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Second-order nondifferentiable multiobjective mixed type fractional programming problems

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Abstract

The motivation behind this article is to generalize a new type of class of nondifferentiable multiobjective fractional programming problem in which each component of objective functions contains a term including the support function of a compact convex set. For a differentiable function, we consider the class of pseudoquasi /strictly pseudoquasi/weak strictly pseudoquasi/quasistrictly pseudo (V, ρ, θ)-bonvex-type-I. Further, we formulate unified (mixed type) dual models and derive duality relations under aforesaid assumptions.

Keywords: Support function, Type-I functions, Second order, Pseudoquasi, Unified dual, Strictly pseudoquasi, Efficient solutions.
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1. Introduction

The optimization theory has found its way into all branches of science and engineering due to its wide range of applications. Duality plays an important role in nonlinear programming. Indeed, when the solution of a problem poses some difficulties, we shall see the solution of its dual problem provides some valuable information about the original problem. In multiobjective programming problems, convexity plays an important role in deriving optimality conditions and duality results. To relax convexity assumptions involved in sufficient optimality conditions and duality theorems, various

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generalized convexity notions have been proposed. Hanson [17] in his study has cited one example which demonstrates the application of second-order duality in somewhat different perspective.

The fractional optimization problem with multiple objective functions have been the subject of intense investigations in the past few years, which have produced a number of optimality and duality results for these problems. Mangasarian [11] formulated a class of second order dual problems of nonlinear programming, and gave the duality results under some conditions which none of these condition imposed convexity requirements on all the functions. Preda [3] introduced (F, α, ρ, d)- convex function which were generalized to second-order (F, α, ρ, d)-V- convex function in [4, 6]. Later on, Yang and Hou [10] generalized the work in [12, 16] in the framework of generalized convexity.

Motivated by various concepts of generalize convexity Liang et al. [18] introduced the concept of (F, α, ρ, d)-convex functions. Hachimi and Aghezzaf [9] generalized convexity results extended the concept further to (F, α, ρ, d)-type I functions and introduced the optimality conditions and derived duality results for multiobjective programming problem. Further, Dubey et al. [1] formulated second-order symmetric duality model and established appropriate duality relations under (G, α_f) -bonvexity conditions. Many researchers have worked related to the second and higher-order multiobjective symmetric fractional programming problems [5, 8, 2, 13, 7, 14, 15, 19, 20, 21].

In this article, we generalize the definitions of (V, ρ, θ)-bonvex-type-I functions for a nondifferentiable multiobjective second -order fractional programming problem. We considered second-order unified dual model and proved duality theorems under (V, ρ, θ)-bonvex-type-I assumptions.

2. Definitions and Preliminaries

Throughout this paper, we use the index sets $K = \{1, 2, ..., k\}$ and $M = \{1, 2, ..., m\}$.

The following convention of vectors in R^n will be following throughout this papers:

$$x \leqq y \Leftrightarrow x_i \leqq y_i,\ i = 1, 2, ..., n,$$

$$x \leq y \Leftrightarrow x \leqq y,\ x \neq y,$$

$$x < y \Leftrightarrow x_i < y_i,\ i = 1, 2, ..., n,$$

$$x = y \Leftrightarrow x_i = y_i,\ i = 1, 2, ..., n.$$

Definition 2.1. Let C be a compact convex set in R^n. The support function of C is defined by

$$s\,(x|C) = \max\{x^T y : y \in C\}.$$

Remark 2.1. A support function $\pi_A : R^n \mapsto R$ of a non-empty closed convex set A in R^n is given by

$$\pi_A(x) = \sup\{x\ .\ a : a \in A\},\quad x \in R^n$$

. Its interpretation is most intuitive when x is a unit vector: by definition, A is contained in the closed half space

$$\{y \in R^n : y\ .\ x \leqq \pi_A(x)\}$$

and there is at least one point of A in the boundary

$$H(x) = \{y \in R^n : y\ .\ x = \pi_A(x)\}$$

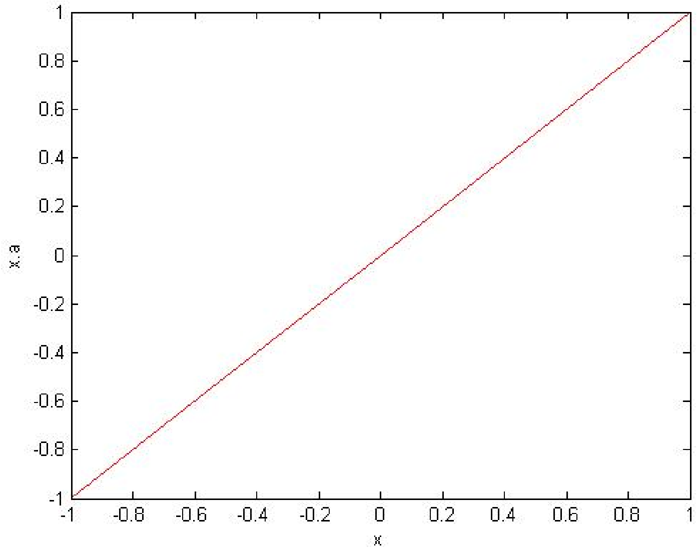

Figure 1: $A = \{a\}$ is $\pi_A(x) = x\ .\ a$

of this half space. The hyperplane $H(x)$ is therefore called a supporting hyperplane with exterior (or outer) unit normal vector x. The word exterior is important here, as the orientation of x plays a role, the set $H(x)$ is in general different from $H(-x)$. Now π_A is the (signed) distance of $H(x)$ from the origin.

Example 1. The support function of a singleton $A = \{a\}$ is $\pi_A(x) = x\ .\ a$.

Example 2. If A is a line segment through the origin with endpoints $-a$ and a then $\pi_A(x) = |x\ .\ a|$.

