2n_0+1})d(x_{2n_0+1}, x_{2n_0+2}) < d(x_{2n_0+1}, x_{2n_0+2}) \Rightarrow \alpha(x_{2n_0}, x_{2n_0+1}) < 1,$$

which is a contradiction. Hence, $u = x_{2n_0+1} = x_{2n_0+2} = gx_{2n_0+1} = gu$. Therefore, u is a common fixed point of f and g. Similarly when $x_{2n_0-1} = x_{2n_0}$ for some $n_0 \in \mathbb{N}$, then also we can deduce that u is a common fixed point of f and g. For the rest of the proof, we can assume that

$$x_n \neq x_{n+1} \text{ for all } n \in \mathbb{N}. \tag{2.5}$$

Set

$$M(x_n, x_m) = \begin{cases} M_{g,f}(x_n, x_m), & \text{if } n \text{ is odd and if } m \text{ is even,} \\ M_{f,g}(x_n, x_m), & \text{if } n \text{ is even and if } m \text{ is odd} \end{cases}$$

for all $m, n \in \mathbb{N}$.

Step 1. We now prove that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.6}$$

First, we claim that $\max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\} = d(x_{2n}, x_{2n+1})$, for all $n \in \mathbb{N}_0$. We argue by a contradiction. Suppose that for some $n \in \mathbb{N}_0$, $\max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\} = d(x_{2n+1}, x_{2n+2})$. For such $n \in \mathbb{N}_0$, we have

$$\begin{aligned} 0 &\leq \eta\big(\alpha(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2}), M(x_{2n}, x_{2n+1})\big) \\ &= \eta\big(\alpha(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2}), d(x_{2n+1}, x_{2n+2})\big) \\ &< d(x_{2n+1}, x_{2n+2}) - \alpha(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2}). \end{aligned}$$

Hence, $\alpha(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2}) < d(x_{2n+1}, x_{2n+2})$, and so $\alpha(x_{2n}, x_{2n+1}) < 1$, which is a contradiction with respect to (2.3). Thus, $\max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\} = d(x_{2n}, x_{2n+1})$, for all $n \in \mathbb{N}_0$. Using (2.3) and Definition 2.1, it follows that

$$\begin{aligned} 0 &\leq \eta\big(\alpha(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2}), M(x_{2n}, x_{2n+1})\big) \\ &= \eta\big(\alpha(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2}), M_{f,g}(x_{2n}, x_{2n+1})\big) \\ &= \eta\big(\alpha(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2}), \max\{d(x_{2n}, x_{2n+1}), d(x_{2n}, fx_{2n}), d(x_{2n+1}, gx_{2n+1})\}\big) \\ &= \eta\big(\alpha(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2}), \max\{d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\}\big) \\ &= \eta\big(\alpha(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2}), \max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\}\big) \\ &= \eta\big(\alpha(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2}), d(x_{2n}, x_{2n+1})\big) \\ &< d(x_{2n}, x_{2n+1}) - \alpha(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2}), \end{aligned} \tag{2.7}$$

for all $n \in \mathbb{N}_0$. Thus, we have

$$d(x_{2n+1}, x_{2n+2}) \leq \alpha(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2}) < d(x_{2n}, x_{2n+1}), \tag{2.8}$$

for all $n \in \mathbb{N}_0$. Similarly, we can obtain that $\max\{d(x_{2n-1}, x_{2n}), d(x_{2n}, x_{2n+1})\} = d(x_{2n-1}, x_{2n})$ for all $n \in \mathbb{N}$. Therefore,

$$\begin{aligned}
0 &\leq \eta\big(\alpha(x_{2n-1}, x_{2n})d(x_{2n}, x_{2n+1}), M(x_{2n-1}, x_{2n})\big) \\
&= \eta\big(\alpha(x_{2n-1}, x_{2n})d(x_{2n}, x_{2n+1}), M_{g,f}(x_{2n-1}, x_{2n})\big) \\
&= \eta\big(\alpha(x_{2n-1}, x_{2n})d(x_{2n}, x_{2n+1}), \max\{d(x_{2n-1}, x_{2n}), d(x_{2n-1}, gx_{2n-1}), d(x_{2n}, fx_{2n})\}\big) \\
&= \eta\big(\alpha(x_{2n-1}, x_{2n})d(x_{2n}, x_{2n+1}), \max\{d(x_{2n-1}, x_{2n}), d(x_{2n-1}, x_{2n}), d(x_{2n}, x_{2n+1})\}\big) \\
&= \eta\big(\alpha(x_{2n-1}, x_{2n})d(x_{2n}, x_{2n+1}), \max\{d(x_{2n-1}, x_{2n}), d(x_{2n}, x_{2n+1})\}\big) \\
&= \eta\big(\alpha(x_{2n-1}, x_{2n})d(x_{2n}, x_{2n+1}), d(x_{2n-1}, x_{2n})\big) \\
&< d(x_{2n-1}, x_{2n}) - \alpha(x_{2n-1}, x_{2n})d(x_{2n}, x_{2n+1}),
\end{aligned}$$

