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Existence of non-trivial solutions for fractional Schrödinger-Poisson systems with subcritical growth

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Abstract

In this paper, we are concerned with the following fractional Schrödinger-Poisson system:

$$\begin{cases} (-\Delta^s)u + u + \lambda\phi u = \mu f(u) + |u|^{p-2}|u|, & x \in \mathbb{R}^3 \\ (-\Delta^t)\phi = u^2, & x \in \mathbb{R}^3 \end{cases}$$

where λ, μ are two parameters, $s, t \in (0, 1]$,$2t + 4s > 3$,$1 < p \leq 2^*_s$ and $f : \mathbb{R} \to \mathbb{R}$ is continuous function. Using some critical point theorems and truncation technique, we obtain the existence and multiplicity of non-trivial solutions with the help of the variational methods.

Keywords: Fractional Schrödinger-Poisson systems, Sublinear nonlinearity, Variational methods

1. Introduction

The aim of this paper is to investigate the existence of non-trivial solutions for the following fractional Schrödinger-Poisson system

$$\begin{cases} (-\Delta^s)u + u + \lambda\phi u = \mu f(u) + |u|^{p-2}|u|, & x \in \mathbb{R}^3 \\ (-\Delta^t)\phi = u^2, & x \in \mathbb{R}^3 \end{cases} \tag{1.1}$$

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where λ, μ are two parameters, $s, t \in (0,1]$,$2t+4s>3$, $1<p\leq 2_s^*$, $(-\Delta^s)$ is the fractional Laplacian and $f(u)$ is continuous function. Where $2_s^* = \frac{6}{3-2s}$.

When $s = t = 1$ the equation (1.1) reduces to Schrödinger-Poisson equation, which describes quantum particles and is related to the study of nonlinear stationary Schrödinger equations interacting with the electromagnetic field generated by the motion [1, 2].

This article was motivated by [3].There the authors show the existence and multiplicity of solutions for the system

$$\begin{cases} (-\Delta)u + u + \lambda\phi u = \mu f(u) + |u|^{p-2}|u|, & x \in \Omega \\ (-\Delta)\phi = u^2, & x \in \Omega \end{cases} \tag{1.2}$$

where Ω is a smooth and bounded domain in $\mathbb{R}^3$, $1 < p \leq 6$, λ, μ are two parameters is a parameter and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. Our purpose is to show that when we consider this system with fractional Laplacian operator instead of the Laplacian, then we obtain the existence and multiplicity of non-trivial solutions for the system.

Fractional Schrödinger-Poisson equations have attracted some attention in recent years. If we only consider the first equation in (1.1) and assume that $\phi = 0$, then it reduces to a fractional Schrödinger equation, which is a fundamental equation in fractional quantum mechanics [4, 5].

Authors in [6] studied the existence of positive solutions and ground state solutions for the following system

$$\begin{cases} (-\Delta)^s u + V(x)u + \phi u = f(u), & \text{in } \mathbb{R}^3, \\ (-\Delta)^t\phi = u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.3}$$

Where $V : \mathbb{R}^3 \to \mathbb{R}^3$ is a continuous periodic potential and positive

Recently, some authors proposed a new approach called perturbation method to study the quasilinear elliptic equations, see [7, 8]

in [9] the existence of infinitely many solutions for the following system was studied by wei

$$\begin{cases} (-\Delta^s)u + V(x)u + \phi u = f(x,u), & \text{in } \mathbb{R}^3, \\ (-\Delta^s)\phi = \gamma_\alpha u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.4}$$

Where $s \in (0,1]$ and γ_α is a positive constant.

Kexue Li in [10] studied the nonlinear fractional Schrödinger-Poisson system

$$\begin{cases} (-\Delta^s)u + u + \phi u = f(x,u), & \text{in } \mathbb{R}^3, \\ (-\Delta^t)\phi = u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.5}$$

And by using the perturbation method and mountain pass theorem, obtained the existence of non-trivial solutions.

