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Coupled fixed points theorems for non-linear contractions in partially ordered modular spaces

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Abstract

The aim of this paper is to determine some coupled coincidence and coupled common fixed point theorems for mixed g-monotone nonlinear contractive mappings in partially ordered modular spaces.

Keywords: Coupled fixed point, coupled coincidence, contraction, mixed monotone mappaing, modular space, partially ordered modular space.
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1. Introduction

The study of modular spaces was initiated by Nakano [24] in 1950 and generalized by Musielak and Orlicz [23], Koshi and Shimogaki [13] and Yamamuro [31] and their collaborators. The monographic exposition of the theory of Orlicz spaces may be found in the book of Krasnosel'skii and Rutickii [12]. Fixed point theory is very useful in solving a variety of problems in control theory, economic theory, nonlinear analysis and so on. The Banach contraction principle is the most famous fixed point theorem. Many authors presented some new results for contractions in partially ordered metric spaces (cf. [1, 2, 3, 5, 6, 7, 8, 19, 26]). The study of fixed points of mappings on complete partial ordered metric spaces was first investigated by Ran and Reurings [27] in 2004, and then by Nieto and Rodrigues-Lupez [25]. Lakshmikantham and Ćirić [18] introduced the notions of mixed g-monotone property and proved coupled fixed point theorems for mixed g-monotone nonlinear contractive mappings in partially ordered complete metric spaces. The theory of fixed points in the content of modular spaces was initiated by Khamsi *et al.* [9] (see also [4, 10, 15, 20, 21, 22, 16, 17, 28, 30]).

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In this paper by using some ideas of [18], we prove some coupled coincidence and coupled common fixed point theorems for mixed g-monotone nonlinear contractive mappings in partially ordered complete modular spaces. Some basic facts and notations about modular spaces are recalled from [14].

Definition 1.1. *Let $\mathcal{X}$ be an arbitrary vector space over $\mathbb{F}(=\mathbb{R}$ or $\mathbb{C})$.*
A functional $\rho : \mathcal{X} \to [0, \infty]$ is called modular if for all $x, y \in \mathcal{X}$,
(i) $\rho(x) = 0$ if and only if $x = 0$,
(ii) $\rho(\alpha x) = \rho(x)$ for every $\alpha \in \mathbb{F}$ with $|\alpha| = 1$,
(iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ if $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$.

Definition 1.2. *If (iii) in definition 1.1 is replaced by*

$$\rho(\alpha x + \beta y) \leq \alpha^s \rho(x) + \beta^s \rho(y),$$

for $\alpha, \beta \geq 0$, $\alpha + \beta = 1$ with an $s \in (0, 1]$, then we say that ρ is a s-convex modular, and if $s = 1$, ρ is called a convex modular.

A modular ρ defines a corresponding modular space, i.e., the vector space $\mathcal{X}_\rho$ given by

$$\mathcal{X}_\rho = \{x \in \mathcal{X} : \quad \rho(\lambda x) \to 0 \text{ as } \lambda \to 0\}.$$

Let ρ be a convex modular, the modular space $\mathcal{X}_\rho$ can be equipped with a norm called the Luxemburg norm, defined by

$$\|x\|_\rho = \inf\left\{\lambda > 0 \quad ; \quad \rho\left(\frac{x}{\lambda}\right) \leq 1\right\}.$$

Definition 1.3. *A modular ρ is said to satisfy the Δ_2–condition if there exists $\kappa > 0$ such that for any $x \in \mathcal{X}_\rho$, we have $\rho(2x) \leq \kappa \rho(x)$.*

Definition 1.4. *Let $\mathcal{X}_\rho$ be a modular space and let $\{x_n\}$ and x be in $\mathcal{X}_\rho$. Then*
(i) $\{x_n\}$ is said to be ρ–convergent to x and write $x_n \xrightarrow{\rho} x$ if $\rho(x_n - x) \to 0$ as $n \to \infty$.
(ii) $\{x_n\}$ is called ρ–Cauchy if $\rho(x_n - x_m) \to 0$ as $n, m \to \infty$.
(iii) A subset $\mathcal{S}$ of $\mathcal{X}_\rho$ is called ρ–complete if any ρ–Cauchy sequence is ρ–convergent to an element of $\mathcal{S}$.
(v) We say the modular ρ has the Fatou property if $\rho(x) \leq \liminf_{n\to\infty} \rho(x_n)$ whenever $x_n \xrightarrow{\rho} x$.