for all $n \geq 1$. Thus, we have

$$d(x_{2n}, x_{2n+1}) \leq \alpha(x_{2n-1}, x_{2n})d(x_{2n}, x_{2n+1}) < d(x_{2n-1}, x_{2n}), \tag{2.9}$$

for all $n \geq 1$. From (2.8) and (2.9), we have $d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$, for all $n \geq 1$. So, there exists some $\epsilon \geq 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = \epsilon$. We shall prove that $\epsilon = 0$. On the contrary, suppose that $\epsilon > 0$. From (2.8) and (2.9), we have $\lim_{n\to\infty} \alpha(x_{n-1}, x_n)d(x_n, x_{n+1}) = \epsilon$. Set $s_n = \alpha(x_{n-1}, x_n)d(x_n, x_{n+1})$ and $t_n = d(x_n, x_{n+1})$. By Definition 1.1-(η3), we have

$$0 \leq \limsup_{n\to\infty} \eta\left(d\left(x_n, x_{n+1}\right), \alpha\left(x_{n-1}, x_n\right) d\left(x_n, x_{n+1}\right)\right) = \limsup_{n\to\infty} \eta\left(t_n, s_n\right) < 0,$$

which is a contradiction. Therefore, $\epsilon = 0$.

Step 2. We now prove

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0. \tag{2.10}$$

On the contrary we consider that

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = a > 0. \tag{2.11}$$

Also, we construct another sequence (y_n) defined as

$$y_0 = x_0, \quad y_1 = gy_0, \; y_2 = fy_1, \ldots, y_{2n} = fx_{2n-1} \quad \text{and} \quad y_{2n+1} = gy_{2n} \ldots,$$

for all $n \in \mathbb{N}$. Now, by (iii), we can derive that

$$x_{2n} = (TS)^n x_0 = (ST)^n x_0 = (ST)^n y_0 = y_{2n}$$

for sufficiently large positive integer n. Also, using similar calculations as in the proof of

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0,$$

we can obtain

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0. \tag{2.12}$$

Further, from (2.4) we have, $\alpha(x_{2n-1}, x_{2n+1}) \geq 1$ and hence,

$$\alpha(x_{2n-1}, x_{2n+1}) = \alpha(x_{2n-1}, y_{2n+1}) = \alpha(fx_{2n-2}, gy_{2n}) = \alpha(fx_{2n-2}, gx_{2n}) \geq 1.$$

On the other hand, we have

$$\begin{aligned}0 &\le \eta\big(\alpha(x_{2n-1}, x_{2n+1})d(gx_{2n-1}, fx_{2n+1}), M_{g,f}(x_{2n-1}, y_{2n+1})\big)\\ &< M_{g,f}(x_{2n-1}, y_{2n+1}) - \alpha(x_{2n-1}, y_{2n+1})d(gx_{2n-1}, fx_{2n+1}).\end{aligned}$$

This implies that

$$\alpha(x_{2n-1}, y_{2n+1})d(gx_{2n-1}, fx_{2n+1}) < M_{g,f}(x_{2n-1}, y_{2n+1}). \tag{2.13}$$

Now using (2.13), we have

$$\begin{aligned}d(x_{2n}, x_{2n+2}) &= d(x_{2n}, y_{2n+2})\\ &= d(gx_{2n-1}, fy_{2n+1})\\ &\le \alpha(x_{2n-1}, y_{2n+1})d(gx_{2n-1}, fy_{2n+1})\\ &\le M_{g,f}(x_{2n-1}, y_{2n+1})\\ &= \max\{d(x_{2n-1}, x_{2n+1}), d(x_{2n-1}, gx_{2n-1}), d(y_{2n+1}, fy_{2n+1})\}\\ &= \max\{d(x_{2n-1}, x_{2n+1}), d(x_{2n-1}, x_{2n}), d(y_{2n+1}, y_{2n+2})\}\\ &= d(x_{2n-1}, x_{2n+1})\\ &= d(fx_{2n-2}, gx_{2n})\\ &\le \alpha(x_{2n-2}, x_{2n})d(fx_{2n-2}, gx_{2n})\\ &\le M_{f,g}(x_{2n-2}, x_{2n})\\ &= \max\{d(x_{2n-2}, x_{2n}), d(x_{2n-2}, fx_{2n-2}), d(x_{2n+1}, gx_{2n})\}\\ &= \max\{d(x_{2n-2}, x_{2n}), d(x_{2n-2}, x_{2n-1}), d(x_{2n+1}, x_{2n+1})\}\\ &= d(x_{2n-2}, x_{2n})\end{aligned} \tag{2.14}$$