Up to our knowledge, there are no result on the existence of multiple solutions for problem (1.1) with the nonlinear term $f(u) + |u|^{p-2}u$. In particular, if f may be not odd, than the associated functional of (1.1) may be nonsymmetric which leads to the significant difficulty in finding multiple solutions. Indeed problem (1.1) with the nonsymmetric term f does possess a variational structure as well, problem (1.1) can be attacked by means of variational methods. Namely, the weak solutions are characterized as critical points of a C^1 functional $I = I(u)$ defined on the fractional sobolev space $H^s(\mathbb{R}^3)$. Here, we obtain a sufficient result ensuring the existence of at least three weak solutions for problem (1.1) whose the nonlinear term f may be nonsymmetric. The following is our result in the case of the subcritical exponent $p \in (1, 2)$.

Theorem 1.1. *Assume that $f \in C(\mathbb{R}, \mathbb{R})$ is not an odd function and the following condition holds:*
(f) there exist three positive constants c_1, c_2, q such that $|f(u)| \leq c_1 + c_2|u|^{q-1}$.
If $q \in (1, 2)$ and $p \in (1, 2)$, then there exist $L > 0$ and an open interval J with $0 \in J$ such that, for every $\mu \in J$, problem (1.1) with $\lambda = \mu$ admits at least three weak solutions whose norms are less or equal to L.

The proof of 1.1 is based on an abstract critical point theorem developed by Anello [11] in finding two local minimum points of the associated functional which is the two weak solutions of problem (1.1). And then, we find the third weak solution different from the earlier two ones using a Mountain Pass Theorem coming from [12].

The reminder of this paper is organized as follows. In section 2 we present a suitable variational framework for our problem. In section 3, we prove Theorem 1.1. Throughout this paper, $C > 0$ will be used indiscriminately to denote a suitable positive constant whose value may change from line to line. Moreover, we use $\|.\|_s$ to denote the usual norm on $L^s(\mathbb{R}^3)$ for $1 < s < +\infty$.

2. Variational setting and preliminaries

For $p \in [1, \infty)$, we denote by $L^p(\mathbb{R}^3)$ the usual Lebesgue space with the norm $\|u\|_p = \left(\int_{\mathbb{R}^3} |u|^p dx\right)^{\frac{1}{p}}$. For any $p \in [1, \infty)$ and $s \in (0, 1)$, we recall some definitions of fractional Sobolev spaces and the fractional Laplacian $(-\Delta)^s$, for more details, we refer to [13]. $H^s(\mathbb{R}^3)$ is defined as follows

$$H^s(\mathbb{R}^3) = \left\{ u \in L^2(\mathbb{R}^3) : \int_{\mathbb{R}^3} (1 + |\xi|^{2s}) |\mathcal{F}u(\xi)|^2 d\xi < \infty \right\}$$

with the norm

$$\|u\|_{H^s} = \left(|\mathcal{F}u(\xi)|^2 + |\xi|^{2s} |\mathcal{F}u(\xi)|^2 \right)^{\frac{1}{2}}, \tag{2.1}$$

where $\mathcal{F}u$ denotes the Fourier transform of u. By $\mathcal{S}(\mathbb{R}^3)$, we denote the Schwartz space of rapidly decaying C^∞ functions in $\mathbb{R}^3$. For $u \in \mathcal{S}(\mathbb{R}^3)$ and $s \in (0,1)$, $(-\Delta)^s$ is defined by

$$(-\Delta)^s f = \mathcal{F}^{-1}(|\xi|^{2s}(\mathcal{F}f)), \ \forall \xi \in \mathbb{R}^3.$$

By Plancherel's theorem, we have $\|\mathcal{F}u\|_2 = \|u\|_2$, $\||\xi|^s \mathcal{F}u\|_2 = \|(-\Delta)^{\frac{s}{2}} u\|$. Then by (2.1), we get the equivalent norm

$$\|u\|_{H^s} = \left(\int_{\mathbb{R}^3} (|(-\Delta)^{\frac{s}{2}} u(x)|^2 + |u(x)|^2) dx \right)^{\frac{1}{2}}.$$

