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Solution of Vacuum Field Equation Based on Physics Metrics in Finsler Geometry and Kretschmann Scalar

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Abstract

The Lemaître-Tolman-Bondi (LTB) model represents an inhomogeneous spherically symmetric universe filled with freely falling dust like matter without pressure. First, we have considered a Finslerian anstaz of (LTB) and have found a Finslerian exact solution of vacuum field equation. We have obtained the $R(t,r)$ and $S(t,r)$ with considering establish a new solution of $R_{\mu\nu} = 0$. Moreover, we attempt to use Finsler geometry as the geometry of spacetime which compute the Kretschmann scalar. An important problem in General Relativity is singularities. The curvature singularities is a point when the scalar curvature blows up diverges. Thus we have determined K_s singularity is at $R = 0$. Our result is the same as Reimannian geometry. We have completed with a brief example of how these solutions can be applied. Second, we have some notes about anstaz of the Schwarzschild and Friedmann- Robertson- Walker (FRW) metrics. We have supposed condition $d\log(F) = d\log(\bar{F})$ and we have obtained $\bar{F}$ is constant along its geodesic and geodesic of F. Moreover we have computed Weyl and Douglas tensors for F^2 and have concluded that $R_{ijk} = 0$ and this conclude that $W_{ijk} = 0$, thus F^2 is the Ads Schwarzschild Finsler metric and therefore F^2 is conformally flat. We have provided a Finslerian extention of Friedmann- Lemaitre- Robertson- Walker metric based on solution of the geodesic equation. Since the vacuum field equation in Finsler spacetime is equivalent to the vanishing of the Ricci scalar, we have obtained the energy- momentum tensor is zero.

Keywords: Einstein's equations, Lemaître–Tolman–Bondi; Kretschmann scalar, Finsler Geometry, Friedmann-Robertson-Walker, Schwarzschild.

1. Introduction

Cosmic structures today have entered the non-linear structure. They can not on all scales be described by a linear perturbation theory on top of the Friedmann-Lemaitre-Robertson-Walker (FLRW) metric

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[1]. Lemaitre-Tolman-Bondi (LTB) solutions are used frequently to describe the collapse or expansion of spherically symmetric inhomogeneous mass distributions in the Universe. The LTB solutions contain as special cases both the Schwazchild and dust FLRW solutions. The LTB metric is the simplest spherically symmetric solution of Einstein equations representing an inhomogeneous dust distribution. It may be written in synchronous coordinates as [2]:

$$F^2 = -y^t y^t + (\frac{{R'}^2}{1+E}) y^r y^r + R^2(r,t)(y^\theta y^\theta + sin\theta y^\varphi y^\varphi),$$

The Kretschmann scalar gives the amount of curvature of spacetime, as a function of position near (and within) a black hole [3]. In many cases one of the most useful ways to check that is by checking for the finiteness of the Kretschmann scalar. The alternative way to describe the black holes within the cosmological surrounding is to use the Lemaitre-Tolman-Bondi (LTB) models for inhomogeneous matter distribution that was fundamentally explored in [4].
There are very few exact solutions of the Einstein equations, but perhaps the most well-known solution was first derived by Schwarzschild. Exact solutions of Einstein's field equations have played important roles in the discussion of physical problems. Obvious examples are the Schwarzschild and Kerr solutions for studying black holes and the Friedman solutions for cosmology [5].
Gravitational field equations describing the geometry of space-time play a fundamental role in modern theoretical physics. There are many methods in mathematical physics for studying such systems of equations. For example, one can use methods of additional symmetries of the system of equations, the Hamiltonian formulation of the theory of dynamical systems, etc. Essentially, all physically interesting cases (FRW cosmology, black hole) belong to Stackel and homogeneous spaces [6]
In contrast, the cosmic fluid in the FRW model is supposed to possess arbitrary high pressure apart from arbitrary high density. But there is no pressure gradient to support the fluid against its self-gravity [7].
The paper is organized as follows. We review some basic matterial of Finsler geometry and list of tools that is needed in context. In section III we establish a new solution of $R^{\mu\nu} = 0$ which appears curious and explains the source of curvature of the well-known Lemaître–Tolman–Bondi(LTB) solution. Moreover, we have analytic solution to the geodesic equations. In the section V we compute Kretschmann scalar and show that the LTB singularity is at $R = 0$. In the section VI we obtain Kretschmann scalar for a one example. The section VII presents opinion of Li and Chang [8] to other words. We consider an ansatz of the Schwarzschild metric F^2 and we show results of vacuum solution. In the section $VIII$, we compute Weyl and Douglas tensors for F^2 and conclude that $R_{ijk} = 0$ and this conclude that $W_{ijk} = 0$, thus F^2 is the Ads Schwarzschild Finsler metric and therefore F^2 is conformally flat. Then we consider Douglas tensor if $D = 0$ we conclude that F^2 is the Schwarzschild metric. In the section IX, we consider $FLRW$ metric that it has the Finslerian structure then if only we suppose that the radial motion of particles, and notice the velocity of a particle $\frac{dr}{dt}$ is small, therefore the energy-momentum tensor is zero.

2. Preliminaries

We begin with a very brief discussion of Finsler geometry.
A Finsler metric is a continuous function $F : TM \to [0, \infty]$ with the following properties.
(1) Regularity: F is smooth on $TM \setminus 0 := \{(x, y) \in TM | y \neq 0\}$.
(2) Positive homogeneity: $F(x, \lambda y) = \lambda F(x, y)$ for all $\lambda > 0$.

(3) Strong convexity: the fundamental tensor $g_{ij} := \frac{\partial}{\partial y^i}\frac{\partial}{\partial y^j}(\frac{1}{2}F^2)$.
is positive definite at all $(x, y) \in TM \setminus 0$ [9].
A positive, zero and negative F correspond to time-like, null and space-like curves, respectively. For a Finsler metric $F = F(x, y)$ its geodesics are characterized by the system of differential equations

$$\frac{d^2x^\mu}{d\tau^2} + 2G^\mu = 0, \tag{2.1}$$

Where the local functions $G^\mu = G^\mu(x, y)$ are called the spray coefficients and are given by

$$G^\mu = \frac{1}{4}g^{\mu\nu}(\frac{\partial^2 F^2}{\partial x^\lambda \partial y^\nu}y^\lambda - \frac{\partial F^2}{\partial x^\nu}), \tag{2.2}$$

Where $y \in T_xM$ and $(g^{\mu\nu}) := (g_{\mu\nu})^{-1}$.
The Cartan tensor quantifies, the deviation of F from being Riemannian, and is defined as

$$A_{ijk} := \frac{\partial}{\partial y^i}\frac{\partial}{\partial y^j}\frac{\partial}{\partial y^k}(\frac{1}{4}F^2), \tag{2.3}$$

Where the components A_{ijk} are minus one-homogeneous symmetric $(0, 3)$-tensor field. In order to calculate the Kretschmamm scalar, we need first to calculate the Christoffel symbols by differentiating the metric of F.

$$\Gamma^\mu_{\nu\sigma} = \frac{1}{2}g^{\mu l}(\frac{\delta g_{l\sigma}}{\delta x^\nu} + \frac{\delta g_{l\nu}}{\delta x^\sigma} - \frac{\delta g_{\sigma\nu}}{\delta x^l}), \tag{2.4}$$

where $\frac{\delta}{\delta x^\mu} = \frac{\partial}{\partial x^\mu} - \frac{\partial G^\rho}{\partial y^\mu}\frac{\partial}{\partial y^\rho}$. In Finsler geometry once the Christoffel symbols are calculated then we calculate the Riemann curvature of Chern connection to be:

$$R^\mu_{\nu\sigma\lambda} = \frac{\delta\Gamma^\mu_{\nu\lambda}}{\delta x^\sigma} - \frac{\delta\Gamma^\mu_{\nu\sigma}}{\delta x^\lambda} + \Gamma^l_{\nu\lambda}\Gamma^\mu_{l\sigma} - \Gamma^l_{\nu\lambda}\Gamma^\mu_{l\lambda}, \tag{2.5}$$