Note that ρ-convergence does not imply ρ-cauchy since ρ does not satisfy the triangle inequality. In fact, this will happen if and only if ρ satisfies the Δ_2–condition.

Remark 1.5. *Note that $\rho(.x)$ is an increasing function, for any $x \in \mathcal{X}$. Suppose $0 < a < b$, then the property (iii) of Definition 1.1 with $y = 0$ shows that $\rho(ax) = \rho\left(\frac{a}{b} bx\right) \leq \rho(bx)$ for all $x \in \mathcal{X}$. Moreover, if ρ is a convex modular on $\mathcal{X}$ and $|\alpha| \leq 1$, then $\rho(\alpha x) \leq \alpha \rho(x)$ and also $\rho(x) \leq \frac{1}{2}\rho(2x)$ for all $x \in \mathcal{X}$.*

Bhaskar and Lakshmikantham [5] introduced the following notions of a mixed monotone mapping and a coupled fixed point.

Definition 1.6. *Let $(\mathcal{X}, \leq)$ be a partially ordered set. The mapping $F : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ is said to has the mixed monotone property if F is monotone non-decreasing in its first argument and is monotone non-increasing in its second argument, that is, for any $x, y \in \mathcal{X}$*

$$x_1, x_2 \in \mathcal{X}, \quad x_1 \leq x_2 \to F(x_1, y) \leq F(x_2, y) \tag{1.1}$$

and

$$y_1, y_2 \in \mathcal{X}, \quad y_1 \leq y_2 \to F(x, y_1) \geq F(x, y_2). \tag{1.2}$$

Definition 1.7. *An element $(x, y) \in \mathcal{X} \times \mathcal{X}$ is called a coupled fixed point of the mapping $F : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ if*

$$F(x, y) = x, \quad F(y, x) = y.$$

The following definition is recalled from [18].

Definition 1.8. *Let $(\mathcal{X}, \leq)$ be a partially ordered set and let $F : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ and $g : \mathcal{X} \to \mathcal{X}$ be mappings. We say F has the mixed g-monotone property if F is monotone g-non-decreasing in its first argument and is monotone g-non-increasing in its second argument, that is, for any $x, y \in \mathcal{X}$*

$$x_1, x_2 \in \mathcal{X}, \quad g(x_1) \leq g(x_2) \to F(x_1, y) \leq F(x_2, y) \tag{1.3}$$

and

$$y_1, y_2 \in \mathcal{X}, \quad g(y_1) \leq g(y_2) \to F(x, y_1) \geq F(x, y_2). \tag{1.4}$$

Note that if g is the identity mapping, then Definition 1.8 reduces to Definition 1.6.

Definition 1.9. *An element $(x, y) \in \mathcal{X} \times \mathcal{X}$ is called a coupled coincidence point of mappings $F : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ and $g : \mathcal{X} \to \mathcal{X}$ if*

$$F(x, y) = g(x), \quad F(y, x) = g(y).$$

Definition 1.10. *Let $\mathcal{X}$ be a non-empty set and let $F : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ and $g : \mathcal{X} \to \mathcal{X}$ be mappings. We say F and g are commutative if for each $x, y \in \mathcal{X}$*

$$g(F(x, y)) = F(g(x), g(y)).$$

2. Coupled fixed point theorems for mixed g-monotone contractions

Let $\mathcal{X}$ be a vector space. Then $(\mathcal{X}, \preceq, \rho)$ is called an ordered modular space if ρ is a modular on $\mathcal{X}$ and $\preceq$ is a partial order on $\mathcal{X}$. Throughout this section, we assume that the modular ρ satisfies the Δ_2–condition with $\kappa > 1$. Also if $\alpha, \beta \in \mathbb{R}^+$ with $\alpha > \beta$, we consider $\alpha_0 \in \mathbb{R}^+$ such that $\frac{\beta}{\alpha} + \frac{1}{\alpha_0} = 1$ and we assume that r is the smallest positive integer in which $\alpha_0 \leq 2^r$.