For $s \in (0,1)$, the fractional Sobolev space $D^{s,2}(\mathbb{R}^3)$ is defined as follows

$$D^{s,2}(\mathbb{R}^3) = \left\{ u \in L^{2_s^*}(\mathbb{R}^3) : |\xi|^s \mathcal{F}u(\xi) \in L^2(\mathbb{R}^3) \right\},$$

which is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|_{D^{s,2}} = \left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx \right)^{\frac{1}{2}} = \left(\int_{\mathbb{R}^3} |\xi|^{2s} |\mathcal{F}u(\xi)|^2 d\xi \right)^{\frac{1}{2}}.$$

Lemma 2.1. (Theorem 2.1 in [14]). *For any $s \in (0, \frac{3}{2})$, $D^{s,2}(\mathbb{R}^3)$ is continuously embedded in $L^{2_s^*}(\mathbb{R}^3)$, i.e., there exists $c_s > 0$ such that*

$$\left(\int_{\mathbb{R}^3} |u|^{2_s^*} dx \right)^{2/2_s^*} \leq c_s \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx, \ u \in D^{s,2}(\mathbb{R}^3).$$

We consider the variational setting of (1.1). From Theorem 6.5 and Corollary 7.2 in [13], it is known that the space $H^s(\mathbb{R}^3)$ is continuously embedded in $L^q(\mathbb{R}^3)$ for any $q \in [1, 2_s^*]$ and the embedding $H^s(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3)$ is locally compact for $q \in [1, 2_s^*)$.

If $2t + 4s > 3$, then $H^s(\mathbb{R}^3) \hookrightarrow L^{\frac{12}{3+2t}}(\mathbb{R}^3)$. For $u \in H^s(\mathbb{R}^3)$, the linear operator $T_u : D^{t,2}(\mathbb{R}^3) \to \mathbb{R}$ defined as

$$T_u(v) = \int_{\mathbb{R}^3} u^2 v dx,$$

By Hölder inequality and Lemma 2.1,

$$|T_u(v)| \leq \|u\|^2_{12/(3+2t)}\|v\|_{2_t^*} \leq C\|u\|^2_{H^s}\|v\|_{D^{t,2}}. \tag{2.2}$$

Set

$$\eta(u,v) = \int_{\mathbb{R}^3} (-\Delta)^{\frac{t}{2}} u \cdot (-\Delta)^{\frac{t}{2}} v dx,\ u,v \in D^{t,2}(\mathbb{R}^3).$$

It is clear that $\eta(u,v)$ is bilinear, bounded and coercive. The Lax-Milgram theorem implies that for every $u \in H^s(\mathbb{R}^3)$, there exists a unique $\phi_u^t \in D^{t,2}(\mathbb{R}^3)$ such that $T_u(v) = \eta(\phi_u, v)$ for any $v \in D^{t,2}(\mathbb{R}^3)$, that is

$$\int_{\mathbb{R}^3} (-\Delta)^{\frac{t}{2}} \phi_u^t (-\Delta)^{\frac{t}{2}} v dx = \int_{\mathbb{R}^3} u^2 v dx. \tag{2.3}$$

Therefore, $(-\Delta)^t \phi_u^t = u^2$ in a weak sense. Moreover,

$$\|\phi_u^t\|_{D^{t,2}} = \|T_u\| \leq C\|u\|^2_{H^s}. \tag{2.4}$$

Since $t \in (0,1]$ and $2t+4s > 3$, then $\frac{12}{3+2t} \in (2, 2_s^*)$. From Lemma 2.1, (2.2) and (2.3), it follows that

$$\|\phi_u^t\|^2_{D^{t,2}} = \int_{\mathbb{R}^3} |(-\Delta)^{\frac{t}{2}} \phi_u^t|^2 dx = \int_{\mathbb{R}^3} u^2 \phi_u^t dx \leq \|u\|^2_{\frac{12}{3+2t}} \|\phi_u^t\|_{2_t^*} \leq C\|u\|^2_{\frac{12}{3+2t}} \|\phi_u^t\|_{D^{t,2}}. \tag{2.5}$$