From the Riemann tensor $R^\lambda{}_{\nu\rho\sigma}$ one can calculate the Riemann curvature tensor $R_{\mu\nu\rho\sigma}$ as follows:

$$R_{\mu\nu\rho\sigma} = g_{\mu\lambda}R^\lambda{}_{\nu\rho\sigma}, \tag{2.6}$$

The inverse of $R_{\mu\nu\rho\sigma}$ can be compute by

$$R^{\mu\nu\rho\sigma} = g^{\mu i}g^{\nu j}g^{\rho k}g^{\sigma l}R_{ijkl}, \tag{2.7}$$

The Kretschmann invariant is

$$K_s = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} \tag{2.8}$$

Because it is a sum of squares of tensor components, this is a quadratic invariant. For the use of a computer algebra system a more detailed writing is meaningful:

$$K_s = g^{\mu i}g^{\nu j}g^{\rho k}g^{\sigma l}R_{ijkl}R_{\mu\nu\rho\sigma}, \tag{2.9}$$

The Ricci tensor $R_{\mu\nu}$, it is the Riemann curvature tensor with the first and third indices contracted,

$$R_{\mu\nu} = g^{\alpha\beta}R_{\alpha\mu\beta\nu} = R^\beta_{\mu\beta\nu},$$

Which is a symmetric tensor,

$$R_{\mu\nu} = R_{\nu\mu},$$

It is straightforward to show that, because of the symmetry relations, the alternative contraction leads to the same Ricci tensor

$$R_{\mu\nu} = -g^{\alpha\beta} R_{\mu\alpha\beta\nu},$$

The Ricci scalar R, it is the Riemann curvature tensor contracted twice.

$$R = g^{\alpha\beta} R_{\alpha\beta} = R^{\beta}_{\beta},$$

By definition, the Ricci scalar is a positively homogeneous function of degree two in $y \in TM$.
For a Finsler metric F, the $S-$curvature is given by following:

$$S = g^{\mu\nu} Ric_{\mu\nu}, \tag{2.10}$$

and, as a consequence, the modified Einstein tensor in Finsler space time can be obtained as

$$G_{\mu\nu} \equiv Ric_{\mu\nu} - \frac{1}{2} g_{\mu\nu} S, \tag{2.11}$$

The covariant derivative of G^{μ}_{ν} in Finsler spacetime is given as

$$G^{\mu}_{\nu|k} = \frac{\delta}{\delta x^k} G^{\mu}_{\nu} + \Gamma^{\mu}_{k\rho} G^{\rho}_{\nu} - \Gamma^{\rho}_{k\nu} G^{\mu}_{\rho}, \tag{2.12}$$

$\Gamma^{\mu}_{k\rho}$ is the Chern connection and $'|'$ to denote the covariant derivative. Here, we consider also the Rici scalar which is expressed entirely in terms of x and y derivatives of spray coefficients G^{μ} as follows:

$$Ric F^2 = 2\frac{\partial G^{\mu}}{\partial x^{\mu}} - y^{\lambda} \frac{\partial^2 G^{\mu}}{\partial x^{\lambda} \partial y^{\mu}} + 2G^{\lambda} \frac{\partial^2 G^{\mu}}{\partial y^{\lambda} \partial y^{\mu}} - \frac{\partial G^{\mu}}{\partial y^{\lambda}} \frac{\partial G^{\lambda}}{\partial y^{\mu}}, \tag{2.13}$$

3. Vacuum solution of Finsler metric by using Ansatz of the LTB

The theory of relativity describes the relation between the curvature of spacetime to the energy of an object. The vacuum field equations of General Relativity are $R_{\mu\nu} = 0$. Exact solutions of Einstein's field equations have played important roles in the discussion of physical problems. Obvious examples are the Schwarzschild and Kerr solutions for studying black hole and the Friedman solutions for cosmology. Since dust is believed to be an appropriate representation of the universe's matter content on the large scale at the present time, LTB solutions have been much used to provide exact models of structures in the universe [10]. First of all we consider an Finslerian ansatz in the form of bellow then we obtain the solutions are similar to the LTB solutions in general relativity.

$$F^2 = -y^t y^t + S^2(r,t) y^r y^r + R^2(r,t) \bar{F}^2(\theta, \varphi, y^{\theta}, y^{\varphi}), \tag{3.1}$$

We begin by calculating the connection symbols based on the spherically symmetric form of (3.1). It will be convenient to introduce the notation and we obtain some results, where the Finsler metric can be derived as

$$g_{\mu\nu} = diag(-1, S^2, R^2 \bar{g}_{\theta\theta}, R^2 \bar{g}_{\varphi\varphi}), \tag{3.2}$$

$$g^{\mu\nu} = diag(-1, \frac{1}{S^2}, \frac{1}{R^2} \bar{g}^{\theta\theta}, \frac{1}{R^2} \bar{g}^{\varphi\varphi}), \tag{3.3}$$

The matrix $g_{\mu\nu}$ is invertible and its inverse is denoted by $g^{\mu\nu} = [g_{\mu\nu}]^{-1}$ and $\bar{g_{ij}}$ and $\bar{g^{ij}}$ are components of the metric derived from $\bar{F}$ and the indices i,j run over the angular coordinates θ,φ .
plugging the Finsler structure (3.1) into (2.2), we obtain that

$$G^t = \frac{S\dot{S}}{2} y^r y^r + \frac{R\dot{R}}{2} \bar{F}^2, \tag{3.4}$$

$$G^r = \frac{S'}{2S}y^r y^r + \frac{\dot{S}}{S}y^r y^t - \frac{RR'}{2S^2}\bar{F}^2, \tag{3.5}$$

$$G^\theta = \frac{\dot{R}}{R}y^t y^\theta + \frac{R'}{R}y^r y^\theta + \bar{G}^\theta, \tag{3.6}$$

$$G^\varphi = \frac{\dot{R}}{R}y^t y^\varphi + \frac{R'}{R}y^r y^\varphi + \bar{G}^\varphi, \tag{3.7}$$

Here dot and prime denote partial derivatives with respect to the parameters t and r respectively. $\bar{G}$ are the geodic spray coefficient derives from $\bar{F}$. Plugging the geodesic coefficients$(3.4)-(3.7)$ into the formula for the Ricci scalar (2.13), we obtain that

$$\begin{aligned} RicF^2 = R^\mu_\mu F^2 = (\frac{-2\ddot{R}}{R} - \frac{\ddot{S}}{S})y^t y^t + (\frac{-2R''}{R} + \frac{2\dot{R}\dot{S}S}{R} \\ + \frac{2R'S'}{RS} + S\ddot{S})y^r y^r + (-\frac{4\dot{R}'}{R} + \frac{4\dot{S}R'}{RS})y^r y^t + (\dot{R}^2 \\ - \frac{R'^2}{S^2} + R\ddot{R} + \frac{\dot{S}\dot{R}R}{S} + \frac{R'S'R}{S^3} - \frac{R''R}{S^2} + \bar{Ric})\bar{F}^2, \end{aligned} \tag{3.8}$$

where $\bar{Ric}$ denotes the Ricci scalar of the Finsler structure $\bar{F}$. Since the vacuum field equation in Finsler spacetime is equivalent to the vanishing of the Ricci scalar and $\bar{F}$ is independent of y^t and y^r, the vanishing of Ricci scalar implies that the terms in each square bracket of (3.8) should vanish as well. These equations are given as

$$\frac{-2\ddot{R}}{R} - \frac{\ddot{S}}{S} = 0, \tag{3.9}$$

$$\frac{-2R''}{R} + \frac{2\dot{R}\dot{S}S}{R} + \frac{2R'S'}{RS} + S\ddot{S} = 0, \tag{3.10}$$

$$\frac{-4\dot{R}'}{R} + \frac{4\dot{S}R'}{RS} = 0, \tag{3.11}$$

$$\dot{R}^2 - \frac{R'^2}{S^2} + R\ddot{R} + \frac{\dot{S}\dot{R}R}{S} + \frac{R'S'R}{S^3} - \frac{R''R}{S^2} + \bar{Ric} = 0, \tag{3.12}$$