Denote by Ψ the family of non-decreasing functions $\psi : [0, +\infty) \to [0, +\infty)$ such that $\psi(t) < t$ and $\lim_{r \to t^+} \psi(r) < t$ for all $t > 0$.

Lemma 2.1. *(Singh and Meade [29]) $\psi : [0, +\infty] \to [0, +\infty]$ is non-decreasing and right continuous, then $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t \geq 0$ if and only if $\psi(t) < t$ for all $t > 0$.*

Lakshmikantham and Ćirić in [18] proved coupled fixed point theorems for mixed g-monotone contractive mappings in partially ordered complete metric spaces. In following we prove similar results in partially ordered complete modular spaces.

Theorem 2.2. *Let $(\mathcal{X}, \preceq, \rho)$ be a complete ordered modular space and $\psi \in \Psi$. Also suppose $F : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ and $g : \mathcal{X} \to \mathcal{X}$ be mappings such that F has the mixed g-monotone property and there exist $\alpha, \beta \in \mathbb{R}^+$ with $\alpha > \beta$ such that*

$$\rho\Big(\alpha(F(x,y) - F(z,w))\Big) \leq \psi\Big[\frac{\rho\Big(\beta(g(x) - g(z))\Big) + \rho\Big(\beta(g(y) - g(w))\Big)}{2\kappa^r}\Big] \tag{2.1}$$

for all $x, y, z, w \in \mathcal{X}$ for which $g(x) \leq g(z)$ and $g(y) \geq g(w)$. Let $F(\mathcal{X} \times \mathcal{X}) \subseteq g(\mathcal{X})$, g is continuous and commutes with F. Also assume either
$(i)F$ is continuous or
$(ii)\mathcal{X}$ has the following properties:

$$(a)\textit{if } \{x_n\} \textit{is a non-decreasing sequence such that } x_n \to x, \textit{then } x_n \leq x, \textit{for all } n, \tag{2.2}$$

$$(b)\textit{if } \{y_n\} \textit{is a non-decreasing sequence such that } y_n \to y, \textit{then } y \leq y_n, \textit{for all } n. \tag{2.3}$$

If there exist $x_0, y_0 \in \mathcal{X}$ such that

$$g(x_0) \leq F(x_0, y_0) \quad \textit{and} \quad g(y_0) \geq F(y_0, x_0), \tag{2.4}$$

then F and g have a coupled coincidence point.

Proof . *Let $x_0, y_0 \in \mathcal{X}$ be such that $g(x_0) \leq F(x_0, y_0)$ and $g(y_0) \geq F(y_0, x_0)$. Since $F(\mathcal{X} \times \mathcal{X}) \subseteq g(\mathcal{X})$, we can choose $x_1, y_1 \in \mathcal{X}$ such that $g(x_1) = F(x_0, y_0)$ and $g(y_1) = F(y_0, x_0)$. Continuing this process we get sequences $\{x_n\}$ and $\{y_n\}$ in $\mathcal{X}$ such that*

$$g(x_{n+1}) = F(x_n, y_n) \quad \textit{and} \quad g(y_{n+1}) = F(y_n, x_n), \quad (n \geq 0). \tag{2.5}$$

By using induction, we show that

$$g(x_n) \leq g(x_{n+1}) \quad \textit{and} \quad g(y_n) \geq g(y_{n+1}), \quad (n \geq 0). \tag{2.6}$$