for all $n \in \mathbb{N}_0$. Thus, we have

$$\begin{aligned}d(x_{2n}, x_{2n+2}) &\le \alpha(x_{2n-2}, x_{2n})d(fx_{2n-2}, gx_{2n})\\ &= \alpha(x_{2n-2}, x_{2n})d(x_{2n-1}, x_{2n+1}) \le d(x_{2n-2}, x_{2n}),\end{aligned} \tag{2.15}$$

for all $n \in \mathbb{N}_0$. From (2.15), we have

$$\lim_{n\to\infty} \alpha(x_{2n-2}, x_{2n})d(x_{2n-1}, x_{2n+1}) = a.$$

Set $s_n = \alpha(x_{2n-2}, x_{2n})d(x_{2n-1}, x_{2n+1})$ and $t_n = d(x_{2n}, x_{2n+2})$. By Definition 1.1-($\eta$3), we have

$$0 \le \limsup_{n\to\infty}\ \eta\left(d\left(x_{2n}, x_{2n+2}\right), \alpha(x_{2n-2}, x_{2n})d(x_{2n-1}, x_{2n+1})\right) = \limsup_{n\to\infty}\ \eta\left(t_n, s_n\right) < 0,$$

which is a contradiction. Therefore, $a = 0$.

Step 3. Here we prove that $x_{2n+1} \neq x_{2m+1}$ and $x_{2n} \neq x_{2m}$ for all $n \neq m$. The discussion naturally splits into the following two cases:

Case-I: if for some $m, n \in \mathbb{N}_0$, with $m > n$, $x_{2n} = x_{2m}$;

Case-II: if for some $m, n \in \mathbb{N}_0$, with $m > n$, $x_{2n+1} = x_{2m+1}$.

In Case-I, by Step 1, the sequence $(d(x_n, x_{n+1}))$ is decreasing, so we have,

$$\begin{aligned}d(x_{2n}, x_{2n+1}) &= d(x_{2n}, fx_{2n})\\ &= d(x_{2m}, fx_{2m})\\ &= d(x_{2m}, x_{2m+1})\\ &< d(x_{2n}, x_{2n+1}),\end{aligned}$$

a contradiction. In Case-II, by Step 1, the sequence $(d(x_n, x_{n+1}))$ is decreasing, thus we get,

$$\begin{aligned} d(x_{2n+2}, x_{2n+1}) &= d(gx_{2n+1}, x_{2n+1}) \\ &= d(gx_{2m+1}, x_{2m+1}) \\ &= d(x_{2m+2}, x_{2m+1}) \\ &< d(x_{2n+2}, x_{2n+1}), \end{aligned}$$

which is a contradiction. Thus, we can assume that $x_n \neq x_m$ for all $n \neq m$.

Step 4. We now prove that (x_n) is a Cauchy sequence. Suppose, on the contrary, that (x_n) is not a Cauchy sequence. Since (x_n) is a sequence in X with distinct elements (that is, $x_n \neq x_m$ for $n \neq m$), and since from Step 1 and Step 2, $d(x_n, x_{n+1})$ and $d(x_n, x_{n+2})$ tend to 0 as $n \to \infty$, using Lemma 3.3 from [12], there exist $\epsilon > 0$ and two subsequences (m_k) and (n_k) of positive integers such that $n_k > m_k > k$ and the following four sequences tend to ϵ as $n \to \infty$

$$d(x_{m_k}, x_{n_k})\ , d(x_{m_k}, x_{n_{k+1}})\ , d(x_{m_{k-1}}, x_{n_k})\ , d(x_{m_{k-1}}, x_{n_{k+1}}). \tag{2.16}$$

Thus, by Step 1, Step 2 and (2.16), we find that

$$\limsup_{k\to\infty} M(x_{m_k}, x_{n_k}) = \epsilon. \tag{2.17}$$

Since the pair (f, g) is α-admissible, we may get $\alpha(x_{m_k}, x_{n_k}) \geq 1$. Regarding (f, g) is a generalized Λ-contraction pair of mappings with respect to η and considering m_k as an odd number and n_k as an even number, we get that

$$\begin{aligned} 0 &\leq \eta\big(\alpha(x_{m_k}, x_{n_k}) d(x_{n_{k+1}}, x_{m_{k+1}}), M(x_{m_k}, x_{n_k})\big) \\ &= \eta\big(\alpha(x_{m_k}, x_{n_k}) d(x_{n_{k+1}}, x_{m_{k+1}}), M_{f,g}(x_{m_k}, x_{n_k})\big) \\ &= \eta\big(\alpha(x_{m_k}, x_{n_k}) d(x_{n_{k+1}}, x_{m_{k+1}}), \max\{d(x_{m_k}, x_{n_k}), d(x_{m_k}, fx_{m_k}), d(x_{n_k}, gx_{n_k})\}\big) \\ &= \eta\big(\alpha(x_{m_k}, x_{n_k}) d(x_{n_{k+1}}, x_{m_{k+1}}), \max\{d(x_{m_k}, x_{n_k}), d(x_{m_k}, x_{m_{k+1}}), d(x_{n_k}, x_{n_{k+1}})\}\big) \\ &< \max\{d(x_{m_k}, x_{n_k}), d(x_{m_k}, x_{m_{k+1}}), d(x_{n_k}, x_{n_{k+1}})\} - \alpha(x_{m_k}, x_{n_k}) d(x_{n_{k+1}}, x_{m_{k+1}}), \end{aligned} \tag{2.18}$$