Consider the following nondifferentiable multiobjective fractional programming problem:

$$\textbf{(MFP)}\ \text{Minimize}\ \Psi(x) = \left(\frac{\phi_1(x) + s(x|C_1)}{\psi_1(x) - s(x|E_1)}, \frac{\phi_2(x) + s(x|C_2)}{\psi_2(x) - s(x|E_2)}, ..., \frac{\phi_k(x) + s(x|C_k)}{\psi_k(x) - s(x|E_k)}\right)^T$$

$$\text{subject to } x \in X_0 = \{x \in X : \pi_j(x) + s(x|D_j) \leqq 0, j \in M\},$$

where $X \subseteq R^n$ is an open set. The functions $\phi,\ \psi : X \to R^k,\ \pi : X \to R^m$ are differentiable on X and $C_i,\ E_i,\ D_j$ are compact convex sets in R^n for $i \in K$ and $j \in M$. Let $\phi_i(x) + s(x|C_i) \geqq 0$ and $\psi_i(x) - s(x|E_i) > 0,\ i \in K$. Next, $\eta : X \times X \to R^n,\ \rho \in R^n$ and $\theta : X \times X \to R^n$.

Definition 2.2. A point $u \in X_0$ is said to be an efficient solution of (MFP) if there exists no $x \in X_0$ such that $\Psi(x) \leqq \Psi(u)$.

Definition 2.3. $\left(\dfrac{\phi_i(.) + (.)^T z_i}{\psi_i(.) - (.)^T v_i},\ \pi_j(.) + (.)^T w_j\right)$ is pseudoquasi (V, ρ, θ)-bonvex-type-I at u of (MFP), if there exist $\eta,\ \rho$ and θ such that, for any $x \in X_0$ and $p \in R^n$, such that

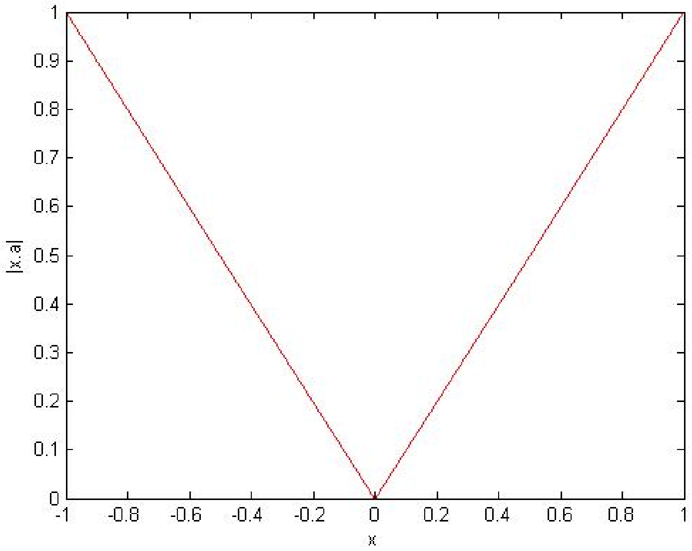

Figure 2: $\pi_A(x) = |x\ .\ a|$

$$\frac{\phi_i(x)+x^Tz_i}{\psi_i(x)-x^Tv_i} < \frac{\phi_i(u)+u^Tz_i}{\psi_i(u)-u^Tv_i} - \frac{1}{2}p^T\nabla^2\bigg(\frac{\phi_i(u)+u^Tz_i}{\psi_i(u)-u^Tv_i}\bigg)p$$

$$\Rightarrow \eta^T(x,u)\bigg\{\nabla\bigg(\frac{\phi_i(u)+u^Tz_i}{\psi_i(u)-u^Tv_i}\bigg) + \nabla^2\bigg(\frac{\phi_i(u)+u^Tz_i}{\psi_i(u)-u^Tv_i}\bigg)p\bigg\} + \rho_i^1||\theta_i^1(x,u)||^2 < 0,\ \forall\ i \in K$$

and

$-\pi_j(u) - u^Tw_j \leqq -\frac{1}{2}p^T\nabla^2\pi_j(u)p$

$$\Rightarrow \eta^T(x,u)\{\nabla\pi_j(u) + w + \nabla^2\pi_j(u)p\} + \rho_j^2||\theta_j^2(x,u)||^2 \leqq 0,\ \forall\ j \in M.$$

Definition 2.4. $\bigg(\frac{\phi_i(.)+(.)^Tz_i}{\psi_i(.)-(.)^Tv_i},\ \pi_j(.)+(.)^Tw_j\bigg)$ is strictly pseudoquasi (V,ρ,θ)-bonvex -type-I at u of (MFP), if there exist $\eta,\ \rho$ and θ such that, for any $x \in X_0$ and $p \in R^n$

$$\frac{\phi_i(x)+x^Tz_i}{\psi_i(x)-x^Tv_i} \leqq \frac{\phi_i(u)+u^Tz_i}{\psi_i(u)-u^Tv_i} - \frac{1}{2}p^T\nabla^2\bigg(\frac{\phi_i(u)+u^Tz_i}{\psi_i(u)-u^Tv_i}\bigg)p$$

$$\Rightarrow \eta^T(x,u)\bigg\{\nabla\bigg(\frac{\phi_i(u)+u^Tz_i}{\psi_i(u)-u^Tv_i}\bigg) + \nabla^2\bigg(\frac{\phi_i(u)+u^Tz_i}{\psi_i(u)-u^Tv_i}\bigg)p\bigg\}\bigg) + \rho_i^1||\theta_i^1(x,u)||^2 < 0,\ \forall\ i \in K$$

and

$-\pi_j(u) - u^Tw_j \leqq -\frac{1}{2}p^T\nabla^2\pi_j(u)p$

$$\Rightarrow \eta^T(x,u)\{\nabla\pi_j(u) + w + \nabla^2\pi_j(u)p\} + \rho_j^2||\theta_j^2(x,u)||^2 \leqq 0,\ \forall\ j \in M.$$