Let $n = 0$, from (2.4) we deduce $g(x_1) = F(x_0, y_0)$ and $g(y_1) = F(y_0, x_0)$, hence (2.6) holds. Now suppose (2.6) holds for $n \geq 0$, since F has the mixed g-monotone property, from (2.5) we get

$$g(x_{n+1}) = F(x_n, y_n) \leq F(x_{n+1}, y_n) \quad \textit{and} \quad F(y_{n+1}, x_n) \leq F(y_n, x_n) = g(y_{n+1}), \tag{2.7}$$

$$g(x_{n+2}) = F(x_{n+1}, y_{n+1}) \geq F(x_{n+1}, y_n) \quad \textit{and} \quad F(y_{n+1}, x_n) \geq F(y_{n+1}, x_{n+1}) = g(y_{n+2}). \tag{2.8}$$

Therefore $g(x_{n+1}) \leq g(x_{n+2})$ and $g(y_{n+1}) \geq g(y_{n+2})$. Hence (2.6) holds for all $n \geq 0$. Put

$$\gamma_n = \rho\Big(\beta(g(x_n) - g(x_{n+1}))\Big) + \rho\Big(\beta(g(y_n) - g(y_{n+1}))\Big),$$

we prove that

$$\gamma_n \leq 2\psi\Big(\frac{\gamma_{n-1}}{2\kappa^r}\Big). \tag{2.9}$$

Since $g(x_{n-1}) \leq g(x_n)$ and $g(y_{n-1}) \geq g(y_n)$, from (2.1) and (2.5) we have

$$\begin{aligned} \rho\Big(\beta(g(x_n) - g(x_{n+1}))\Big) &\leq \rho\Big(\alpha(g(x_n) - g(x_{n+1}))\Big) = \rho\Big(\alpha(F(x_{n-1}, y_{n-1}) - F(x_n, y_n))\Big) \\ &\leq \psi\Big[\frac{\rho\Big(\beta(g(x_{n-1}) - g(x_n))\Big) + \rho\Big(\beta(g(y_{n-1}) - g(y_n))\Big)}{2\kappa^r}\Big] \\ &= \psi\Big(\frac{\gamma_{n-1}}{2\kappa^r}\Big). \end{aligned} \tag{2.10}$$

Similarly, we get

$$\begin{aligned}\rho\Big(\beta(g(y_n)-g(y_{n+1}))\Big) \leq \rho\Big(\alpha(g(y_n)-g(y_{n+1}))\Big) &= \rho\Big(\beta(F(y_n,x_n)-F(y_{n-1},x_{n-1}))\Big)\\ &\leq \psi\Big[\frac{\rho\Big(\beta(g(y_{n-1})-g(y_n))\Big)+\rho\Big(\beta(g(x_{n-1})-g(x_n))\Big)}{2\kappa^r}\Big]\\ &= \psi\Big(\frac{\gamma_{n-1}}{2\kappa^r}\Big). \end{aligned} \tag{2.11}$$

Adding (2.10) and (2.11) we obtain (2.9). Since $\psi(t) < t$ *for* $t > 0$*, hence (2.9) implies that the sequence* $\{\gamma_n\}$ *is monotone decreasing. Therefore there is* $\gamma \geq 0$ *such that* $\lim_{n\to\infty}\gamma_n = \gamma$*. We show that* $\gamma = 0$*. Suppose that* $\gamma > 0$*, then (2.9) implies that*

$$\gamma = \lim_{n\to\infty}\gamma_n \leq 2lim_{n\to\infty}\psi\Big(\frac{\gamma_{n-1}}{2\kappa^r}\Big) < 2\frac{\gamma}{2\kappa^r} = \frac{\gamma}{\kappa^r},$$

and this is a contradiction. Hence $\gamma = 0$*, so*

$$\lim_{n\to\infty}\Big[\rho\Big(\beta(g(x_n)-g(x_{n+1}))\Big)+\rho\Big(\beta(g(y_n)-g(y_{n+1}))\Big)\Big] = 0. \tag{2.12}$$