From equation (3.11) we obtain that

$$\frac{-4\dot{R}'}{R} = \frac{-4\dot{S}R'}{RS}, \tag{3.13}$$

We remove the same variable from both sides of equation (3.13) at the same time, we obtain

$$\dot{R}'S = \dot{S}R', \tag{3.14}$$

$$\frac{\dot{R}'}{R'} = \frac{\dot{S}}{S}, \tag{3.15}$$

Integrating this and using the formula $\int \frac{dx}{x} = \ln x$ we get,

$$\int \frac{\dot{R}'}{R'}dt = \int \frac{\dot{S}}{S}dt, \tag{3.16}$$

Upon evaluating each of these integrals we should get a constant of integration for each integral since we really are doing two integrals. Since there is no reason to think that the constants of integration will be the same from each integral we use different constants for each integral. Now, both c and k are unknown constants and so the sum of two unknown constants is just an unknown constant and we acknowledge that by simply writing the sum as a c. Moreover, c is arbitrary, we can choose $c = \ln(\sqrt{1+E})$. So, the integral is then,

$$\ln R^{'} = \ln S + \ln(\sqrt{1+E}), \tag{3.17}$$

Apply the product rule for logarithms, we have

$$\ln R^{'} = \ln S(\sqrt{1+E}), \tag{3.18}$$

After simplifying equation (3.18), we get

$$\frac{R^{'}}{\sqrt{1+E}} = S, \tag{3.19}$$

Equations (3.1) and (3.19) show that F^2 is similar to the LTB solution

$$F^2 = -y^t y^t + (\frac{R^{'2}}{1+E})y^r y^r + R^2(r,t)\bar{F}^2(\theta, \varphi, y^\theta, y^\varphi) \tag{3.20}$$

Noticing that $\bar{Ric}$ is independent of r and t, thus equation (3.12) is satisfied if and only if $\bar{Ric}$ is constant. This means that the two- dimensional Finsler space $\bar{F}$ is constant flag curvature space. Therefore $\bar{Ric} = \lambda$. From equation (3.9), we obtain

$$\frac{-2\ddot{R}}{R} = \frac{\ddot{S}}{S}, \tag{3.21}$$

From simplifying equation (3.21), we obtain

$$\ddot{R} = \frac{-\ddot{S}R}{2S}, \tag{3.22}$$

With using equation (3.10) and multiplying the two sides of equation by $\frac{R^2}{2S^2}$, we have

$$\frac{-RR^{''}}{S^2} + \frac{\dot{R}\dot{S}R}{S} + \frac{RR^{'}S^{'}}{S^3} = \frac{-R^2\ddot{S}}{2S}, \tag{3.23}$$

By substituting equation (3.22) into equation (3.23), we get

$$\frac{-RR^{''}}{S^2} + \frac{\dot{R}\dot{S}R}{S} + \frac{RR^{'}S^{'}}{S^3} = R\ddot{R}, \tag{3.24}$$

Equations (3.12) and (3.24) conclude that

$$\dot{R}^2 - \frac{R^{'2}}{S^2} + 2R\ddot{R} + \lambda = 0, \tag{3.25}$$

By plugging equation (3.19) into equation (3.25) we get following equations

$$\dot{R}^2 - \frac{R^{'2}(1+E)}{R^{'2}} + 2R\ddot{R} + \lambda = 0, \tag{3.26}$$

$$\dot{R}^2 - (1+E) + 2R\ddot{R} + \lambda = 0, \tag{3.27}$$

$$\dot{R}^2 + 2R\ddot{R} = E, \tag{3.28}$$

Therefore Here angular distance R, depending on the value of t and $E(r)$, is given by

$$R(t,r) = \sqrt{E(r)}t, \tag{3.29}$$

By considering

$$\lambda = 1, \tag{3.30}$$

Thus our LTB metric is,

$$F^2 = -y^t y^t + \frac{E'^2 t^2}{4E(1+E)} y^r y^r + E(r) t^2 \bar{F}^2(\theta, \varphi, y^\theta, y^\varphi), \tag{3.31}$$

4. Analytic solution to the geodesic equations of the finsler geometry

In this section, for studying the geodesics of the test particles in the finsler geometry, the equations for the geodesic sprays coefficients (3.4) – (3.7) plug into the geodesic equation (2.1), we obtain the geodesic equation of Finsler spacetime (3.1),

$$\frac{d^2t}{d\tau^2} + \frac{S\dot{S}}{2} y^r y^r + \frac{R\dot{R}}{2} \bar{F}^2 = 0, \tag{4.1}$$

$$\frac{d^2r}{d\tau^2} + \frac{S'}{2S} y^r y^r + \frac{\dot{S}}{S} y^r y^t - \frac{RR'}{2S^2} \bar{F}^2 = 0, \tag{4.2}$$

$$\frac{d^2\theta}{d\tau^2} + \frac{\dot{R}}{R} y^t y^\theta + \frac{R'}{R} y^r y^\theta + \bar{G}^\theta = 0, \tag{4.3}$$

$$\frac{d^2\varphi}{d\tau^2} + \frac{\dot{R}}{R} y^t y^\varphi + \frac{R'}{R} y^r y^\varphi + \bar{G}^\varphi = 0, \tag{4.4}$$

Noticing that $\bar{F}$ is independent of r and t, thus equations (4.1) and (4.2) are satisfied if and only if $\bar{F}$ is constant along the geodesic.

5. Kretschmann Scalar

What is the nature of the singularity ?

Definition 1. *The point z is called a singular point or singularity of K if K is not analytic at z but every neighborhood of z contains at least one point at which K is analytic.*

The Kretschmann scalar gives the amount of curvature of spacetime, as a function of position near (and within) a black hole. [1]. The derivation of the Kretschmann scalar is simple in principle, but requires tedious algebraic computation in practice. From the specified metric, we compute, first, the connection, which is not itself a tensor. Next, we compute the Riemann tensor and Kretschmann scalar components. The non-vanishing Christoffel symbols for the metric (3.1) are given by

$$\Gamma^t_{rr} = S\dot{S}, \quad \Gamma^t_{\theta\theta} = R\dot{R}\bar{g}_{\theta\theta}, \quad \Gamma^t_{\varphi\varphi} = R\dot{R}\bar{g}_{\varphi\varphi}, \quad \Gamma^r_{tr} = \frac{\dot{S}}{S}, \quad \Gamma^r_{rt} = \frac{\dot{S}}{S}, \quad \Gamma^r_{rr} = \frac{S'}{S}$$