Definition 2.5. $\bigg(\frac{\phi_i(.)+(.)^Tz_i}{\psi_i(.)-(.)^Tv_i},\ \pi_j(.)+(.)^Tw_j\bigg)$ is weak strictly pseudoquasi (V,ρ,θ)-bonvex -type-I at u of (MFP), if there exist $\eta,\ \rho$ and θ such that, for any $x \in X_0$ and $p \in R^n$

$$\frac{\phi_i(x)+x^T z_i}{\psi_i(x)-x^T v_i} \leqq \frac{\phi_i(u)+u^T z_i}{\psi_i(u)-u^T v_i} - \frac{1}{2}p^T\nabla^2\left(\frac{\phi_i(u)+u^T z_i}{\psi_i(u)-u^T v_i}\right)p$$

$$\Rightarrow \eta^T(x,u)\left\{\nabla\left(\frac{\phi_i(u)+u^T z_i}{\psi_i(u)-u^T v_i}\right)+\nabla^2\left(\frac{\phi_i(u)+u^T z_i}{\psi_i(u)-u^T v_i}\right)p\right\}+\rho_i^1||\theta_i^1(x,u)||^2<0,\ \forall\ i\in K$$

and

$-\pi_j(u)-u^T w_j \leqq -\frac{1}{2}p^T\nabla^2\pi_j(u)p$

$$\Rightarrow \eta^T(x,u)\{\nabla\pi_j(u)+w+\nabla^2\pi_j(u)p\}+\rho_j^2||\theta_j^2(x,u)||^2\leqq 0,\ \forall\ j\in M.$$

Definition 2.6. $\left(\frac{\phi_i(.)+(.)^T z_i}{\psi_i(.)-(.)^T v_i},\ \pi_j(.)+(.)^T w_j\right)$ is quasistrictly pseudo (V,ρ,θ)-bonvex -type-I at u of (MFP), if there exist $\eta,\ \rho$ and θ such that, for any $x\in X_0$ and $p\in R^n$

$$\frac{\phi_i(x)+x^T z_i}{\psi_i(x)-x^T v_i} \leqq \frac{\phi_i(u)+u^T z_i}{\psi_i(u)-u^T v_i} - \frac{1}{2}p^T\nabla^2\left(\frac{\phi_i(u)+u^T z_i}{\psi_i(u)-u^T v_i}\right)p$$

$$\Rightarrow \eta^T(x,u)\left\{\nabla\left(\frac{\phi_i(u)+u^T z_i}{\psi_i(u)-u^T v_i}\right)+\nabla^2\left(\frac{\phi_i(u)+u^T z_i}{\psi_i(u)-u^T v_i}\right)p\right\}\Bigg)+\rho_i^1||\theta_i^1(x,u)||^2\leqq 0,\ \forall\ i\in K$$

and

$-\pi_j(u)-u^T w_j \leqq -\frac{1}{2}p^T\nabla^2\pi_j(u)p$

$$\Rightarrow \eta^T(x,u)\{\nabla\pi_j(u)+w+\nabla^2\pi_j(u)p\}+\rho_j^2||\theta_j^2(x,u)||^2<0,\ \forall\ j\in M.$$

We consider the Karush-Kuhn-Tucker necessary optimality conditions for the nondifferentiable multiobjective fractional programming problem (MFP) involving support functions in the objective and constraint functions.

Theorem 2.1 (K-K-T-type necessary condition). Assume that u is an efficient solution of (MFP) at which the Kuhn-Tucker constraint qualification is satisfied on X. Then there exist $0<\bar{\lambda}\in R^k,\ 0\leqq \bar{y}_j\in R^m,\ \bar{z}_i\in R^n,\ \bar{v}_i,\ \bar{w}_j\in R^n,\ i\in K,\ j\in M$ such that

$$\sum_{i=1}^{k}\bar{\lambda}_i\nabla\left(\frac{\phi_i(u)+u^T\bar{z}_i}{\psi_i(u)-u^T\bar{v}_i}\right)+\sum_{j=1}^{m}\bar{y}_j\nabla(\pi_j(u)+u^T\bar{w}_j)=0,$$

$$\sum_{j=1}^{m}\bar{y}_j(\pi_j(u)+u^T\bar{w}_j)=0,$$

$$u^T\bar{z}_i=S(u|C_i),\ u^T\bar{v}_i=S(u|E_i),\ u^T\bar{w}_j=S(u|D_j),$$

$$\bar{z}_i\in C_i,\ \bar{v}_i\in D_i,\ \bar{w}_j\in E_j,\ i\in K,\ j\in M.$$

3. Unified second-order duality model-I:

In this section, we consider the following unified second order dual for (MFP) and derive weak, strong and strict converse duality theorems.

$$\textbf{(MDP):}\ \text{Maximize}\ \left(\frac{\phi_1(y)+y^Tz_1}{\psi_1(y)-y^Tv_1}-\frac{1}{2}p^T\nabla^2\left(\frac{\phi_1(y)-y^Tz_1}{\psi_1(y)-y^Tv_1}\right)p+\sum_{j\in J_0}\mu_j\{\pi_j(y)+y^Tw_j-\frac{1}{2}p^T\nabla^2\pi_j(y)p\}\right.$$

$$\left.,...,\frac{\phi_k(y)+y^Tz_k}{\psi_k(y)-y^Tv_k}-\frac{1}{2}p^T\nabla^2\left(\frac{\phi_k(y)-y^Tz_k}{\psi_k(y)-y^Tv_k}\right)p+\sum_{j\in J_0}\mu_j\{\pi_j(y)+y^Tw_j-\frac{1}{2}p^T\nabla^2\pi_j(y)p\}\right)$$

subject to $y \in X$,

$$\sum_{i=1}^{k}\lambda_i\left\{\nabla\left(\frac{\phi_i(y)+y^Tz_i}{\psi_i(y)-y^Tv_i}\right)+\nabla^2\left(\frac{\phi_i(y)+y^Tz_i}{\psi_i(y)-y^Tv_i}\right)p\right\}$$