Now we prove that $\{g(x_n)\}$ *and* $\{g(y_n)\}$ *are cauchy sequences. Suppose, to the contrary, that at least one of* $\{g(x_n)\}$ *or* $\{g(y_n)\}$ *is not a cauchy sequence. Then there exists an* $\varepsilon > 0$ *and two subsequences* $\{m_l\}$ *and* $\{n_l\}$ *of integers such that* $m_l > n_l \geq l$ *and*

$$\delta_l = \rho\Big(\beta(g(x_{n_l})-g(x_{m_l}))\Big)+\rho\Big(\beta(g(y_{n_l})-g(y_{m_l}))\Big) \geq \frac{\varepsilon}{\kappa^r} \quad (l\in\mathbb{N}), \tag{2.13}$$

and

$$\rho\Big(\beta(g(x_{n_l})-g(x_{m_l-1}))\Big)+\rho\Big(\beta(g(y_{n_l})-g(y_{m_l-1}))\Big) < \frac{\varepsilon}{\kappa^r}. \tag{2.14}$$

We choose m_l *in such a way that it is the smallest integer with* $m_l > n_l$ *for which (2.13) holds. From*

(2.13), (2.14) and the condition (iii) of Definition 1.1 we have

$$
\begin{aligned}
\frac{\varepsilon}{\kappa^r} \leq \delta_l &= \rho\Big(\frac{\beta}{\alpha}\Big[\alpha(g(x_{n_l}) - g(x_{m_l-1}))\Big] + \frac{1}{\alpha_0}\Big[\alpha_0\beta(g(x_{m_l-1}) - g(x_{m_l}))\Big]\Big) \\
&\quad + \rho\Big(\frac{\beta}{\alpha}\Big[\alpha(g(y_{n_l}) - g(y_{m_l-1}))\Big] + \frac{1}{\alpha_0}\Big[\alpha_0\beta(g(y_{m_l-1}) - g(y_{m_l}))\Big]\Big) \\
&\leq \rho\Big(\alpha(g(x_{n_l}) - g(x_{m_l-1}))\Big) + \rho\Big(\alpha_0\beta(g(x_{m_l-1}) - g(x_{m_l}))\Big) \\
&\quad + \rho\Big(\alpha(g(y_{n_l}) - g(y_{m_l-1}))\Big) + \rho\Big(\alpha_0\beta(g(y_{m_l-1}) - g(y_{m_l}))\Big) \\
&= \rho\Big(\alpha(F(x_{n_l-1}, y_{n_l-1}) - F(x_{m_l-2}, y_{m_l-2}))\Big) + \rho\Big(\alpha_0\beta(g(x_{m_l-1}) - g(x_{m_l}))\Big) \\
&\quad + \rho\Big(\alpha(F(y_{n_l-1}, x_{n_l-1}) - F(y_{m_l-2}, x_{m_l-2}))\Big) + \rho\Big(\alpha_0\beta(g(y_{m_l-1}) - g(y_{m_l}))\Big) \\
&\leq \psi\Big[\frac{\rho\Big(\beta(g(x_{n_l-1}) - g(x_{m_l-2}))\Big) + \rho\Big(\beta(g(y_{n_l-1}) - g(y_{m_l-2}))\Big)}{2\kappa^r}\Big] \\
&\quad + \psi\Big[\frac{\rho\Big(\beta(g(y_{n_l-1}) - g(y_{m_l-2}))\Big) + \rho\Big(\beta(g(x_{n_l-1}) - g(x_{m_l-2}))\Big)}{2\kappa^r}\Big] \\
&\quad + \rho\Big(\alpha_0\beta(g(x_{m_l-1}) - g(x_{m_l}))\Big) + \rho\Big(\alpha_0\beta(g(y_{m_l-1}) - g(y_{m_l}))\Big) \\
&< \frac{\rho\Big(\beta(g(x_{n_l-1}) - g(x_{m_l-2}))\Big) + \rho\Big(\beta(g(y_{n_l-1}) - g(y_{m_l-2}))\Big)}{2\kappa^r} \\
&\quad + \frac{\rho\Big(\beta(g(y_{n_l-1}) - g(y_{m_l-2}))\Big) + \rho\Big(\beta(g(x_{n_l-1}) - g(x_{m_l-2}))\Big)}{2\kappa^r} \\
&\quad + \rho\Big(\alpha_0\beta(g(x_{m_l-1}) - g(x_{m_l}))\Big) + \rho\Big(\alpha_0\beta(g(y_{m_l-1}) - g(y_{m_l}))\Big) \\
&\leq \frac{\varepsilon}{2\kappa^r} + \frac{\varepsilon}{2\kappa^r} + \rho\Big(\alpha_0\beta(g(x_{m_l-1}) - g(x_{m_l}))\Big) + \rho\Big(\alpha_0\beta(g(y_{m_l-1}) - g(y_{m_l}))\Big) \\
&< \frac{\varepsilon}{\kappa^r} + \kappa^r\Big[\rho\Big(\beta(g(x_{m_l-1}) - g(x_{m_l}))\Big) + \rho\Big(\beta(g(y_{m_l-1}) - g(y_{m_l}))\Big)\Big],
\end{aligned}
$$