for all $k \in \mathbb{N}$. Consequently, we have

$$\begin{aligned} 0 &< d(x_{m_{k+1}}, x_{n_{k+1}}) \leq \alpha(x_{m_k}, x_{n_k}) d(x_{m_{k+1}}, x_{n_{k+1}}) \\ &< \max\{d(x_{m_k}, x_{n_k}), d(x_{m_k}, x_{m_{k+1}}), d(x_{n_k}, x_{n_{k+1}})\}, \end{aligned} \tag{2.19}$$

for all $k \in \mathbb{N}$. From (2.19), together with (2.16) and (2.17), we get

$$\lim_{k\to\infty} \alpha(x_{m_k}, x_{n_k}) d(x_{m_{k+1}}, x_{n_{k+1}}) = \epsilon.$$

Set $s_n = M(x_{m_k}, x_{n_k})$ and $t_n = \alpha(x_{m_k}, x_{n_k}) d(x_{m_{k+1}}, x_{n_{k+1}})$. By Definition 1.1-($\eta 3$) and the relation (2.17), we can conclude that

$$0 \leq \limsup_{k\to\infty} \eta\Big(\alpha(x_{m_k}, x_{n_k}) d(x_{m_{k+1}}, x_{n_{k+1}}), M(x_{m_k}, x_{n_k})\Big) < 0,$$

which is a contradiction. Therefore, (x_n) is a Cauchy sequence. Since X is a complete Branciari metric space, there exists $u \in X$ such that (x_n) converges to u. Thus,

$$\lim_{n\to\infty} d(x_n, u) = 0. \tag{2.20}$$

Step 5. We claim that u is a common fixed point of f and g. Since, f and g are continuous, by (2.20), we have

$$\lim_{n\to\infty} d(x_{2n+1}, fu) = \lim_{n\to\infty} d(fx_{2n}, fu) = 0,$$

and

$$\lim_{n\to\infty} d(x_{2n}, gu) = \lim_{n\to\infty} d(gx_{2n-1}, gu) = 0.$$

By Proposition 1.4, we conclude that $fu = u = gu$. Hence, u is a common fixed point of f and g. □

Our next result involves a semi-generalized Λ-contraction pair of mappings and here we note that down.

Theorem 2.3. *Let (X, d) be a complete Branciari metric space, $f, g : X \to X$ be two self-maps and $\eta \in \Lambda$. Suppose that*

(i) (f, g) is a semi-generalized Λ-contraction pair of mappings with respect to η;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$, $\alpha(x_0, gx_0) \geq 1$ and $\alpha(x_0, fgx_0) \geq 1$;
(iii) for every $x, y \in X$, $\alpha(x, y) = \alpha(y, x)$;
(iv) both of f and g are continuous and for any sufficiently large $n \in \mathbb{Z}^+$, $(fg)^n x_0 = (gf)^n x_0$.

Then there exists a common fixed point $u \in X$ of f and g.

Proof . We omit the proof. It is similar to the proof of Theorem 2.2. □

3. Some Examples

This section takes care of constructive examples which authenticates our obtained Theorem 2.2 concerning a generalized Λ-contraction pair of self-maps.

Example 3.1. *We consider the complete metric space $X = \{0, \frac{1}{n} : n \in \mathbb{N}, n \geq 2\}$ endowed with the Branciari metric,*

$$d(x,y) = \begin{cases} 0, & \text{if } x = y; \\ 2, & \text{if } x, y \in \{\frac{1}{n} : n \in \mathbb{N}, n \geq 2\}; \\ \frac{1}{2n}, & \text{if } x = \frac{1}{n}, y = 0,\ x = 0, y = \frac{1}{n}. \end{cases}$$

It can be easily verified that d is not a metric but it is a Branciari metric. Now we define mappings $f : X \to X$ such that

$$fx = \begin{cases} 0, & \text{if } x = 0; \\ \frac{1}{n}, & \text{if } x = \frac{1}{n}, \end{cases}$$

and $g : X \to X$ such that $gx = 0$ for all $x \in X$. We also consider $\eta(t, s) = \lambda s - t$, $t, s \in [0, \infty)$, as the required simulation function, where $\lambda = \frac{9}{10}$ and define $\alpha : X \times X \to [0, \infty)$ as

$$\alpha(x,y) = \begin{cases} \frac{1}{2}, & \text{if } x = y; \\ 0, & \text{if } x = \frac{1}{n},\ y = \frac{1}{m} \text{ with } n \neq m; \\ \frac{1}{n}, & x = \frac{1}{n},\ y = 0 \text{ or } x = 0,\ y = \frac{1}{n}. \end{cases}$$

Therefore, we have,

$$\eta(\alpha(x,y)d(fx,gy), M_{f,g}(x,y)) = \lambda M_{f,g}(x,y) - \alpha(x,y)d(fx,gy). \tag{3.1}$$

Here we have three cases.