Then

$$\|\phi_u^t\|_{D^{t,2}} \leq C\|u\|^2_{\frac{12}{3+2t}}. \tag{2.6}$$

For $x \in \mathbb{R}^3$, we have

$$\phi_u^t(x) = c_t \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|^{3-2t}} dy, \tag{2.7}$$

which is the Riesz potential [15], where

$$c_t = \frac{\Gamma(\frac{3-2t}{2})}{\pi^{3/2} 2^{2t} \Gamma(t)}.$$

Substituting ϕ_u^t in (1.1), we have the fractional Schrödinger equation

$$(-\Delta)^s u + u + \lambda \phi_u^t u = \mu f(u) + |u|^{p-2}|u|,\ x \in \mathbb{R}^3, \tag{2.8}$$

The energy functional $I : H^s(\mathbb{R}^3) \to \mathbb{R}$ corresponding to problem (2.8) is defined by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3} (|(-\Delta)^{\frac{s}{2}} u(x)|^2 + |u(x)|^2) dx + \frac{\lambda}{4}\int_{\mathbb{R}^3} \phi_u^t u^2 dx - \mu \int_{\mathbb{R}^3} F(u) dx - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p dx. \tag{2.9}$$

It is easy to see that I is well defined in $H^s(\mathbb{R}^3)$ and $I \in C^1(H^s(\mathbb{R}^3), \mathbb{R})$, and

$$\langle I'(u), v\rangle = \int_{\mathbb{R}^3} \left((-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v + uv + \lambda \phi_u^t uv - \mu f(u) v - |u|^{p-2} uv\right) dx,\quad v \in H^s(\mathbb{R}^3). \tag{2.10}$$

Definition 2.2.

(1) We call $(u,\phi)\in H^s(\mathbb{R}^3)\times D^{t,2}(\mathbb{R}^3)$ *is a weak solution of* (1.1) *if* u *is a weak solution of (2.8).*

(2) We call u *is a weak solution of (2.8) if*

$$\int_{\mathbb{R}^3}\left((-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}v+uv+\lambda\phi_u^t uv-\mu f(u)v-|u|^{p-2}uv\right)dx=0,$$

for any $v\in H^s(\mathbb{R}^3)$.

Definition 2.3. *We say a* C^1 *functional* I *satisfies Palais-Smale condition* $\big($*(PS) condition for short*$\big)$ *if any sequence* $\{u_n\}\subset H^s(\mathbb{R}^3)$ *such that*

$$I(u_n)\ \textit{being bounded,}\ I'(u_n)\to 0,\quad \textit{as}\ n\to 0 \tag{2.11}$$

admits a convergent subsequence, and such a sequence is called a Palais-Smale sequence $\big($*(PS) sequence*$\big)$.

Now we define the following integral momentums

$$\Psi(u):=\frac{1}{2}\|u\|^2-\frac{1}{p}\int_{\mathbb{R}^3}|u|^p dx,\qquad\qquad \Phi(u):=\frac{1}{4}\int_{\mathbb{R}^3}\phi_u^t u^2dx-\int_{\mathbb{R}^3}F(u)dx \tag{2.12}$$

Theorem 2.4. *([16]).Let* E *be a real Banach space and* $I\in C^1(E,\mathbb{R})$ *satisfy the (PS)-condition. If* I *is bounded from below, then* $c=\inf_E I$ *is a critical value of* I.

Theorem 2.5. *([11]).Let* E *be a reflexive Banach space and* Φ,Ψ *be two sequentially weakly lower semicontinuous real functionals defined on* E*. Suppose* Ψ *is (strongly) continuous. Moreover, assume that there exists* $x_1,x_2,...,x_n\in E$*,* $r_1,...,r_n>0$*, with* $r_i+r_j<\|x_i-x_j\|$ *for all* $i,j\in\{1,...,n\}$ *with* $i\neq j$*, such that foe all* $i\in\{1,...,n\}$*,*

(a) the functional $r\to\inf_{\|x\|=r}\Psi(x+x_i)$ *is continuous in* $\mathbb{R}^+$*,*

(b) $\Psi(x_i)<\inf_{\|x\|=r_i}\Psi(x+x_i)$.

Then there exists $\rho^*>0$ *such that for every* $\rho>\rho^*$ *the functional* $\rho\Psi+\Phi$ *admits at least* n *distinct local minimum points* $y_1,...,y_n$ *such that* $\|x-i-y_j\|<r_i$ *for all* $i=1,...,n$.