$$+\sum_{j=1}^{m}\mu_j\{\nabla\pi_j(y)+w_j+\nabla^2\pi_j(y)p\}=0, \tag{3.1}$$

$$\sum_{j\in J_\beta}\mu_j\{\pi_j(y)+y^Tw_j-\frac{1}{2}p^T\nabla^2\pi_j(y)p\}\geqq 0,\ \ \beta=1,...,r, \tag{3.2}$$

$$\lambda_i\geqq 0,\quad \sum_{i=1}^{k}\lambda_i=1, \tag{3.3}$$

$$\mu_j\geqq 0,\ z_i\in C_i,\ v_i\in E_i,\ w_j\in D_j,\quad \text{for } i\in K,\ j\in M, \tag{3.4}$$

where $J_\delta \subseteq N$, $\delta = 0, 1, ..., r$ with $\bigcup_{\delta=0}^{r} J_\delta = N$ and $J_{\delta_1} \bigcap J_{\delta_2}$ if $\delta_1 \neq \delta_2$. It may be noted that $J_0 = N$ and $J_\beta = \phi(1 \leqq \beta \leqq r)$, we obtain Wolfe type dual. If $J_0 = \phi$, $J_1 = N$ and $J_\beta = \phi$ $(2 \leqq \beta \leqq r)$, then (MDP) reduces to Mond-Weir Type dual. Let Z_0 be set feasible solution of (MDP).

Theorem 3.1 (Weak Duality). Let $x \in X_0$ and $(y, \lambda, v, \mu, z, w, p) \in Z_0$. Let$\forall\ i \in K$ and $\forall\ j \in M$, such that

(i) $\left(\dfrac{\phi_i(.)+(.)^Tz_i}{\psi_i(.)+(.)^Tv_i}+\mu_{J_0}^T\left(\pi_jJ_0+(.)^Tw_jJ_0\right)e,\ \{\pi_j(.)+(.)^Tw_j\}_{J_\beta}^{\mu}\right)$ is weak strictly pseudo quasi (V,ρ,θ)-bonvex-type I at y,

(ii) $\displaystyle\sum_{i=1}^{k}\lambda_i\rho_i^1||\theta_i^1(x,u)||^2+\sum_{j=1}^{m}\mu_j\rho_j^2||\theta_j^2(x,u)||^2\geqq 0.$

Then, the following cannot hold

$$\frac{\phi_i(x)+s(x|C_i)}{\psi_i(x)-s(x|E_i)}\leqq\frac{\phi_i(y)+y^Tz_i}{\psi_i(y)-y^Tv_i}-\frac{1}{2}p^T\nabla^2\left(\frac{\phi_i(y)+y^Tz_i}{\psi_i(y)-y^Tv_i}\right)p$$

$$+\sum_{j\in J_0}\mu_j\{\pi_j(y)+y^Tw_j-\frac{1}{2}p^T\nabla^2\pi_j(y)p\},\quad \text{for all } i\in K \tag{3.5}$$

and

$$\frac{\phi_j(x)+s(x|C_j)}{\psi_j(x)-s(x|E_j)} < \frac{\phi_j(y)+y^T z_j}{\psi_j(x)-y^T v_j} - \frac{1}{2}p^T\nabla^2\Big(\frac{\phi_j(y)+y^T z_j}{\psi_j(y)-y^T v_j}\Big)p + \sum_{j\in J_0}\mu_j\{\pi_j(y)+y^T w_j - \frac{1}{2}p^T\nabla^2\pi_j(y)p\},\quad \text{for some } j\in K. \tag{3.6}$$

Proof Suppose inequalities (3.5) and (3.6) hold. As $x^T z_i \leqq s(x|C_i)$, $x^T v_i \leqq s(x|E_i)$, $\forall i\in K$ and $\sum_{j\in J_0}\mu_j(\pi_j(x)+x^T w_j)\leqq 0$, using the inequalities and the dual constraint (3.2), hypothesis (i) gives

$$\eta^T(x,u)\Big\{\nabla\Big(\frac{\phi_i(y)+y^T z_i}{\psi_i(y)-y^T v_i}\Big)+\nabla^2\Big(\frac{\phi_i(y)+y^T z_i}{\psi_i(y)-y^T v_i}\Big)p + \sum_{j\in J_0}\mu_j\{\nabla\pi_j(y)+w_j+\nabla^2\pi_j(y)p\}e\Big\} < -\rho_i^1||\theta_i^1(x,u)||^2,\quad \forall\ i\in K$$

and

$$\eta^T(x,u)\sum_{j\in J_\beta}\mu_j\{\nabla\pi_j(y)+w_j+\nabla^2\pi_j(y)p\}+\rho_\beta^2||\theta_j^2(x,u)||^2\leqq 0,\ \beta=1,...,r.$$

Since $\lambda\geqq 0$, $\lambda^T e=1$, it gives that

$$\eta^T(x,u)\Big\{\sum_{i=1}^k\lambda_i\Big\{\nabla\Big(\frac{\phi_i(y)+y^T z_i}{\psi_i(y)-y^T v_i}\Big)+\nabla^2\Big(\frac{\phi_i(y)+y^T z_i}{\psi_i(y)-y^T v_i}\Big)p\Big\} + \sum_{j\in J_0}\mu_j\{\nabla\pi_j(y)+w_j+\nabla^2\pi_j(y)p\}\Big\} < -\sum_{i=1}^k\lambda_i\rho_i^1||\theta_i^1(x,u)||^2$$

and

$$\eta^T(x,u)\sum_{j\in J_\beta}\mu_j\{\nabla\pi_j(y)+w_j+\nabla^2\pi_j(y)p\}\leqq -\rho_\beta^2||\theta_j^2(x,u)||^2,\ \beta=1,...,r.$$