$$\Gamma^r_{\theta\theta} = \frac{-R'R}{S^2}\bar{g}_{\theta\theta}, \quad \Gamma^r_{\varphi\varphi} = \frac{-R'R}{S^2}\bar{g}_{\varphi\varphi}, \quad \Gamma^\theta_{t\theta} = \frac{\dot{R}}{R}, \quad \Gamma^\theta_{r\theta} = \frac{R'}{R}, \quad \Gamma^\theta_{\theta r} = \frac{R'}{R}, \quad \Gamma^\theta_{\theta t} = \frac{\dot{R}}{R},$$
$$\Gamma^\theta_{\theta\theta} = \frac{1}{2}\bar{g}^{\theta\theta}\frac{\partial \bar{g}_{\theta\theta}}{\partial\theta} \quad \Gamma^\theta_{\theta\varphi} = \frac{1}{2}\bar{g}^{\theta\theta}\frac{\partial \bar{g}_{\theta\theta}}{\partial\varphi}, \quad \Gamma^\theta_{\varphi\theta} = \frac{1}{2}\bar{g}^{\theta\theta}\frac{\partial \bar{g}_{\theta\theta}}{\partial\varphi} \quad \Gamma^\theta_{\varphi\varphi} = \frac{-1}{2}\bar{g}^{\theta\theta}\frac{\partial \bar{g}_{\varphi\varphi}}{\partial\theta}$$
$$\Gamma^\varphi_{t\varphi} = \frac{\dot{R}}{R}, \quad \Gamma^\varphi_{r\varphi} = \frac{R'}{R}, \quad \Gamma^\varphi_{\varphi r} = \frac{R'}{R}, \quad \Gamma^\varphi_{\varphi t} = \frac{\dot{R}}{R}, \quad \Gamma^\varphi_{\theta\theta} = \frac{-1}{2}\bar{g}^{\varphi\varphi}\frac{\partial \bar{g}_{\theta\theta}}{\partial\varphi} \quad \Gamma^\varphi_{\theta\varphi} = \frac{1}{2}\bar{g}^{\varphi\varphi}\frac{\partial \bar{g}_{\varphi\varphi}}{\partial\theta}$$
$$\Gamma^\varphi_{\varphi\theta} = \frac{1}{2}\bar{g}^{\varphi\varphi}\frac{\partial \bar{g}_{\varphi\varphi}}{\partial\theta} \quad \Gamma^\varphi_{\varphi\varphi} = \frac{1}{2}\bar{g}^{\varphi\varphi}\frac{\partial \bar{g}_{\varphi\varphi}}{\partial\varphi}, \tag{5.1}$$

With the connection in hand, we are in a position to compute the Riemann tensor itself:

$$R_{t\nu\rho\sigma} = g_{tt}R^t{}_{\nu\rho\sigma} = (-1)R^t{}_{\nu\rho\sigma}, \tag{5.2}$$

$$R_{r\nu\rho\sigma} = g_{rr}R^r{}_{\nu\rho\sigma} = S^2R^r{}_{\nu\rho\sigma}, \tag{5.3}$$
$$R_{\theta\nu\rho\sigma} = g_{\theta\theta}R^\theta{}_{\nu\rho\sigma} = (R^2\bar{g}_{\theta\theta})R^\theta{}_{\nu\rho\sigma}, \tag{5.4}$$
$$R_{\varphi\nu\rho\sigma} = g_{\varphi\varphi}R^\varphi{}_{\nu\rho\sigma} = (R^2\bar{g}_{\varphi\varphi})R^\varphi{}_{\nu\rho\sigma}, \tag{5.5}$$

We use maple program and manual computing and obtain a second scalar i.e: Kretschmann scalar as follows:

$$\begin{aligned} K_s =& \frac{20(E(r))^2 - 9((E(r))')^2 + 20E(r)}{2R^4} \\ &+ \frac{1}{2R^4}\bar{g}^{\theta\theta}\bar{g}^{\varphi\varphi}(\partial_\theta\bar{g}^{\theta\theta}\partial_\varphi\bar{g}_{\theta\theta} - \partial_\varphi\bar{g}^{\theta\theta}\partial_\theta\bar{g}_{\theta\theta})^2 \\ &+ \frac{1}{2R^4}\bar{g}^{\theta\theta}\bar{g}^{\varphi\varphi}(\partial_\theta\bar{g}^{\varphi\varphi}\partial_\varphi\bar{g}_{\varphi\varphi} - \partial_\varphi\bar{g}^{\varphi\varphi}\partial_\theta\bar{g}_{\varphi\varphi})^2 \\ &+ \frac{2}{R^4}(\bar{g}^{\varphi\varphi})^2(R^\theta{}_{\varphi\theta\varphi})^2 + \frac{2}{R^4}(\bar{g}^{\theta\theta})^2(R^\varphi{}_{\theta\varphi\theta})^2, \end{aligned} \tag{5.6}$$

Therefore the LTB solution has singularity at $R = 0$.

6. example

For example, we make subspace $\bar{F}$ to be a "Finslerian sphere" F_{FS}. Then, the exterior metric of vacuum field solution was given as We finished with a brief example of how these solutions can be applied. Now, we consider the Finsler structure in the form

$$\begin{aligned} F^2 =& -y^ty^t + S^2(r,t)y^ry^r \\ &+ R^2(r,t)(\sqrt{y^\theta y^\theta + y^\varphi y^\varphi} - \sin\theta y^\varphi)^2, \end{aligned} \tag{6.1}$$

Where

$$\bar{F}^2 = y^\theta y^\theta + y^\varphi y^\varphi + \sin^2\theta y^\varphi - 2\sqrt{y^\theta y^\theta + y^\varphi y^\varphi}\sin\theta y^\varphi,$$
$$\bar{g}_{\theta\theta} = R^2(t,r)(\frac{-\sin\theta(y^\varphi)^3}{(y^\theta y^\theta + y^\varphi y^\varphi)^{\frac{3}{2}}} + 1),$$

$$\bar{g}^{\theta\theta} = \frac{(y^\theta y^\theta + y^\varphi y^\varphi)^{\frac{3}{2}}}{R^2(-\sin\theta(y^\varphi)^3 + (y^\theta y^\theta + y^\varphi y^\varphi)^{\frac{3}{2}})},$$

$$\bar{g}_{\varphi\varphi} = \frac{R^2((2-\cos^2\theta)((y^\theta)^2 + (y^\varphi)^2)^{\frac{3}{2}} - \sin\theta y^\varphi(2(y^\varphi)^3 + 3(y^\theta)^2))}{(y^\theta y^\theta + y^\varphi y^\varphi)^{\frac{3}{2}}},$$

$$\bar{g}^{\varphi\varphi} = \frac{-(y^\theta y^\theta + y^\varphi y^\varphi)^{\frac{3}{2}}}{R^2((\cos^2\theta - 2)(y^\theta y^\theta + y^\varphi y^\varphi)^{\frac{3}{2}} + \sin\theta y^\varphi(2(y^\varphi)^3 + 3(y^\theta)^2))},$$

F^2 is Finslerian metric, since the fundamental form $\bar{g}$ is function of $(\theta, \varphi, y^\theta, y^\varphi)$. Therefore, we can compute the Kretschman scalar by using the equation (5.6). It is convenient to consider the orbit of particle confined to the equatorial plane $\theta = \frac{\pi}{2}$.

$$\begin{aligned} K_s =& \frac{20(E(r))^2 - 9((E(r))^{'})^2 + 20E(r)}{2R^4} \\ &+ \frac{Q}{(-((y^\theta)^2 + (y^\varphi)^2)^{\frac{3}{2}} + (y^\varphi)^3)^2 R^8}, \end{aligned} \tag{6.2}$$

Where

$$\begin{aligned} Q =& (4(-((y^\theta)^2 + (y^\varphi)^2)^{\frac{3}{2}}(\frac{-1}{2} + R^2(\frac{R'^2}{S^2} - \dot{R}^2))) \\ &+ R^2(y^\varphi)^3(\frac{R'^2}{S^2} - \dot{R}^2))^2(-R^4(y^\varphi)^3((y^\theta)^2 + (y^\varphi)^2)^{\frac{3}{2}} \\ &+ (R^4 + \frac{1}{2})(y^\varphi)^6 + \frac{(R^4+1)(y^\varphi)^2}{2}(3(y^\varphi)^4 \\ &+ 3(y^\varphi)^2(y^\theta)^2 + (y^\theta)^4), \end{aligned} \tag{6.3}$$

Which represents that the curvature singularity is located at $R = 0$.