Theorem 2.6. *([12]). Let* X *be a Banach space and assume that* I *satisfies the following conditions:*

(H1) there exist numbers a,r,R *such that* $0<r<R$ *and* $I(x)\geq a$ *for every* $x\in A:=\{x\in X: r<\|x\|<R\}$;

(H2) $I(0)\leq a$ *and* $I(e)\leq a$ *for some* e *with* $\|e\|\geq R$.

If I *satisfies the Palais-Smale compactness condition, then there exists a critical point* x*, in* X*, different from* 0 *and* e*, with critical value* $c\geq a$*; moreover,* $x\in A$ *when* $c=a$*, where* c *is characterized by*

$$\inf_{\gamma\in\Gamma}\sup_{t\in[0,1]}I(\gamma(t))=c,\qquad \Gamma=\{\gamma\in C([0,1],X):\gamma(0)=0\qquad \textit{and}\qquad \gamma(1)=e\}$$

3. Proof of Theorem 1.1

In this section, under the condition that f may be not odd function, we give the result of the existence of at least three solutions for problem (1.1) (i.e., Theorem 1.1). In what follows, we will give the proof of Theorem 1.1.

Proof . Firstly, with the help of Theorem 2.5, we show that problem (1.1) has at least two weak solutions. It follows from the conditions of Theorem 1.1 and 2.4-2.7 that Ψ,Φ are two well-defined differentiable and sequentially weakly lower semicontinuous functionals. Moreover, Ψ is strongly continuous and coercive. Hence, it is clear that Ψ is bounded from below in $H^s(\mathbb{R}^3)$. Next we show Ψ satisfies the (PS)-condition. Assume that $\{w_n\} \subset H^s(\mathbb{R}^3)$ such that $\{\Psi(w_n)\}$ is bounded and $\Psi'(w_n) \to 0$ as $n \to \infty$. Then there exist positive constants $C, C_p > 0$ such that

$$C > \Psi(w_n) \geq \frac{1}{2}\|w_n\|^2 - \frac{1}{p}C_p\|w_n\|^p$$

In view of the above inequality, we know that $\|w_n\|$ is bounded. Hence, there exists $w_0 \in H^s(\mathbb{R}^3)$ such that $w_n \to w_0$. It follows from the definition of Ψ that

$$\langle \Psi'(w_n) - \Psi'(w_0), w_n - w_0\rangle = \|w_n - w_0\|^2 - \int_{\mathbb{R}^3} (|w_n|^{p-2}w_n - |w_0|^{p-2}w_0)(w_n - w_0). \tag{3.1}$$

Noting that, by sobolev embedding theorem and Höldre inequality, we have

$$\begin{aligned} &|\int_{\mathbb{R}^3} (|w_n|^{p-2}w_n - |w_0|^{p-2}w_0)(w_n - w_0) \\ &\leq \|w_n\|_6^{\frac{p-1}{6}} \|w_n - w_0\|_{\frac{6}{7-p}}^{\frac{7-p}{6}} + |\int_{\mathbb{R}^3} |w_0|^{p-2}w_0(w_n - w_0)| \to 0 \end{aligned} \tag{3.2}$$

as $n \to \infty$. Moreover, it is easy to see that $\langle \Psi'(w_n) - \Psi'(w_0), w_n - w_0\rangle \to 0$ as $n \to \infty$. Combining 3.1 with 3.2, we deduce that $w_n \to w_0$ as $n \to \infty$. Therefore, Ψ satisfies the (PS)-condition. Theorem 2.4 implies that there exists w satisfying $\Psi(w) = \inf_{H^s(\mathbb{R}^3)} \Psi$ and $\Psi'(w) = 0$. We claim that $w \neq 0$. Let $\overline{w} \in H^s(\mathbb{R}^3)/\{0\}$, then $\Psi(s\overline{w}) < 0$ for sufficiently small positive number s.