Expanding the above expressions, it follows that

$$\eta^T(x,u)\Big(\sum_{i=1}^k\lambda_i\Big\{\nabla\Big(\frac{\phi_i(y)+y^T z_i}{\psi_i(y)-y^T v_i}\Big)+\nabla^2\Big(\frac{\phi_i(y)+y^T z_i}{\psi_i(y)-y^T v_i}\Big)p\Big\}+\sum_{j=1}^m\mu_j\{\nabla\pi_j(y)+w_j+\nabla^2\pi_j(y)p\}\Big)$$

$$=\eta^T(x,u)\Big(\sum_{i=1}^k\lambda_i\Big\{\nabla\Big(\frac{\phi_i(y)+y^T z_i}{\psi_i(y)-y^T v_i}\Big)+\nabla^2\Big(\frac{\phi_i(y)+y^T z_i}{\psi_i(y)-y^T v_i}\Big)p\Big\}+\sum_{j\in J_0}\mu_j\{\nabla\pi_j(y)+w_j+\nabla^2\pi_j(y)p\} + \sum_{j\in J_1}\mu_j\{\nabla\pi_j(y)+w_j+\nabla^2\pi_j(y)p\}+...+\sum_{j\in J_r}\mu_j\{\nabla\pi_j(y)+w_j+\nabla^2\pi_j(y)p\}\Big)$$

or

$$\eta^T(x,u)\Big(\sum_{i=1}^k\lambda_i\Big\{\nabla\Big(\frac{\phi_i(y)+y^T z_i}{\psi_i(y)-y^T v_i}\Big)+\nabla^2\Big(\frac{\phi_i(y)+y^T z_i}{\psi_i(y)-y^T v_i}\Big)p\Big\}+\sum_{j=1}^m\mu_j\{\nabla\pi_j(y)+w_j+\nabla^2\pi_j(y)p\}\Big)$$

$$\leqq \eta^T(x,u)\Bigg(\sum_{i=1}^{k}\lambda_i\Bigg\{\nabla\Bigg(\frac{\phi_i(y)+y^Tz_i}{\psi_i(y)-y^Tv_i}\Bigg)+\nabla^2\Bigg(\frac{\phi_i(y)+y^Tz_i}{\psi_i(y)-y^Tv_i}\Bigg)p\Bigg\}+\sum_{j\in J_0}\mu_j\{\nabla\pi_j(y)+w_j+\nabla^2\pi_j(y)p\}\Bigg)$$

$$+\eta^T(x,u)\Bigg(\sum_{j\in J_1}\mu_j\{\nabla\pi_j(y)+w_j+\nabla^2\pi_j(y)p\}\Bigg)+...+\eta^T(x,u)\Bigg(\sum_{j\in J_r}\mu_j\{\nabla\pi_j(y)+w_j+\nabla^2\pi_j(y)p\}\Bigg)$$

$$<-\Bigg(\sum_{i=1}^{k}\lambda_i\rho_i^1||\theta_i^1(x,u)||^2+\sum_{j=1}^{m}\mu_j\rho_j^2||\theta_j^2(x,u)||^2\Bigg).$$

By hypothesis (ii), we have

$$\eta^T(x,u)\Bigg(\sum_{i=1}^{k}\lambda_i\Bigg\{\nabla\Bigg(\frac{\phi_i(y)+y^Tz_i}{\psi_i(y)-y^Tv_i}\Bigg)+\nabla^2\Bigg(\frac{\phi_i(y)+y^Tz_i}{\psi_i(y)-y^Tv_i}\Bigg)p\Bigg\}$$
$$+\sum_{j=1}^{m}\mu_j\{\nabla\pi_j(y)+w_j+\nabla^2\pi_j(y)p\}\Bigg)<0,$$

which contradicts the dual constraint (3.1). Hence, the result.

Theorem 3.2 (Weak Duality).Let $u\in X_0$ and $(y,\lambda,\mu,v,z,w,p)\in Z_0$. Let$\forall\ i\in K$ and $\forall\ j\in M$, such that

(i) $\left(\frac{\phi_i(.)+(.)^Tz_i}{\psi_i(.)+(.)^Tv_i}+\mu_{J_0}^T\left(\pi_jJ_0+(.)^Tw_jJ_0\right)e,\{\pi(.)+(.)^Tw\}_{J_\beta}^{\mu}\right)$ be pseudo quasi (V,ρ,θ)-bonvex type-I at y,

(ii) $\sum_{i=1}^{k}\lambda_i\rho_i^1||\theta_i^1(x,u)||^2+\sum_{j=1}^{m}\mu_j\rho_j^2||\theta_j^2(x,u)||^2\geqq 0.$

Then, the following cannot hold

$$\frac{\phi_i(x)+s(x|C_i)}{\psi_i(x)-s(x|E_i)}\leqq\frac{\phi_i(y)+y^Tz_i}{\psi_i(y)-y^Tv_i}-\frac{1}{2}p^T\nabla^2\Bigg(\frac{\phi_i(y)+y^Tz_i}{\psi_i(y)-y^Tv_i}\Bigg)p$$
$$+\sum_{j\in J_0}\mu_j\{\pi_j(y)+y^Tw_j-\frac{1}{2}p^T\nabla^2\pi_j(y)p\},\quad \text{for all } i\in K \tag{3.7}$$

and

$$\frac{\phi_j(x)+s(x|C_j)}{\psi_j(x)-s(x|E_j)}<\frac{\phi_j(y)+y^Tz_j}{\psi_j(x)-y^Tv_j}-\frac{1}{2}p^T\nabla^2\Bigg(\frac{\phi_j(y)+y^Tz_j}{\psi_j(y)-y^Tv_j}\Bigg)p$$
$$+\sum_{j\in J_0}\mu_j\{\pi_j(y)+y^Tw_j-\frac{1}{2}p^T\nabla^2\pi_j(y)p\},\text{for some } j\in K. \tag{3.8}$$

Proof The proof follows on the lines of Theorem 3.1.