Case-I: *when $x = y$;*

Subcase-I: $x = 0 = y$;
In this case, we obtain

$$\begin{aligned}M_{f,g}(x,y) &= \max\{d(x,y), d(x,fx), d(y,gy)\}\\ &= \max\{d(0,0), d(0,f0), d(0,g0)\}\\ &= \max\{d(0,0), d(0,0), d(0,0)\}\\ &=0,\end{aligned}$$

and

$$d(f0, g0) = 0.$$

Putting these values in (3.1), *we obtain,*

$$\begin{aligned}\eta(\alpha(x,y)d(fx,gy), M_{f,g}(x,y)) = \eta(0,0) &=0\\ &\geq 0.\end{aligned}$$

This is the trivial case.
Subcase-II: $x = \frac{1}{n} = y$;
We get,

$$\begin{aligned}M_{f,g}(x,y) &= \max\{d(x,y), d(x,fx), d(y,gy)\}\\ &= \max\left\{d\left(\frac{1}{n},\frac{1}{n}\right), d\left(\frac{1}{n}, f\frac{1}{n}\right), d\left(\frac{1}{n}, g\frac{1}{n}\right)\right\}\\ &= \max\left\{d\left(\frac{1}{n},\frac{1}{n}\right), d\left(\frac{1}{n},\frac{1}{n}\right), d\left(\frac{1}{n},0\right)\right\}\\ &= \max\left\{0, \frac{1}{2n}\right\}\\ &=\frac{1}{2n},\end{aligned}$$

and

$$d\left(f\frac{1}{n}, g\frac{1}{n}\right) = d\left(\frac{1}{n}, 0\right) = \frac{1}{2n}.$$

Putting the values in (3.1), *we obtain,*

$$\begin{aligned}\eta(\alpha(x,y)d(fx,gy), M_{f,g}(x,y)) &=\eta\left(\frac{1}{4n}, \frac{1}{2n}\right)\\ &=\frac{9}{10}.\frac{1}{2n} - \frac{1}{4n}\\ &=\frac{1}{4n}\left(\frac{9}{5} - 1\right)\\ &\geq 0.\end{aligned}$$

Case-II: $x = \frac{1}{n}$, $y = 0$ *or* $x = 0$, $y = \frac{1}{n}$;
Subcase-I: $x = \frac{1}{n}$, $y = 0$;

For this case, we have,

$$\begin{aligned} M_{f,g}(x,y) &= \max\{d(x,y), d(x,fx), d(y,gy)\} \\ &= \max\left\{d\left(\frac{1}{n},0\right), d\left(\frac{1}{n}, f\frac{1}{n}\right), d(0,g0)\right\} \\ &= \max\left\{\frac{1}{2n}, d\left(\frac{1}{n},\frac{1}{n}\right), d(0,0)\right\} \\ &= \frac{1}{2n}, \end{aligned}$$

and

$$d\left(f\frac{1}{n}, g0\right) = d\left(\frac{1}{n},0\right) = \frac{1}{2n}.$$

From (3.1)*, we obtain,*

$$\begin{aligned} \eta(\alpha(x,y)d(fx,gy), M_{f,g}(x,y)) &= \eta\left(\frac{1}{n}.\frac{1}{2n}, \frac{1}{2n}\right) \\ &= \frac{9}{10}.\frac{1}{2n} - \frac{1}{2n^2} \\ &= \frac{1}{2n}\left(\frac{9}{10} - \frac{1}{n}\right) \\ &\geq 0, \quad [as \;\; n \geq 2]. \end{aligned}$$

Subcase-II: $x = 0,\ y = \frac{1}{n}$;
We get,

$$\begin{aligned} M_{f,g}(x,y) &= \max\{d(x,y), d(x,fx), d(y,gy)\} \\ &= \max\left\{d\left(0,\frac{1}{n}\right), d(0,f0), d\left(\frac{1}{n}, g\frac{1}{n}\right)\right\} \\ &= \max\left\{\frac{1}{2n}, d(0,0), d\left(\frac{1}{n},0\right)\right\} \\ &= \max\left\{\frac{1}{2n}, 0\right\} \\ &= \frac{1}{2n}, \end{aligned}$$

and

$$d\left(f0, g\frac{1}{n}\right) = d(0,0) = 0.$$

Hence taking care of (3.1)*, we obtain,*

$$\begin{aligned} \eta(\alpha(x,y)d(fx,gy), M_{f,g}(x,y)) &= \eta\left(0, \frac{1}{2n}\right) \\ &= \frac{9}{10}.\frac{1}{2n} \\ &\geq 0. \end{aligned}$$