Therefore $\Psi(w) = \inf_{H^s(\mathbb{R}^3)} < 0$. Our claim is true. Moreover, the standard elliptic estimates imply that $w \in L^\infty(\mathbb{R}^3)$ and $v > 0$ follows from the strong maximum principle. we can easily deduce from [17] and the definition of Ψ that w and $-w$ are the unique two global minimum points of the functional Ψ over $H^s(\mathbb{R}^3)$. It is shown in [18] that, if $u_0 \in H^s(\mathbb{R}^3)$, then the real function

$$r \to \inf_{\|u\|=r} \Psi(u - u_0) = \frac{r^2}{2} + \frac{\|u_0\|^2}{2} - \sup_{\|u\|=r} \left(\langle u, u_0\rangle + \frac{1}{p}\int_{\mathbb{R}^3} |u - u_0|^p dx \right)$$

is continuous in $\mathbb{R}^+$. Now we claim that, for each fixed $r \in (0, 2\|w\|)$,

$$\inf_{\|u\|=r} \Psi(u \pm w) > \inf_{H^s(\mathbb{R}^3)} \Psi = \Psi(\pm w) \tag{3.3}$$

If not, without loss of generality, suppose that there exists $r_0 \in (0, 2\|w\|)$ such that

$$\inf_{\|u\|=r_0} \Psi(u + w) = \inf_{H^s(\mathbb{R}^3)} \Psi \tag{3.4}$$

The argument is same as the case where

$$\inf_{\|u\|=r_0} \Psi(u - w) = \inf_{H^s(\mathbb{R}^3)} \Psi$$

So we only consider the case 3.4. It follows from 3.4 that there exists a sequence $\{u_n\} \subset H^s(\mathbb{R}^3)$ with $\|u_n\| = r_0$ such that

$$\lim_{n\to\infty} \Psi(u_n + w) = \inf_{H^s(\mathbb{R}^3)} \Psi$$

From this, up to subsequence, again denoted by $\{u_n\}$, we have $u_n \to u^*$ in $H^s(\mathbb{R}^3)$. Therefore, by using Sobolev embedding theorem we have

$$\begin{aligned} 0 &= \lim_{n\to+\infty} \Psi(u_n + w) - \Psi(w) \\ &= \lim_{n\to+\infty} \left(\frac{r_0^2}{2} + \frac{\|w\|^2}{2} + \langle u_n, w\rangle - \frac{1}{p}\int_{\mathbb{R}^3} |u_n + w|^p dx \right) - \Psi(w) \\ &= \frac{r_0^2}{2} + \langle u^*, w\rangle + \frac{1}{p}\int_{\mathbb{R}^3} (|w|^p - |u^* + w|^p) dx. \end{aligned} \tag{3.5}$$

If we put $B_1 = \{x \in \mathbb{R}^3 : u^* \neq 0\}$, then $|B_1| > 0$. Or else, by (3.5), it would be $r_0 = 0$, against the choice of r_0. On the other hand, if we put $B_2 = \{x \in \mathbb{R}^3 : u^* \neq -2w\}$, then $|B_2| > 0$. Otherwise, again by (3.5), it would be $r_0 = 2\|w\|$, which contradicts with the choice of r_0. Consequently, the function $u^* + w$ is different from w and $-w$. So it follows from Fatou lemma,Sobolev embedding theorem and $\Psi(w) = \inf_{H^S(\mathbb{R}^3)} \Psi$

$$\begin{aligned} \Psi(w) &= \lim_{n\to+\infty} \Psi(u_n + w) \\ &= \lim_{n\to+\infty} \int_{\mathbb{R}^3} [\frac{1}{2}|(-\Delta)^{\frac{s}{2}}(u_n + w)|^2 + |u_n + w|^2 - \frac{1}{p}|u_n + w|^p] dx \\ &\geq \int_{\mathbb{R}^3} [\frac{1}{2}|(-\Delta)^{\frac{s}{2}}(u^* + w)|^2 + |u^* + w|^2 - \frac{1}{p}|u^* + w|^p] dx > \Psi(w), \end{aligned} \tag{3.6}$$

which is impossible. Hence, 3.3 holds. Fix $r \in (0, \|w\|)$, then it is easy to check that all the hypotheses of theorem 2.5 are fulfilled if we take $n = 2$, $r_1 = r_2 = r$ and $x_1 = -x_2 = w$. Hence there exists