taking the limit as $k \to \infty$, by (2.12) and Δ_2-condition we get

$$
\lim_{l\to\infty} \delta_l = \frac{\varepsilon}{\kappa^r}. \tag{2.15}
$$

Similarly we have

$$\begin{aligned}
\delta_l &= \rho\Big(\beta(g(x_{n_l}) - g(x_{m_l}))\Big) + \rho\Big(\beta(g(y_{n_l}) - g(y_{m_l}))\Big)\\
&\leq \rho\Big(\alpha(g(x_{n_l}) - g(x_{n_l+1}))\Big) + \rho\Big(\alpha_0\beta(g(x_{n_l+1}) - g(x_{m_l}))\Big)\\
&\quad + \rho\Big(\alpha(g(y_{n_l}) - g(y_{n_l+1}))\Big) + \rho\Big(\alpha_0\beta(g(y_{n_l+1}) - g(y_{m_l}))\Big)\\
&< \rho\Big(\alpha(g(x_{n_l}) - g(x_{n_l+1}))\Big) + \rho\Big(\alpha(g(y_{n_l}) - g(y_{n_l+1}))\Big)\\
&\quad + \rho\Big(2^r\beta(g(x_{n_l+1}) - g(x_{m_l}))\Big) + \rho\Big(2^r\beta(g(y_{n_l+1}) - g(y_{m_l}))\Big)\\
&< \rho\Big(\alpha(g(x_{n_l}) - g(x_{n_l+1}))\Big) + \rho\Big(\alpha(g(y_{n_l}) - g(y_{n_l+1}))\Big)\\
&\quad + \kappa^r\rho\Big(\beta(g(x_{n_l+1}) - g(x_{m_l}))\Big) + \kappa^r\rho\Big(\beta(g(y_{n_l+1}) - g(y_{m_l}))\Big)\\
&\leq \rho\Big(\alpha(g(x_{n_l}) - g(x_{n_l+1}))\Big) + \rho\Big(\alpha(g(y_{n_l}) - g(y_{n_l+1}))\Big)\\
&\quad + \kappa^r\rho\Big(\alpha(g(x_{n_l+1}) - g(x_{m_l+1}))\Big) + \kappa^r\rho\Big(\alpha_0\beta(g(x_{m_l+1}) - g(x_{m_l}))\Big)\\
&\quad + \kappa^r\rho\Big(\alpha(g(y_{n_l+1}) - g(y_{m_l+1}))\Big) + \kappa^r\rho\Big(\alpha_0\beta(g(y_{m_l+1}) - g(y_{m_l}))\Big),
\end{aligned}$$

moreover

$$\begin{aligned}
&\rho\Big(\alpha_0\beta(g(x_{m_l+1}) - g(x_{m_l}))\Big) + \rho\Big(\alpha_0\beta(g(y_{m_l+1}) - g(y_{m_l}))\Big)\\
&< \rho\Big(2^r\beta(g(x_{m_l+1}) - g(x_{m_l}))\Big) + \rho\Big(2^r\beta(g(y_{m_l+1}) - g(y_{m_l}))\Big)\\
&< \kappa^r\rho\Big(\beta(g(x_{m_l+1}) - g(x_{m_l}))\Big) + \kappa^r\rho\Big(\beta(g(y_{m_l+1}) - g(y_{m_l}))\Big) = \kappa^r\gamma_{m_l}.
\end{aligned}$$