Theorem 3.3 (Strong Duality Theorem). Let $\bar{u}$ be an efficient solution of (MFP) and let the Kuhn-Tucker constraint qualification are satisfied. Then there exist $\bar{\lambda}\in R^k,\ \bar{y}\in R^m,\ \bar{z_i}\in R^n,\ \bar{v_i}\in R^n$ and $\bar{w_j}\in R^n,\ i\in K,\ j\in M$, such that $(\bar{u},\bar{z},\bar{v},\bar{y},\bar{\lambda},\bar{w},\bar{p}=0)$ is a feasible solution of

(MDP) and the (MFP)and (MDP) have equal values. Furthermore, if the assumptions of Theorem 2.1 hold for all feasible solutions of (MFP) and (MDP), then $(\bar{u}, \bar{z}, \bar{v}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{p} = 0)$ is an efficient solution of (MDP).

Proof . From the given conditions in the statement and using Theorem 2.1, there exist $\bar{\lambda} \in R^k,\ \bar{y} \in R^m,\ \bar{z}_i \in R^n,\ \bar{v}_i \in R^n$ and $\bar{w}_j \in R^n,\ \ i \in K,\ \ j \in M$, such that

$$\sum_{i=1}^{k} \bar{\lambda}_i \nabla \left(\frac{\phi_i(\bar{u}) + \bar{u}^T \bar{z}_i}{\psi_i(\bar{u}) - \bar{u}^T \bar{v}_i} \right) + \sum_{j=1}^{m} \bar{y}_j \nabla (\pi_j(\bar{u}) + \bar{u}^T \bar{w}_j) = 0, \tag{3.9}$$

$$\sum_{j=1}^{m} \bar{y}_j (\pi_j(\bar{u}) + \bar{u}^T \bar{w}_j) = 0, \tag{3.10}$$

$$\bar{u}^T \bar{z}_i = S(\bar{u}|C_i),\ \bar{u}^T \bar{v}_i = S(\bar{u}|D_i),\ \bar{u}^T \bar{w}_j = S(\bar{u}|E_j), \tag{3.11}$$

$$\bar{z}_i \in C_i,\ \bar{v}_i \in D_i,\ \bar{w}_j \in E_j, \tag{3.12}$$

$$\bar{\lambda}_i > 0,\ \sum_{i=1}^{k} \bar{\lambda}_i = 1,\ \bar{y}_j \geqq 0,\ i \in K,\ j \in M. \tag{3.13}$$

Hence, $(\bar{u}, \bar{z}, \bar{v}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{p} = 0)$ satisfy all the constraints of (MDP) and remaining part of proof is obvious. Hence the result. □

Theorem 3.4 **(Strict Converse Duality)**. Let $u \in X_0$ and $(\bar{y}, \bar{\lambda}, \bar{\mu}, \bar{v}, \bar{z}, \bar{w}, \bar{p}) \in Z_0$, such that,$\forall\ i \in K$ and $\forall\ j \in M$,

(i) $\displaystyle \sum_{i=1}^{k} \bar{\lambda}_i \left\{ \frac{\phi_i(u) + u^T \bar{z}_i}{\psi_i(u) - u^T \bar{v}_i} \right\} \leqq \sum_{i=1}^{k} \bar{\lambda}_i \left\{ \frac{\phi_i(\bar{y}) + \bar{y}^T \bar{z}_i}{\psi_i(\bar{y}) - \bar{y}^T \bar{v}_i} - \frac{1}{2} p^T \nabla^2 \left(\frac{\phi_i(\bar{y}) + \bar{y}^T \bar{z}_i}{\psi_i(\bar{y}) - \bar{y}^T \bar{v}_i} \right) p \right\} + \sum_{j \in J_0} \bar{\mu}_j \{ \pi_j(\bar{y})$ $+ \bar{y}^T \bar{w}_j - \tfrac{1}{2} p^T \nabla^2 \pi_j(\bar{y}) p \}$,

(ii) $\displaystyle \sum_{i=1}^{k} \lambda_i \rho_i^1 ||\theta_i^1(x, u)||^2 + \sum_{j=1}^{m} \mu_j \rho_j^2 ||\theta_j^2(x, u)||^2 \geqq 0$,

(iii) $\displaystyle \left(\sum_{i=1}^{k} \bar{\lambda}_i \left\{ \frac{\phi_i(.) + (.)^T \bar{z}_i}{\psi_i(.) - (.)^T \bar{v}_i} \right\} + \sum_{j \in J_0} \bar{\mu}_j\ \{\pi_j + (.)^T \bar{w}_j\},\ \{\pi(.) + (.)^T \bar{w}\}_{J_\beta}^{\bar{\mu}} \right)$ is strictly pseudoquasi (V, ρ, d)-bonvex type I at $\bar{y}$.

Then, $u = \bar{y}$.

Proof Suppose on contradiction that is $u \neq \bar{y}$. The dual constraint (3.2) and the hypothesis (iii), for $\beta = 1, ..., r$ yield

$$\eta^T(\bar{y}, u) \sum_{j \in J_\beta} \bar{\mu}_j \{ \nabla \pi_j(\bar{y}) + \nabla^2 \pi_j(\bar{y}) p + \bar{w}_j \} \leqq -\rho_\beta^2 ||\theta_j^2(\bar{y}, u)||^2. \tag{3.14}$$

By the dual constraint (3.1), we have

$$\eta^T(\bar{y},u)\Bigg(\sum_{i=1}^{k}\bar{\lambda}_i\Big\{\nabla\Big(\frac{\phi_i(u)+u^T\bar{z}_i}{\psi_i(u)-u^T\bar{v}_i}\Big)+\nabla^2\Big(\frac{\phi_i(u)+u^T\bar{z}_i}{\psi_i(u)-u^T\bar{v}_i}\Big)p\Big\}$$
$$+\sum_{j=1}^{m}\bar{\mu_j}\Big\{\nabla\pi_j(\bar{y})+\nabla^2\pi_j(\bar{y})p+\bar{w_j}\Big\}\Bigg)=0.$$