$\rho^* > 0$ such that for all $\rho > \rho^*$ the functional $\rho\Psi + \Phi$ contains at least two distinct local minimum points u_1^ρ, u_2^ρ satisfying

$$\max\{\|u_1^\rho - w\|, \|u_2^\rho + w\|\} < r.$$

Indeed, such minimum points are critical points of the same functional. Hence, if we put $\mu_1^* = \frac{1}{\rho^*}$, we have that, for all $\mu \in (0, \mu_1^*)$, functional $\Psi + \mu\Phi$ admits at least two critical points u_1^*, u_2^* such that

$$\max\{\|u_1^*\|, \|u_2^*\|\} \leq 2[\|w\| + r].$$

That is, problem (1.1) admits at least two weak solutions for $\lambda = \mu \in (0, \mu_1^*)$.It is easy to see that, for $\mu = 0$, the same conclusion holds and the two weak solutions are exactly w and $-w$. Now, consider the functional

$$\Phi_1(u) = -\Phi(u) = \int_{\mathbb{R}^3} F(u)dx - \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^t u^2 dx$$

defined for all $u \in H^s(\mathbb{R}^3)$. Repeating the same above argument applied to Φ_1 and Ψ, we easily deduce that there exists μ_2^* such that the previous conclusion holds for all $\mu \in (-\mu_2^*, 0)$. Put $\sigma = 2[\|w\| + r]$ and $J = (-\mu_2^*, \mu_1^*)$, then problem (1.1) contains at least two weak solutions whose norms are less or equal that σ.

Secondly, using a Mountain Pass Theorem (see Theorem 2.6), we will find the third weak solution for problem (1.1). We show that $\Psi + \mu\Phi$ satisfies the (PS)-condition. Assume that u_n is a sequence in $H^s(\mathbb{R}^3)$ such that $\{\Psi(u_n) + \mu\Phi(u_n)\}$ is bounded and $\Psi'(u_n) + \mu\Phi'(u_n) \to 0$ as $n \to +\infty$.

By (2.9),(2.10) and (2.12), there exist two positive constant a_1, a_2 such that

$$\begin{aligned} a_1 + a_2\|u_n\| &\geq \Psi(u_n) + \mu\Phi(u_n) - \frac{1}{4}[\Psi'(u_n)u_n + \mu\Phi'(u_n)u_n] \\ &\geq \frac{1}{4}\|u_n\|^2 - (\frac{1}{p} - \frac{1}{4})\int_{\mathbb{R}^3} |u_n|^p dx + \mu\int_{\mathbb{R}^3} (\frac{1}{4}f(u_n)u_n - F(u_n))dx \\ &\geq \frac{1}{4}\|u_n\|^2 - (\frac{1}{p} - \frac{1}{4})C\|u\|^p + \mu(c_1C\|u\| + c_2C\|u\|^q) \end{aligned}$$

for $\mu \in J$. in view of the earlier inequality, using the fact that $p, q < 2$ we conclude that $\{u_n\}$ is bounded in $H^S(\mathbb{R}^3)$ for $\mu \in J$ and, up to subsequence,

$$\begin{array}{lll} u_n \rightharpoonup u & \text{in} & H^s(\mathbb{R}^3) \\ u_n \to u & \text{in} & L^s(\mathbb{R}^3) \quad \text{for} \quad 1 \leq s < 6 \\ u_n(x) \to u(x) & \text{a.e. in} & \mathbb{R}^3 \end{array} \tag{3.7}$$

Hence, $\Psi'(u_n)(u_n - u) + \mu\Phi'(u_n)(u_n - u) = o_n(1)$, that is

$$\langle u_n, u_n - u\rangle + \int_{\mathbb{R}^3} (\mu\phi_{u_n}^t u_n - \mu f(u_n) - |u_n|^{p-2}u_n(u_n - u))dx = o_n(1) \tag{3.8}$$