7. Vacuum solution of Finsler metric of the Schwarzschild metric

First of all we consider an ansatz of the Schwarzschild metric and we show results of vacuum solution are different from of [8].
Then, we consider FRW metric and compute the geodesic equations.
We consider an ansatz of the Schwarzschild metric which was considered in [8] of the form

$$F^2 = B(r)y^t y^t - A(r)y^r y^r - r^2\bar{F}^2(\theta, \varphi, y^\theta, y^\varphi), \tag{7.1}$$

and we obtain some results, where the Finsler metric can be derived as

$$g_{\mu\nu} = diag(B, -A, -r^2\bar{g}_{ij}), \tag{7.2}$$

$$g^{\mu\nu} = diag(B^{-1}, -A^{-1}, -r^{-2}\bar{g}^{ij}), \tag{7.3}$$

where $\bar{g_{ij}}$ and $\bar{g^{ij}}$ are components of the metric derived from $\bar{F}$.
Plugging the Finsler structure (7.1) into equation (2.2), we obtain

$$G^t = \frac{B'}{2B}y^t y^r, \tag{7.4}$$

$$G^r = \frac{A'}{4A} y^r y^r + \frac{B'}{4B} y^t y^t - \frac{r}{2A} \bar{F}^2, \tag{7.5}$$

$$G^\theta = \bar{G}^\theta + \frac{1}{r} y^\theta y^r, \tag{7.6}$$

and

$$G^\varphi = \bar{G}^\varphi + \frac{1}{r} y^\varphi y^r. \tag{7.7}$$

We consider variations of $\bar{F}$ along the geodesic, therefore we have

$$\frac{d\bar{F}}{d\bar{\tau}} = \frac{\delta \bar{F}}{\delta x^i} \frac{dx^i}{d\bar{\tau}} + \bar{F} \frac{\partial \bar{F}}{\partial y^i} \frac{\delta y^i}{\bar{F} d\bar{\tau}}, \tag{7.8}$$

By simplifying equation (7.8) as following,

$$\begin{aligned} \frac{d\bar{F}}{d\bar{\tau}} &= \frac{\partial \bar{F}}{\partial x^i} \frac{dx^i}{d\bar{\tau}} - \frac{1}{2} \frac{\partial \bar{G}^j}{\partial y^i} \frac{\partial \bar{F}}{\partial y^j} \frac{dx^i}{d\bar{\tau}} + \bar{F} \frac{\partial \bar{F}}{\partial y^i} \Big(\frac{dy^i + \frac{1}{2} \frac{\partial G^i}{\partial y^j} dx^j}{\bar{F} d\bar{\tau}} \Big) \\ &= \frac{\partial \bar{F}}{\partial x^i} \frac{dx^i}{d\bar{\tau}} - \frac{1}{2} \frac{\partial \bar{G}^j}{\partial y^i} \frac{\partial \bar{F}}{\partial y^j} \frac{dx^i}{d\bar{\tau}} + \frac{\partial \bar{F}}{\partial y^i} \frac{dy^i}{d\bar{\tau}} + \frac{1}{2} \frac{\partial \bar{F}}{\partial y^i} \frac{\partial \bar{G}^i}{\partial y^j} \frac{dx^j}{d\bar{\tau}} \\ &= \frac{\partial \bar{F}}{\partial x^i} \frac{dx^i}{d\bar{\tau}} + \frac{\partial \bar{F}}{\partial y^i} \frac{dy^i}{d\bar{\tau}}, \end{aligned} \tag{7.9}$$

where

$$\bar{G}_i = \frac{1}{4} \Big(\frac{\partial^2 \bar{F}^2}{\partial x^k \partial y^i} y^k - \frac{\partial \bar{F}^2}{\partial x^i} \Big), \tag{7.10}$$

by multiplying y^i in the above equation, we obtain the following equations,

$$y^i \bar{G}_i = \frac{1}{2} \bar{F} \frac{\partial \bar{F}}{\partial x^i} y^i, \tag{7.11}$$

and

$$\frac{2 y^i \bar{G}_i}{\bar{F}} = \frac{\partial \bar{F}}{\partial x^i} y^i, \tag{7.12}$$

as we know,

$$\frac{d\bar{F}}{d\bar{\tau}} = \frac{\partial \bar{F}}{\partial y^i} \frac{dy^i}{d\bar{\tau}}, \tag{7.13}$$

and

$$\begin{aligned} \frac{\partial \bar{F}}{\partial y^i} \frac{d^2 x^i}{d\bar{\tau}^2} &= \frac{\partial \bar{F}}{\partial y^\theta} (-2\bar{G}^\theta) + \frac{\partial \bar{F}}{\partial y^\varphi} (-2\bar{G}^\varphi) = -2 \frac{\partial \bar{F}}{\partial y^i} \bar{G}^i \\ &= -2 \frac{\partial \bar{F}}{\partial y^i} \bar{g}^{ij} \bar{G}_j = -2 \frac{\partial \bar{F}}{\partial y^i} \frac{y^i y^j}{\bar{F}^2} \bar{G}_j = -2 y^j \frac{\bar{G}_j}{\bar{F}}. \end{aligned} \tag{7.14}$$

By using the equations (7.12) and (7.14), one rewrite the equation (7.9) as,

$$\frac{d\bar{F}}{d\bar{\tau}} = 0. \tag{7.15}$$

Therefore $\bar{F}$ is constant along its geodesic. We know that

$$\frac{d\bar{F}}{d\tau} = \frac{d\bar{F}}{d\bar{\tau}} \frac{d\bar{\tau}}{d\tau} = 0, \tag{7.16}$$

hence $\bar{F}$ is constant along the geodesic of F. According to

$$d\tau = Fdt,$$
$$d\bar{\tau} = \bar{F}dt, \tag{7.17}$$

Then

$$d\log(F) = d\log(\bar{F}). \tag{7.18}$$

Now,we consider F and $\bar{F}$ satisfied in equation (7.18) and we apply this condition in the following calculations. Plugging the geodesic spray coefficients G^r into the geodesic equation (2.1), we obtain

$$\frac{d^2t}{d\tau^2} + \frac{B'}{B}\frac{dr}{d\tau}\frac{dt}{d\tau} = 0, \tag{7.19}$$

The solution of equation (7.19) is

$$B\frac{dt}{d\tau} = 1. \tag{7.20}$$

$$\frac{d^2r}{d\tau^2} + \frac{B'}{2A}(\frac{dt}{d\tau})^2 + \frac{A'}{2A}(\frac{dr}{d\tau})^2 - \frac{r}{A}\bar{F}^2 = 0, \tag{7.21}$$