Expanding the above expressions with (3.14) give

$$\eta^T(\bar{y},u)\Bigg(\sum_{i=1}^{k}\bar{\lambda}_i\Big\{\nabla\Big(\frac{\phi_i(u)+u^T\bar{z}_i}{\psi_i(u)-u^T\bar{v}_i}\Big)+\nabla^2\Big(\frac{\phi_i(u)+u^T\bar{z}_i}{\psi_i(u)-u^T\bar{v}_i}p\Big)\Big\}+\sum_{j\in J_0}\bar{\mu_j}\Big\{\nabla\pi_j(\bar{y})+\nabla^2\pi_j(\bar{y})p+\bar{w_j}\Big\}\Bigg)$$
$$\geqq-\eta^T(\bar{y},u)\Bigg(\sum_{j\in J_1}\bar{\mu_j}\{\nabla\pi_j(\bar{y})+\nabla^2\pi_j(\bar{y})p+\bar{w_j}\}\Big)-...-\eta^T(\bar{y},u)\Big(\sum_{j\in J_r}\bar{\mu_j}\{\nabla\pi_j(\bar{y})+\nabla^2\pi_j(\bar{y})p+\bar{w_j}\}\Bigg)$$

or

$$\eta^T(\bar{y},u)\Bigg(\sum_{i=1}^{k}\bar{\lambda}_i\Big\{\nabla\Big(\frac{\phi_i(u)+u^T\bar{z}_i}{\psi_i(u)-u^T\bar{v}_i}\Big)+\nabla^2\Big(\frac{\phi_i(u)+u^T\bar{z}_i}{\psi_i(u)-u^T\bar{v}_i}p\Big)\Big\}$$
$$+\sum_{j\in J_0}\bar{\mu_j}\Big\{\nabla\pi_j(\bar{y})+\nabla^2\pi_j(\bar{y})p+\bar{w_j}\Big\}\Bigg)\geqq\sum_{j=1}^{m}\mu_j\rho_j^2||\theta_j^2(x,u)||^2,$$

by assumption (ii), we have

$$\eta^T(\bar{y},u)\Bigg(\sum_{i=1}^{k}\bar{\lambda}_i\Big\{\nabla\Big(\frac{\phi_i(u)+u^T\bar{z}_i}{\psi_i(u)-u^T\bar{v}_i}\Big)+\nabla^2\Big(\frac{\phi_i(u)+u^T\bar{z}_i}{\psi_i(u)-u^T\bar{v}_i}\Big)p\Big\}$$
$$+\sum_{j\in J_0}\bar{\mu_j}\Big\{\nabla\pi_j(\bar{y})+\nabla^2\pi_j(\bar{y})p+\bar{w_j}\Big\}\Bigg)\geqq-\rho_i^1||\theta_i^1(x,u)||^2.$$

It follows that

$$\eta^T(\bar{y},u)\Bigg(\sum_{i=1}^{k}\bar{\lambda}_i\Big\{\nabla\Big(\frac{\phi_i(u)+u^T\bar{z}_i}{\psi_i(u)-u^T\bar{v}_i}\Big)+\nabla^2\Big(\frac{\phi_i(u)+u^T\bar{z}_i}{\psi_i(u)-u^T\bar{v}_i}\Big)p\Big\}$$
$$+\sum_{j\in J_0}\bar{\mu_j}\Big\{\nabla\pi_j(\bar{y})+\nabla^2\pi_j(\bar{y})p+\bar{w_j}\Big\}\Bigg)\geqq-\rho_i^1||\theta_i^1(x,u)||^2.$$

Further, from hypothesis (iii) in view of $\sum_{j\in J_0}\mu_j\{\pi_j(u)+u^T\bar{w_j}\}\leqq 0$ yields

$$\sum_{i=1}^{k}\bar{\lambda}_i\Big\{\frac{\phi_i(u)+u^T\bar{z}_i}{\psi_i(u)-u^T\bar{v}_i}\Big\}>\sum_{i=1}^{k}\bar{\lambda}_i\Big\{\frac{\phi_i(\bar{y})+\bar{y}^T\bar{z}_i}{\psi_i(\bar{y})-\bar{y}^T\bar{v}_i}-\frac{1}{2}p^T\nabla^2\Big(\frac{\phi_i(\bar{y})+\bar{y}^T\bar{z}_i}{\psi_i(\bar{y})-\bar{y}^T\bar{v}_i}\Big)p\Big\}$$
$$+\sum_{j\in J_0}\bar{\mu_j}\{\pi_j(\bar{y})+\bar{y}^T\bar{w}_j-\frac{1}{2}\bar{p}^T\nabla^2\pi_j(\bar{y})\bar{p}\},$$

which contradicts hypothesis (i). Hence, the result.

Theorem 3.5 **(Strict Converse Duality)**. Let $u \in X_0$ and $(\bar{y}, \bar{\lambda}, \bar{\mu}, \bar{v}, \bar{z}, \bar{w}, \bar{p}) \in Z_0$, such that,$\forall\ i \in K$ and $\forall\ j \in M$,

$$(i)\quad \sum_{i=1}^{k} \bar{\lambda}_i \left\{ \frac{\phi_i(u) + u^T \bar{z}_i}{\psi_i(u) - u^T \bar{v}_i} \right\} \leqq \sum_{i=1}^{k} \bar{\lambda}_i \left\{ \frac{\phi_i(\bar{y}) + \bar{y}^T \bar{z}_i}{\psi_i(\bar{y}) - \bar{y}^T \bar{v}_i} - \frac{1}{2} p^T \nabla^2 \left(\frac{\phi_i(\bar{y}) + \bar{y}^T \bar{z}_i}{\psi_i(\bar{y}) - \bar{y}^T \bar{v}_i} \right) p \right\} + \sum_{j \in J_0} \bar{\mu_j} \{ \pi_j(\bar{y}) + \bar{y}^T \bar{w_j} - \frac{1}{2} p^T \nabla^2 \pi_j(\bar{y}) p \},$$

$$(ii)\quad \sum_{i=1}^{k} \lambda_i \rho_i^1 ||\theta_i^1(x, u)||^2 + \sum_{j=1}^{m} \mu_j \rho_j^2 ||\theta_j^2(x, u)||^2 \geqq 0,$$