From 2.4, we have

$$\int_{\mathbb{R}^3} \mu\phi_{u_n}^t u_n(u_n - u)dx = o_n(1) \tag{3.9}$$

Using (f), (3.7) and $1 < p < 2$, we obtain

$$\int_{\mathbb{R}^3} \mu f(u_n)(u_n - u)dx = o_n(1) \qquad \text{and} \qquad \int_{\mathbb{R}^3} |u_n|^{p-1}u_n(u_n - u)dx = o_n(1) \tag{3.10}$$

Therefore, combining 3.9, 3.10 with 3.8, we get

$$\langle u_n, u_n - u\rangle = o_n(1) \tag{3.11}$$

With the help of the fact that $\langle u_n, u_n - u\rangle = o_n(1)$, we deduce that $u_n \to u$ in $H^s(\mathbb{R}^3)$. In fact, the two weak solutions u_1^*, u_2^* turn out to be the global minimum points for the restriction of the functional $\Psi + \mu\Phi$ to the set $B_r(w)$ and $B_r(-w)$, respectively. Therefore, by applying Theorem 2.6, for every $\mu \in J$, we can get a critical points u_3^* of $\Psi + \mu\Phi$ different from u_1^* and u_2^* such that

$$\Psi(u_3^*) + \mu\Phi(u_3^*) = c(\mu) \tag{3.12}$$

where

$$c(\mu) = \inf_{\gamma\in\Gamma_\mu} \sup_{t\in[0,1]} (\Psi(\gamma(t)) + \mu\Phi(\gamma(t))),$$

and

$$\Gamma_\mu = \{\gamma \in C([0,1], H^s(\mathbb{R}^3)) : \gamma(0) = u_1^* \qquad \text{and} \qquad \gamma(1) = u_2^*\}$$

Note that, for every $\mu \in J$ and $t \in [0,1]$, if we take

$$\gamma_0(t) = tu_2^* + (1-t)u_1^*$$

then $\gamma_0 \in \Gamma_\mu$ and $\|\gamma_0(t)\| \le \sigma$. Consequently, for $\mu \in J$ we have

$$\begin{aligned} c(\mu) &= \inf_{\gamma\in\Gamma_\mu} \sup_{t\in[0,1]} (\Psi(\gamma(t)) + \mu\Phi(\gamma(t))) \\ &\le \sup_{t\in[0,1]} (\Psi(\gamma_0(t)) + \mu\Phi(\gamma_0(t))) \\ &\le \frac{\sigma^2}{2} + |\mu|(\frac{1}{4}C\|\gamma_0(t)\|^4 + \sup_{t\in[0,1]} \int_{\mathbb{R}^3} |F(\gamma_0(t))|dx) \\ &\le \frac{\sigma^2}{2} + (\mu_1^* + \mu_2^*)(\frac{1}{4}C\sigma^4 + c_1C\sigma + c_2C\sigma^q) := C^* \end{aligned} \tag{3.13}$$

It follows from 3.12 and (3.13) that

$$\begin{aligned} C^* \geq c(\mu) &= \Psi(u_3^*) + \mu\Phi(u_3^*) - \frac{1}{4}[\Psi'(u_3^*)u_3^* + \mu\Phi'(u_3^*)u_3^*] \\ &\geq \frac{1}{4}\|u_3^*\|^2 - (\frac{1}{p} - \frac{1}{4})\int_{\mathbb{R}^3} |u_3^*|^p dx + \mu\int_{\mathbb{R}^3} (\frac{1}{4}f(u_3^*)u_3^* - F(u_3^*))dx \\ &\geq \frac{1}{4}\|u_3^*\|^2 - (\frac{1}{p} - \frac{1}{4})C\|u_3^*\|^p + \mu(c_1 C\|u_3^*\| + c_2 C\|u_3^*\|^q). \end{aligned} \tag{3.14}$$

Since $p, q < 2$, for $\mu \in J$, there exists a positive constant C_2 such that $\|u_3^*\| \leq C_2$. Therefore, let $L = \max\{\sigma, C_2\}$, then we have $\max\{\|u_1^*\|, \|u_2^*\|, \|u_3^*\|\} \leq L$ for all $\mu \in J$. This completes the proof of Theorem 1.1. □

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