Case-III: $x = \frac{1}{n}$, $y = \frac{1}{m}$ *with* $n \neq m$;
So, we obtain,

$$\begin{aligned} M_{f,g}(x,y) =& \max\left\{d\left(\frac{1}{n},\frac{1}{m}\right), d\left(\frac{1}{n}, f\frac{1}{n}\right), d\left(\frac{1}{m}, g\frac{1}{m}\right)\right\} \\ =& \max\left\{2, d\left(\frac{1}{n},\frac{1}{n}\right), d\left(\frac{1}{m},0\right)\right\} \\ =& \max\left\{2, 0, \frac{1}{2m}\right\} \\ =& 2, \end{aligned}$$

and

$$d\left(f\frac{1}{n}, g\frac{1}{m}\right) = d\left(\frac{1}{n},0\right) = \frac{1}{2n}.$$

Putting the values in (3.1), *we obtain,*

$$\begin{aligned} \eta(\alpha(x,y)d(fx,gy), M_{f,g}(x,y)) =& \eta\,(0,2) \\ =& \frac{9}{10}.2 \\ \geq& 0. \end{aligned}$$

So condition (*a*) *of Theorem 2.2 is satisfied. Similarly, one can check for condition* (*b*). *We skip the verification. Therefore,* f *and* g *satisfy both the hypotheses of Theorem 2.2 and using the theorem,* f *and* g *have a common fixed point and it is* $w = 0 \in X$.

Example 3.2. *Consider* $X = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}\}$ *and define the generalized metric* d *on* X *as follows:*

$$\begin{aligned} d\left(\frac{1}{2},\frac{2}{3}\right) =& d\left(\frac{3}{4},\frac{4}{5}\right) = 0.2, \\ d\left(\frac{1}{2},\frac{4}{5}\right) =& d\left(\frac{2}{3},\frac{3}{4}\right) = 0.3, \\ d\left(\frac{1}{2},\frac{3}{4}\right) =& d\left(\frac{2}{3},\frac{4}{5}\right) = 0.6, \\ d(x,x) =& 0 \quad \textit{for all} \quad x \in X. \end{aligned}$$

We define mappings $f, g : X \to X$ *such that* $fx = \frac{3}{4}$ *for all* $x \in X$ *and* $g : X \to X$ *such that*

$$gx = \begin{cases} \frac{3}{4}, & \textit{if } x = \frac{1}{2}, \frac{3}{4}; \\ \frac{4}{5}, & \textit{if } x = \frac{2}{3}; \\ \frac{2}{3}, & \textit{if } x = \frac{4}{5}. \end{cases}$$

We also consider $\eta(t,s) = \frac{s}{s+1} - t$, $t, s \in [0,\infty)$, *as the required simulation function and* $\alpha(x,y) = 1$, $x, y \in X$. *Therefore, we have,*

$$\eta(\alpha(x,y)d(fx,gy), M_{f,g}(x,y)) = \frac{M_{f,g}(x,y)}{M_{f,g}(x,y)+1} - \alpha(x,y)d(fx,gy) \tag{3.2}$$

To verify criteria (a), we have to consider the following cases.

Case-Ia: $x = \frac{1}{2}, \quad y = \frac{2}{3}.$

For this case, we get,

$$\begin{aligned} M_{f,g}\left(\frac{1}{2}, \frac{2}{3}\right) =& \max\left\{d\left(\frac{1}{2}, \frac{2}{3}\right), d\left(\frac{1}{2}, f\frac{1}{2}\right), d\left(\frac{2}{3}, g\frac{2}{3}\right)\right\} \\ =& \max\left\{d\left(\frac{1}{2}, \frac{2}{3}\right), d\left(\frac{1}{2}, \frac{3}{4}\right), d\left(\frac{2}{3}, \frac{4}{5}\right)\right\} \\ =& \max\{0.2, 0.6, 0.6\} \\ =& 0.6, \end{aligned}$$

and

$$d\left(f\frac{1}{2}, g\frac{2}{3}\right) = d\left(\frac{3}{4}, \frac{4}{5}\right) = 0.2.$$

Using these in (3.2)*, we obtain,*

$$\begin{aligned} \eta(\alpha(x,y)d(fx,gy), M_{f,g}(x,y)) =& \eta(0.2, 0.6) \\ =& 0.175 \\ \geq& 0. \end{aligned}$$

Case-IIa: $x = \frac{1}{2}, \quad y = \frac{3}{4}.$

Therefore we have,

$$\begin{aligned} M_{f,g}\left(\frac{1}{2}, \frac{3}{4}\right) =& \max\left\{d\left(\frac{1}{2}, \frac{3}{4}\right), d\left(\frac{1}{2}, f\frac{1}{2}\right), d\left(\frac{3}{4}, g\frac{3}{4}\right)\right\} \\ =& \max\left\{d\left(\frac{1}{2}, \frac{3}{4}\right), d\left(\frac{1}{2}, \frac{3}{4}\right), d\left(\frac{3}{4}, \frac{3}{4}\right)\right\} \\ =& \max\{0.6, 0.6, 0\} \\ =& 0.6, \end{aligned}$$

and

$$d\left(f\frac{1}{2}, g\frac{3}{4}\right) = d\left(\frac{3}{4}, \frac{3}{4}\right) = 0.$$