With simplifying as following

$$(\frac{d^2r}{d\tau^2} + \frac{B'}{2A}(\frac{dt}{d\tau})^2 + \frac{A'}{2A}(\frac{dr}{d\tau})^2 - \frac{r}{A}\bar{F}^2 = 0) \times 2A(\frac{dr}{d\tau}), \tag{7.22}$$

$$A'(\frac{dr}{d\tau})^3 + 2A\frac{dr}{d\tau}\frac{d^2r}{d\tau^2} - 2r\frac{dr}{2A}d\tau\bar{F}^2 - \frac{B'}{B^2}\frac{dr}{d\tau} = 0, \tag{7.23}$$

$$\frac{dA}{dr}\frac{dr}{d\tau}(\frac{dr}{d\tau})^2 + A(2\frac{dr}{d\tau}\frac{d^2r}{d\tau^2}) - \bar{F}^2\frac{d}{dr}(r^2)\frac{dr}{d\tau} - \frac{d}{dr}(B^{-1})\frac{dr}{d\tau} = 0 \tag{7.24}$$

$$\frac{dA}{d\tau}(\frac{dr}{d\tau})^2 + A\frac{d}{d\tau}(\frac{dr}{d\tau})^2 - \bar{F}^2\frac{d}{d\tau}(r^2) - \frac{d}{d\tau}(B^{-1}) = 0 \tag{7.25}$$

$$\frac{d}{d\tau}[A(\frac{dr}{d\tau})^2 - \bar{F}^2r^2 - \frac{1}{B}] = 0, \tag{7.26}$$

By replacing equation (7.20) in equation (7.26), we have the following equation

$$\frac{d}{d\tau}[A(\frac{dr}{d\tau})^2 - \bar{F}^2r^2 - B(\frac{dt}{d\tau})^2] = 0, \tag{7.27}$$

By regarding the equation (2.1), we have

$$\frac{d}{d\tau}(-2r^2\bar{F}^2 - F^2) = 0, \tag{7.28}$$

$$-4r\frac{dr}{d\tau}\bar{F}^2 - \frac{dF^2}{d\tau} = 0, \tag{7.29}$$

$$\frac{dF^2}{d\tau} = -4r\frac{dr}{d\tau}\bar{F}^2, \tag{7.30}$$

And

$$\frac{dF^2}{d\tau} = -4r\frac{dr}{dt}\frac{dt}{d\tau}\bar{F}^2, \tag{7.31}$$

Because $\frac{dF}{d\tau} = 0$ thus $\frac{dr}{dt}$ is small.
Plugging the geodesic spray coefficients G^t into the geodesic equation (2.1), we gain

$$\frac{d^2t}{dt^2} + \frac{B'}{B}\frac{dr}{d\tau}\frac{dt}{d\tau} = 0, \tag{7.32}$$

$$\frac{\frac{d^2t}{d\tau^2}}{\frac{dt}{d\tau}} d\tau = -\frac{B'}{B} dr, \tag{7.33}$$

And

$$\ln\frac{dt}{d\tau} = -\ln B, \tag{7.34}$$

We consider

$$F^2 = By^t y^t - Ay^r y^r - r^2 F_{FS}^2, \tag{7.35}$$

Where $B = (1 - \frac{2GM}{\lambda r})$ and $A = (\lambda - \frac{2GM}{r})^{-1}$, and the "Finslerian sphere" F_{FS} is of the form of

$$F_{FS} = \frac{\sqrt{(1-\varepsilon^2\sin^2\theta)y^\theta y^\theta + \sin^2\theta y^\varphi y^\varphi}}{1-\varepsilon^2\sin^2\theta} - \frac{\varepsilon\sin^2\theta y^\varphi}{1-\varepsilon^2\sin^2\theta}, \tag{7.36}$$

Where $0 \le \varepsilon < 1$. If we consider the orbit of a particle confined to the equatorial plane $\theta = \frac{\pi}{2}$, Finslerian sphere simplifies as,

$$F_{FS} = \frac{\frac{d\varphi}{d\tau}}{1-\varepsilon^2} - \frac{\varepsilon\frac{d\varphi}{d\tau}}{1-\varepsilon^2} = \frac{\frac{d\varphi}{d\tau}}{1+\varepsilon}, \tag{7.37}$$

By using $\frac{dF_{FS}}{d\tau} = 0$, we obtain

$$\frac{d\varphi}{d\tau} = J_\pm, \tag{7.38}$$

Where $J_\pm = J(1 \pm \varepsilon)$. Plugging equation (7.38) into the equation (7.35) and consider $F = 0$ for massless particles we obtain

$$(1 - \frac{2GM}{r})(\frac{dt}{d\tau})^2 - (1 - \frac{2GM}{r})^{-1}(\frac{dr}{d\tau})^2 - \frac{r^2}{(1\pm\varepsilon)^2}(\frac{d\varphi}{d\tau})^2 = 0, \tag{7.39}$$

By using equations (7.39) and (7.38) we have

$$(\frac{dr}{d\tau})^2 - 1 = -\frac{r^2}{(1\pm\varepsilon)^2}(1 - \frac{2GM}{r})(\frac{d\varphi}{d\tau})^2, \tag{7.40}$$

$$(\frac{dr}{d\varphi})^2 - \frac{1}{(\frac{d\varphi}{d\tau})^2} = -\frac{r^2}{(1\pm\varepsilon)^2}(1 - \frac{2GM}{r}), \tag{7.41}$$

Plugging equation (7.38) into the equation (7.41) we obtain

$$(\frac{dr}{d\varphi})^2 - \frac{1}{J^2(1\pm\varepsilon)^2} = -\frac{r^2}{(1\pm\varepsilon)^2}(1 - \frac{2GM}{r}), \tag{7.42}$$

After simplifying equation (7.42), we gain

$$\frac{dr}{\sqrt{(\frac{\sqrt{G^2M^2J^2+1}}{J})^2 - (r - GM)^2}} = \frac{1}{(1\pm\varepsilon)} d\varphi, \tag{7.43}$$

We consider $r - GM = \frac{\sqrt{G^2M^2J^2+1}}{J}\sin\alpha$ and solve the equation (7.43) we obtain

$$r = \frac{\sqrt{G^2M^2J^2+1}}{J}\sin((\frac{1}{(1\pm\varepsilon)})\varphi + c) + GM, \tag{7.44}$$

8. Weyl and Douglas tensor

Let F be a Finsler metric on an n-dimensional manifold M and G^i be the geodesic coefficient of F, $x \in M$ and $y \in T_xM - \{0\}$, the Riemannian curvature tensor is defined by $R^i_{jk} = \frac{\delta G^i_j}{\delta x^k} - \frac{\delta G^i_k}{\delta x^j}$. We know $H^h_{ijk} = \frac{\partial R^h_{jk}}{\partial y^i}$, $H_{ij} = H^k_{ijk}$ and $H_i = \frac{(nH_{ki}+H_{ik})y^k}{n-1}$ provided $n \neq 1$. The Weyl tensor can be expressed, in four dimensions, as [11]

$$W^i_{jk} = R^i_{jk} + \frac{(H_{jk} - H_{kj})y^i + \delta^i_j H_k - \delta^i_k H_j}{n+1}, \tag{8.1}$$

we know from the

$$G^t_t = \frac{A'}{rA^2} - \frac{1}{rA^2} + \frac{\lambda}{r^2}, \tag{8.2}$$

$$G^r_r = \frac{-B'}{rAB} - \frac{1}{rA^2} + \frac{\lambda}{r^2}, \tag{8.3}$$

$$G^\theta_\theta = G^\varphi_\varphi = \frac{B}{2AB} - \frac{B'}{2rAB} + \frac{A'}{2rA^2} + \frac{B'}{4AB}(\frac{A'}{A} + \frac{B'}{B}), \tag{8.4}$$

According to definition of H^h_{ijk} since H^h_{ijk} is independent of y therefore $H^h_{ijk} = 0$, $H_{ij} = 0$ and $H_i = 0$. Thus $W^i_{jk} = R^i_{jk}$. By using the equation (8.1), we obtain

$$R^t_{rt} = -\frac{\partial G^t_t}{\partial r} = 0, \quad R^r_{rt} = -\frac{\partial G^r_r}{\partial t} = 0,$$

$$R^\theta_{\theta r} = -\frac{\partial G^\theta_\theta}{\partial r} = 0, \quad R^\varphi_{\varphi r} = -\frac{\partial G^\varphi_\varphi}{\partial r} = 0,$$

Thus G^t_t =constant, G^r_r =constant, G^θ_θ =constant and G^φ_φ =constant.