(iii) $\left(\sum_{i=1}^{k} \bar{\lambda}_i \left\{ \frac{\phi_i(.) + (.)^T \bar{z}_i}{\psi_i(.) - (.)^T \bar{v}_i} \right\} + \sum_{j \in J_0} \bar{\mu_j}\ \{ \pi_j + (.)^T \bar{w_j} \},\ \{ \pi(.) + (.)^T \bar{w} \}_{J_\beta}^{\bar{\mu}} \right)$ is quasistrictly pseudo (V, ρ, d)-bonvex type I at $\bar{y}$.

Then, $u = \bar{y}$.

Proof Proof follows on the lines of theorem 3.4.

Lemma 3.1 $u \in X$ is an efficient solution for (MFP) if and only if there exists $\bar{\nu_i} \in R_+^k$ such that u is an efficient solution for $(\text{MFP})_{\bar{\nu}_i}$, where $\bar{\nu_i} = \dfrac{\phi_i(u) + S(u|C_i)}{\psi_i(u) - S(u|E_i)},\ i = 1, 2, ..., k.$

With the help of above lemma (lemma 3.1), we can reformulate a nondifferentiable multiobjective programming problem (MFP) as :

$$\textbf{(MFP)}_{\bar{\nu}_i}\ \text{Minimize}\ \pi(x) = \Big(\phi_1(x) + s(x|C_1) - \nu_1(\psi_1(x) - s(x|E_1)), \phi_2(x) + s(x|C_2) - \nu_2(\psi_2(x) - s(x|E_2)), ..., \phi_k(x) + s(x|C_k) - \nu_k(\psi_k(x) - s(x|E_k)) \Big)^T$$

subject to $x \in X_0 = \{ x \in X : \pi_j(x) + s(x|D_j) \leqq 0, j \in M \}$.

4. Unified second-order duality model-II:

In this section, we formulate the following unified second order dual for $(\text{MFP})_{\bar{\nu}}$ as:

$$\textbf{(MDP)}_{\bar{\nu}_i}\textbf{:}\ \text{Maximize}\ \Big(\phi_1(y) + y^T z_1 - \nu_1(\psi_1(y) - y^T v_1) - \frac{1}{2} p^T \nabla^2 \Big(\phi_1(y) + y^T z_1 - \nu_1(\psi_1(y) - y^T v_1) \Big) p + \sum_{j \in J_0} \mu_j \{ \pi_j(y) + y^T w_j - \frac{1}{2} p^T \nabla^2 \pi_j(y) p \}, ..., \phi_k(y) + y^T z_k - \nu_k(\psi_k(y) - y^T v_k) - \tfrac{1}{2} p^T \nabla^2 \Big(\phi_k(y) + y^T z_k - \nu_k(\psi_k(y) - y^T v_k) \Big) p + \sum_{j \in J_0} \mu_j \{ \pi_j(y) + y^T w_j - \frac{1}{2} p^T \nabla^2 \pi_j(y) p \} \Big)$$

subject to $\quad y \in X$,

$$\sum_{i=1}^{k} \lambda_i \Big\{ \nabla(\phi_i(y) + y^T z_i - \nu_i(\psi_i(y) - y^T v_i)) + \nabla^2 \Big(\phi_i(y) + y^T z_i - \nu_i(\psi_i(y) - y^T v_i) \Big) p \Big\} + \sum_{j=1}^{m} \mu_j \{ \nabla \pi_j(y) + w_j + \nabla^2 \pi_j(y) p \} = 0, \tag{4.1}$$

$$\sum_{j \in J_\beta} \mu_j \{ \pi_j(y) + y^T w_j - \frac{1}{2} p^T \nabla^2 \pi_j(y) p \} \geqq 0, \ \beta = 1, ..., r, \tag{4.2}$$

$$\phi_i(y) + y^T z_i - \nu_i(\psi_i(y) - y^T v_i) \geqq 0, \ \forall \ i, \tag{4.3}$$

$$\lambda_i \geqq 0, \quad \sum_{i=1}^{k} \lambda_i = 1, \tag{4.4}$$

$$\mu_j \geqq 0, \ z_i \in C_i, \ v_i \in E_i, \ w_j \in D_j, \quad \text{for } i \in K, \ j \in M, \tag{4.5}$$

where $J_\delta \subseteq N$, $\delta = 0, 1, ..., r$ with $\bigcup_{\delta=0}^{r} J_\delta = N$ and $J_{\delta_1} \bigcap J_{\delta_2}$ if $\delta_1 \neq \delta_2$. It may be noted that $J_0 = N$ and $J_\beta = \phi(1 \leqq \beta \leqq r)$, we obtain Wolfe type dual. If $J_0 = \phi$, $J_1 = N$ and $J_\beta = \phi(2 \leqq \beta \leqq r)$, then $(\text{MDP})_{\bar{\nu}}$ reduces to Mond-Weir Type dual.

For the above models $\Big($see $(\text{MFP})_{\bar{\nu}}$ and $(\text{MDP})_{\bar{\nu}}\Big)$, proof follows on the same lines as theorems $(3.1) - (3.5)$ under the aforesaid assumptions.

5. Conclusion

In this article, we have considered a mixed (unified) type nondifferentiable second order fractional dual model and prove duality theorems under weak strictly pseudo quasi (V, ρ, θ)-bonvex/pseudo quasi (V, ρ, θ)-bonvex/strictly pseudoquasi (V, ρ, θ)-bonvex type-I assumptions. The present work can further be extended to nondifferentiable higher order fractional programming arbitrary over cones. This will orient the future task of the authors/researchers.

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