However, considering (3.2)*, we get*

$$\begin{aligned} \eta(\alpha(x,y)d(fx,gy), M_{f,g}(x,y)) =& \eta(0, 0.6) \\ =& 0.375 \\ \geq& 0. \end{aligned}$$

Case-IIIa: $x = \frac{1}{2}, \quad y = \frac{4}{5}.$

In this case, we have

$$\begin{aligned} M_{f,g}\left(\frac{1}{2}, \frac{4}{5}\right) =& \max\left\{d\left(\frac{1}{2}, \frac{4}{5}\right), d\left(\frac{1}{2}, f\frac{1}{2}\right), d\left(\frac{4}{5}, g\frac{4}{5}\right)\right\} \\ =& \max\left\{d\left(\frac{1}{2}, \frac{4}{5}\right), d\left(\frac{1}{2}, \frac{3}{4}\right), d\left(\frac{4}{5}, \frac{2}{3}\right)\right\} \\ =& \max\{0.3, 0.6, 0.6\} \\ =& 0.6, \end{aligned}$$

and

$$d\left(f\frac{1}{2}, g\frac{4}{5}\right) = d\left(\frac{3}{4}, \frac{2}{3}\right) = 0.3.$$

Using the values in (3.2), *we obtain,*

$$\begin{aligned}\eta(\alpha(x,y)d(fx,gy), M_{f,g}(x,y)) =& \eta(0.3, 0.6)\\ =& 0.075\\ \geq& 0.\end{aligned}$$

Case-IVa: $x = \frac{2}{3}, \quad y = \frac{3}{4}$.

Hence we have,

$$\begin{aligned}M_{f,g}\left(\frac{2}{3}, \frac{3}{4}\right) =& \max\left\{d\left(\frac{2}{3}, \frac{3}{4}\right), d\left(\frac{2}{3}, f\frac{2}{3}\right), d\left(\frac{3}{4}, g\frac{3}{4}\right)\right\}\\ =& \max\left\{d\left(\frac{2}{3}, \frac{3}{4}\right), d\left(\frac{2}{3}, \frac{3}{4}\right), d\left(\frac{3}{4}, \frac{3}{4}\right)\right\}\\ =& \max\{0.3, 0\}\\ =& 0.3,\end{aligned}$$

and

$$d\left(f\frac{2}{3}, g\frac{3}{4}\right) = d\left(\frac{3}{4}, \frac{3}{4}\right) = 0.$$

Indeed, from (3.2), *we obtain,*

$$\begin{aligned}\eta(\alpha(x,y)d(fx,gy), M_{f,g}(x,y)) =& \eta(0, 0.3)\\ =& 0.375\\ \geq& 0.\end{aligned}$$

Case-Va: $x = \frac{2}{3}, \quad y = \frac{4}{5}$.

For this case, we get,

$$\begin{aligned}M_{T,S}\left(\frac{2}{3}, \frac{4}{5}\right) =& \max\left\{d\left(\frac{2}{3}, \frac{4}{5}\right), d\left(\frac{2}{3}, f\frac{2}{3}\right), d\left(\frac{4}{5}, g\frac{4}{5}\right)\right\}\\ =& \max\left\{d\left(\frac{2}{3}, \frac{4}{5}\right), d\left(\frac{2}{3}, \frac{3}{4}\right), d\left(\frac{4}{5}, \frac{2}{3}\right)\right\}\\ =& \max\{0.6, 0.3, 0.6\}\\ =& 0.6,\end{aligned}$$

and

$$d\left(f\frac{2}{3}, g\frac{4}{5}\right) = d\left(\frac{3}{4}, \frac{2}{3}\right) = 0.3.$$

Using these in (3.2), *we obtain,*

$$\begin{aligned}\eta(\alpha(x,y)d(fx,gy), M_{f,g}(x,y)) =& \eta(0.3, 0.6)\\ =& 0.075\\ \geq& 0.\end{aligned}$$

Case-VIa: $x = \frac{3}{4}, \quad y = \frac{4}{5}$.