And other R^i_{jk} are equal zero. Since $G^t_t = \alpha$ that α is constant therefore we have

$$\frac{A'}{rA^2} - \frac{1}{rA^2} + \frac{\lambda}{r^2} = \alpha, \tag{8.5}$$

If we solve the equation (8.5) as follows

$$A'A^{-2} - \frac{1}{r}A^{-1} = (\alpha r - \frac{\lambda}{r}),$$

Therefore we have

$$A = (\lambda - \frac{1}{3}\alpha r^2 + \frac{c}{r})^{-1}, \tag{8.6}$$

Since $G^\theta_\theta = \gamma$ that γ is constant therefore we have

$$\frac{B}{2AB} - \frac{B'}{2rAB} + \frac{A'}{2rA^2} + \frac{B'}{4AB}(\frac{A'}{A} + \frac{B'}{B}) = \gamma, \tag{8.7}$$

From the equation (8.5) and equation (8.7) we conclude that

$$\frac{-B'}{rAB} - \frac{1}{rA^2} + \frac{\lambda}{r^2} = \beta = constant, \tag{8.8}$$

We solved equation (8.8) as follows

$$\frac{B'}{B} = -\beta r A + \frac{\lambda}{r}A - \frac{1}{r},$$

We use the equation (8.6) and we suppose that $\alpha = \beta$ therefore we obtain

$$B = (\lambda - \frac{1}{3}\alpha r^2 + \frac{c}{r}), \tag{8.9}$$

From equation (8.6) and equation (8.9) we conclude

$$B = A^{-1}, \tag{8.10}$$

Therefore, if $W = 0$ and $\alpha = \beta$ =constant we conclude that

$$F^2 = (\lambda - \frac{1}{3}\alpha r^2 + \frac{c}{r})y^t y^t - (\lambda - \frac{1}{3}\alpha r^2 + \frac{c}{r})^{-1} y^r y^r - r^2 \bar{F}^2, \tag{8.11}$$

Thus F^2 is the Ads Schwarzschild Finsler metric [12]. If F^2 express as follows

$$F^2 = (\lambda - \frac{2M}{r} + \frac{r^2}{b^2})y^t y^t - (\lambda - \frac{2M}{r} + \frac{r^2}{b^2})^{-1} y^r y^r - r^2 \bar{F}^2, \tag{8.12}$$

We conclude that $R^i_{jk} = 0$ and this conclude that $W^i_{jk} = 0$, thus F^2 is the Ads Schwarzschild Finsler metric and F^2 is conformally flat. Now we consider Douglas tensor as follows

$$D^h_{ijk} = G^h_{ijk} - \frac{y^h G_{ijk} + \sigma_(i,j,k)\{\delta^h_i G_{jk}\}}{n+1}, \tag{8.13}$$

Where

$$G^i = \frac{1}{2}\Gamma^i_{jk} y^i y^j, \quad G^h_{ijk} = \frac{\partial G^h_{jk}}{\partial y^i}, \quad G_{ij} = G^r_{ijr}, \quad G_{ijk} = \frac{\partial G_{ij}}{\partial y^k}, \tag{8.14}$$

If tensor Douglas $D = 0$ thus $D^h_{ijk} = 0 \Rightarrow G^h_{ijk} = 0 \Rightarrow G^i_i = 0$. Therefore $G^t_t = 0$ and we can conclude that

$$\frac{A'}{rA^2} - \frac{1}{r^2 A} + \frac{\lambda}{r^2} = 0,$$

Therefore we have

$$A = (\lambda + \frac{c}{r})^{-1}, \tag{8.15}$$

Also we know that $G^r_r = 0$ therefore

$$\frac{-\acute{B}}{rAB} - \frac{1}{r^2 A} + \frac{\lambda}{r^2} = 0,$$

$$\frac{B'}{B} = \frac{\lambda A}{r} - \frac{1}{r},$$

we use the equation (8.15) and we obtain

$$B = (\lambda + \frac{c}{r}), \tag{8.16}$$

From the equations (8.15) and (8.16) we conclude

$$B = A^{-1}, \tag{8.17}$$

Therefore if $D = 0$ we conclude that F^2 is the Schwarzschild metric.

9. The Friedmann-Lemaître-Robertson-Walker ($FLRW$) solution

By using an ansatz of the $FLRW$ metric that it has the Finslerian structure

$$F^2=\frac{dt}{d\tau}\frac{dt}{d\tau}-\frac{a^2(t)}{1-kr^2}\frac{dr}{d\tau}\frac{dr}{d\tau}-r^2a^2(t)\bar{F}^2(\theta,\varphi,\frac{d\theta}{d\tau}\frac{d\varphi}{d\tau}), \tag{9.1}$$

We have found an exact solution of the vacuum field equation, where geodesic spray coefficients can be derived as

$$G^t=\frac{1}{4}(\frac{2aa^{'}}{1-kr^2}\frac{dr}{d\tau}\frac{dr}{d\tau}+2aa^{'}r^2\bar{F}^2), \tag{9.2}$$

$$G^r=\frac{a^{'}}{a}\frac{dr}{d\tau}\frac{dt}{d\tau}+\frac{1}{2}\frac{kr}{1-kr^2}\frac{dr}{d\tau}\frac{dr}{d\tau}-\frac{1}{2}r(1-kr^2)\bar{F}^2, \tag{9.3}$$

$$G^\theta=\bar{G}^\theta+\frac{a^{'}}{a}\frac{d\theta}{d\tau}\frac{dt}{d\tau}+\frac{1}{r}\frac{d\theta}{d\tau}\frac{dr}{d\tau}, \tag{9.4}$$

$$G^\varphi=\bar{G}^\varphi+\frac{a^{'}}{a}\frac{d\varphi}{d\tau}\frac{dt}{d\tau}+\frac{1}{r}\frac{d\varphi}{d\tau}\frac{dr}{d\tau}, \tag{9.5}$$

Now, we compute Ricci scalar of equation (9.1).

$$RicF^2=2\frac{\partial G^\mu}{\partial x^\mu}-y^\lambda\frac{\partial^2G^\mu}{\partial x^\lambda\partial y^\mu}+2G^\lambda\frac{\partial^2G^\mu}{\partial y^\lambda\partial y^\mu}-\frac{\partial G^\mu}{\partial y^\lambda}\frac{\partial G^\lambda}{\partial y^\mu}, \tag{9.6}$$

Where $\bar{Ric}$ denotes the Ricci scalar of the Finsler structure $\bar{F}$. Since the vacuum field equation in Finsler spacetime is equivalent to the vanishing of the Ricci scalar thus

$$\begin{aligned}&2\frac{\partial G^t}{\partial t}-y^\lambda\frac{\partial}{\partial x^\lambda}\frac{\partial G^t}{\partial y^t}+2G^\lambda\frac{\partial}{\partial y^\lambda}\frac{\partial G^t}{\partial y^t}-\frac{\partial G^t}{\partial y^\lambda}\frac{\partial G^\lambda}{\partial y^t}\\&=\frac{aa^{''}}{1-kr^2}y^ry^r+aa^{''}r^2\bar{F}^2,\end{aligned} \tag{9.7}$$

$$\begin{aligned}&2\frac{\partial G^r}{\partial r}-y^\lambda\frac{\partial}{\partial x^\lambda}\frac{\partial G^r}{\partial y^r}+2G^\lambda\frac{\partial}{\partial y^\lambda}\frac{\partial G^r}{\partial y^r}-\frac{\partial G^r}{\partial y^\lambda}\frac{\partial G^\lambda}{\partial y^r}\\&=-\frac{a^{''}}{a}y^ty^t+r^2(a^{'2}+k)\bar{F}^2,\end{aligned} \tag{9.8}$$

and

$$\begin{aligned}&2\frac{\partial G^\theta}{\partial \theta}-y^\lambda\frac{\partial}{\partial x^\lambda}\frac{\partial G^\theta}{\partial y^\theta}+2G^\lambda\frac{\partial}{\partial y^\lambda}\frac{\partial G^\theta}{\partial y^\theta}-\frac{\partial G^\theta}{\partial y^\lambda}\frac{\partial G^\lambda}{\partial y^\theta}\\&+2\frac{\partial G^\varphi}{\partial \varphi}-y^\lambda\frac{\partial}{\partial x^\lambda}\frac{\partial G^\varphi}{\partial y^\varphi}+2G^\lambda\frac{\partial}{\partial y^\lambda}\frac{\partial G^\varphi}{\partial y^\varphi}-\frac{\partial G^\varphi}{\partial y^\lambda}\frac{\partial G^\lambda}{\partial y^\varphi}\\&=\bar{Ric}\bar{F}^2-2\frac{a^{''}}{a}y^ty^t+2\frac{a^{'2}+k}{1-kr^2}y^ry^r,\end{aligned} \tag{9.9}$$

Then we rewrite equation (2.13) as

$$\begin{aligned}RicF^2=&-3\frac{a^{''}}{a}y^ty^t+\frac{aa^{''}+2a^{'2}+2k}{1-kr^2}y^ry^r+\\&(\bar{Ric}+r^2(aa^{''}+a^{'2}+k))\bar{F}^2,\end{aligned} \tag{9.10}$$

Because of vanishing of the Ricci scalar, we have $\bar{Ric} = \lambda$

$$\frac{a''}{a} = 0, \quad \frac{aa'' + 2a'^2 + 2k}{1 - kr^2} = 0,$$
$$\lambda + r^2(aa'' + a'^2 + k) = 0, \tag{9.11}$$

$$a'' = 0 \Rightarrow a(t) = \alpha t + \beta, \tag{9.12}$$

$$\frac{aa'' + 2a'^2 + 2k}{1 - kr^2} = 0 \Rightarrow k = -\alpha^2, \tag{9.13}$$

$$r^2(\alpha^2 + k) + \lambda = 0,$$
$$\alpha^2 + k = 0,$$
$$\Rightarrow \lambda = 0, \tag{9.14}$$

Thus we obtain,

$$F^2 = \frac{dt}{d\tau}\frac{dt}{d\tau} - \frac{(\alpha t + \beta)^2}{1 + \alpha^2 r^2}\frac{dr}{d\tau}\frac{dr}{d\tau} - r^2(\alpha t + \beta)^2 \bar{F}^2, \tag{9.15}$$

Where $\bar{F}$ is Ricci flat and $\bar{Ric} = \lambda$. Now, we consider geodesic equation

$$\frac{d^2 x^\mu}{d\tau^2} + 2G^\mu = 0, \tag{9.16}$$

$$\frac{d^2 t}{d\tau^2} + 2G^t = 0,$$
$$\frac{d^2 t}{d\tau^2} + \left(\frac{2aa'}{1 - kr^2}\left(\frac{dr}{dt}\right)^2\left(\frac{dt}{d\tau}\right)^2 + 2aa' r^2 \bar{F}^2\right) = 0, \tag{9.17}$$

If, we only consider the radial motion of particles, and notice the velocity of a particle $\frac{dr}{dt}$ is small, we obtain

$$\frac{d^2 t}{d\tau^2} = \frac{d}{d\tau}\left(\frac{dt}{d\tau}\right) = 0, \tag{9.18}$$

$$\frac{dt}{d\tau} = constant = A, \tag{9.19}$$

From the another geodesic equation, we gain

$$\frac{d^2 r}{d\tau^2} + 2G^r = 0,$$
$$\frac{d^2 r}{dt^2}\left(\frac{dt}{d\tau}\right)^2 - r(1 - kr^2)\bar{F}^2 = 0, \tag{9.20}$$

Therefore

$$\frac{d^2 r}{dt^2} A = 0 \Rightarrow \frac{d^2 r}{dt^2} = 0, \tag{9.21}$$

On the other hand from the equation of motion of a test particle we obtain

$$\frac{d^2 r}{dt^2} = -\frac{\partial \varphi}{\partial r} = \frac{2GM}{r^2}, \tag{9.22}$$

From the equations (9.21) and (9.22) we obtain

$$\frac{2GM}{r^2} = 0, \tag{9.23}$$

Therefore $r \to \infty$ or $M = 0$.
We consider the gravitational field equation in the given Finsler spacetime [9] should be of the form

$$G^{\mu}_{\nu} = 8\pi_F G T^{\mu}_{\nu}, \tag{9.24}$$

We use the modified Einstein tensor [8] as follows

$$G_{\mu\nu} \equiv Ric_{\mu\nu} - \frac{1}{2} g_{\mu\nu} S, \tag{9.25}$$

And consider $G^k_k = g^{kl} G_{kl}$ we obtain

$$G^t_t = 2\frac{a^{'2}}{a^2} + \frac{2k}{a^2} + \frac{\lambda}{r^2 a^2}, \tag{9.26}$$

$$G^r_r = 2\frac{a^{''}}{a} + \frac{\lambda}{r^2 a^2}, \tag{9.27}$$

and

$$G^{\theta}_{\theta} = G^{\varphi}_{\varphi} = 2\frac{a^{''}}{a} + \frac{a^{'2} + k}{a^2}, \tag{9.28}$$

as we know that the energy-momentum tensor to be of the form

$$T^{\mu}_{\nu} = diag(\rho, -p, -p, -p) \tag{9.29}$$

we use equations (9.24) and(9.26), obtain

$$\begin{gathered} G^t_t = 8\pi_F G T^t_t \\ 2\frac{a^{'2}}{a^2} + \frac{2k}{a^2} + \frac{\lambda}{r^2 a^2} = 8\pi_F G \rho, \end{gathered} \tag{9.30}$$

According to the equations (9.12)-(9.14) we have

$$\rho = 0, \tag{9.31}$$

Moreover

$$\begin{gathered} G^{\theta}_{\theta} = 8\pi_F G T^{\theta}_{\theta} \\ 2\frac{a^{''}}{a} + \frac{a^{'2} + k}{a^2} = 8\pi_F G(-p), \end{gathered} \tag{9.32}$$

If we notice to the equations (9.12)-(9.14) we obtain

$$p = 0, \tag{9.33}$$

Thus ρ and p are $p = 0$, $\rho = 0$ therefore the energy-momentum tensor is zero.

10. conclusions

In this paper, we investigated the LTB solutions for (3.1) containing dust with $p = 0$. We have obtained the $R(t,r)$ and $S(t,r)$ with considering establish a new solution of $R_{\mu\nu} = 0$. Our solutions show that two dimensional subspace $\bar{F}$ has constant Ricci curvature.Then, we compute the covariant derivative Einstein tensor and show that it is not conserved in the Finsler spacetime. Moreover, we obtained the Kretschman scalar for the considering anstaz. Finally, we determined K_s singularity is at $R = 0$ and we showed one example. Singularity theorems have been one of the most important developments in the theory of classical general relativity . Also, we consider an ansatz of the Schwarzschild metric and we show results of vacuum solution are different from of [8].We consider Weyl tensor and we conclude that the Finsler metric is the Ads Schwarzschild and conformally flat.

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