Here we obtain,

$$\begin{aligned} M_{f,g}\left(\frac{3}{4},\frac{4}{5}\right) =& \max\left\{d\left(\frac{3}{4},\frac{4}{5}\right), d\left(\frac{3}{4}, f\frac{3}{4}\right), d\left(\frac{4}{5}, g\frac{4}{5}\right)\right\} \\ =& \max\left\{d\left(\frac{3}{4},\frac{4}{5}\right), d\left(\frac{3}{4},\frac{3}{4}\right), d\left(\frac{4}{5},\frac{2}{3}\right)\right\} \\ =& \max\{0.2, 0, 0.6\} \\ =& 0.6, \end{aligned}$$

and

$$d\left(f\frac{3}{4}, g\frac{4}{5}\right) = d\left(\frac{3}{4},\frac{2}{3}\right) = 0.3.$$

Putting the values in (3.2)*, we get,*

$$\begin{aligned} \eta(\alpha(x,y)d(fx,gy), M_{f,g}(x,y)) =& \eta(0.3, 0.6) \\ =& 0.075 \\ \geq& 0. \end{aligned}$$

So condition (a) of Theorem 2.2 is satisfied. Similarly, we can check for condition (b). We skip the calculation here. Therefore, f and g satisfy both the hypotheses of Theorem 2.2 and using the theorem, f and g have a unique common fixed point and it is $w = \frac{3}{4} \in X$.

4. Application

Fixed point and common fixed point results for various contractions in different metric setting are hugely investigated and have been found diverse applications in differential equations, integral equations and thermostat models (see [24, 10, 28] and references therein). The common fixed point theorem secured in this article makes way for an interesting application on complete Branciari metric spaces to warrant the existence and uniqueness of a common solution of the subsequent integral equations.

Theorem 4.1. *Consider the integral equations*

$$x(t) = g(t) + \int_0^1 K_1(t, s, x(s))ds, \quad t \in [0, 1], \tag{4.1}$$

$$x(t) = g(t) + \int_0^1 K_2(t, s, x(s))ds, \quad t \in [0, 1]. \tag{4.2}$$

Suppose that

(1) $K_1, K_2 : [0,1]^2 \times \mathbb{R} \to \mathbb{R}$ *and* $g : [0,1] \to \mathbb{R}$ *are members of* $L^1([0,1])$;

(2) *there exists* $\lambda \in [0,1)$ *such that for* $t, s \in [0,1]$ *and* $u, v \in \mathbb{R}$,

$$|K_1(t, s, u) - K_2(t, s, v)| \leq \lambda |u - v|.$$

Then the Integral Equations (4.1) and (4.2) have a unique solution in $C([0,1])$.

Proof . Let $X = C([0,1])$. We define $d : X \times X \to [0,\infty)$ by

$$d(f,g) = ||f-g||_\infty = \max_{s\in[0,1]} |f(s)-g(s)|.$$

Then (X,d) is a metric space and hence is a Branciari metric space. We define $\alpha : X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1, & \text{if } x,y \in [0,1], \\ 0, & \text{otherwise.} \end{cases}$$

Let $S,T : X \to X$ be

$$T(x(t)) = g(t) + \int_0^1 K_1(t,s,x(s))ds, \quad t \in [0,1],$$

$$S(x(t)) = g(t) + \int_0^1 K_2(t,s,x(s))ds, \quad t \in [0,1].$$

We mention that Integral Equations (4.1) and (4.2) have a unique common solution if and only if the operators T and S have a common fixed point. Thus we have,

$$\begin{aligned} d(T,S) = ||Tx(t) - Sy(t)|| &= \Big|\int_0^1 (K_1(t,s,x(s)) - K_2(t,s,y(s)))ds\Big| \\ &\leq \int_0^1 |K_1(t,s,x(s)) - K_2(t,s,y(s))|ds \\ &\leq \int_0^1 \lambda|x(s)-y(s)|ds \\ &= \lambda||x-y||_\infty \\ &= \lambda d(x,y) \\ &\leq \lambda M_{T,S}(x,y) \\ \Rightarrow \lambda M_{T,S}(x,y) - d(T,S) &\geq 0. \end{aligned} \tag{4.3}$$

Again,

$$\begin{aligned} d(S,T) = ||Sx(t) - Ty(t)|| &= \Big|\int_0^1 (K_2(t,s,x(s)) - K_1(t,s,y(s)))ds\Big| \\ &\leq \int_0^1 |K_2(t,s,x(s)) - K_1(t,s,y(s))|ds \\ &= \int_0^1 |K_1(t,s,y(s)) - K_2(t,s,x(s))|ds \\ &\leq \int_0^1 \lambda|y(s)-x(s)|ds \\ &\leq \int_0^1 \lambda|x(s)-y(s)|ds \\ &= \lambda||x-y||_\infty \\ &= \lambda d(x,y) \\ &\leq \lambda M_{S,T}(x,y) \\ \Rightarrow \lambda M_{S,T}(x,y) - d(S,T) &\geq 0. \end{aligned} \tag{4.4}$$

We consider the simulation function as $\eta(t,s)=\lambda s-t$. Then from (4.3) and (4.4), and considering $\alpha(x,y)=1$ we have, for all $T,S\in X$

$$\eta\Big(\alpha(x,y)d(T,S),M_{T,S}(x,y)\Big)\geq 0 \quad\text{and}\quad \eta\Big(\alpha(x,y)d(S,T),M_{S,T}(x,y)\Big)\geq 0.$$

Then by Theorem 2.2, the Integral Equations (4.1) and (4.2) have a unique solution. □

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