Consequently

$$\delta_l \leq \kappa^r\gamma_{n_l} + \kappa^{2r}\gamma_{m_l} + \kappa^r\Big[\rho\Big(\alpha(g(x_{n_l+1}) - g(x_{m_l+1}))\Big) + \rho\Big(\alpha(g(y_{n_l+1}) - g(y_{m_l+1}))\Big)\Big] \tag{2.16}$$

on the other hand $g(x_{n_l}) \leq g(x_{m_l})$ *and* $g(y_{n_l}) \geq g(y_{m_l})$, *hence we have*

$$\begin{aligned}
\rho\Big(\alpha(g(x_{n_l+1}) - g(x_{m_l+1}))\Big) &= \rho\Big(\alpha(F(x_{n_l}, y_{n_l}) - F(x_{m_l}, y_{m_l}))\Big)\\
&\leq \psi\Big[\frac{\rho\Big(\beta(g(x_{m_l}) - g(x_{m_l}))\Big) + \rho\Big(\beta(g(y_{m_l}) - g(y_{m_l}))\Big)}{2\kappa^r}\Big]\\
&= \psi\Big(\frac{\delta_l}{2\kappa^r}\Big), \qquad (2.17)
\end{aligned}$$

and similarly

$$\begin{aligned}
\rho\Big(\alpha(g(y_{n_l+1}) - g(y_{m_l+1}))\Big) &= \rho\Big(\alpha(F(y_{n_l}, x_{n_l}) - F(y_{m_l}, x_{m_l}))\Big)\\
&\leq \psi\Big[\frac{\rho\Big(\beta(g(y_{m_l}) - g(y_{m_l}))\Big) + \rho\Big(\beta(g(x_{m_l}) - g(x_{m_l}))\Big)}{2\kappa^r}\Big]\\
&= \psi\Big(\frac{\delta_l}{2\kappa^r}\Big). \qquad (2.18)
\end{aligned}$$

Inserting (2.17) and (2.18) in (2.16) we get

$$\delta_l \leq \kappa^r \gamma_{n_l} + \kappa^{2r}\gamma_{m_l} + 2\kappa^r \psi\Big(\frac{\delta_l}{2\kappa^r}\Big). \tag{2.19}$$

Letting $k \to \infty$ from (2.15) we obtain

$$\frac{\varepsilon}{\kappa^r} \leq 2\kappa^r \lim_{k\to\infty} \psi\Big(\frac{\delta_l}{2\kappa^r}\Big) < \frac{\varepsilon}{\kappa^r}, \tag{2.20}$$

and this is a contradiction. Therefore $\{g(x_n)\}$ and $\{g(y_n)\}$ are cauchy sequences. Since $\mathcal{X}$ is complete, there exist $x, y \in \mathcal{X}$ such that

$$\lim_{n\to\infty} g(x_n) = x \quad \textit{and} \quad \lim_{n\to\infty} g(y_n) = y. \tag{2.21}$$

g is continuous hence

$$\lim_{n\to\infty} g(g(x_n)) = g(x) \quad \textit{and} \quad \lim_{n\to\infty} g(g(y_n)) = g(y). \tag{2.22}$$

From (2.5) and commutativity of F and g we have

$$g(g(x_{n+1})) = g(F(x_n, y_n)) = F(g(x_n), g(y_n)), \tag{2.23}$$
$$g(g(y_{n+1})) = g(F(y_n, x_n)) = F(g(y_n), g(x_n)). \tag{2.24}$$

Finally, we claim that $g(x) = F(x, y)$ and $g(y) = F(y, x)$. Suppose that the assumption (i) holds that is F is continuous, then taking the limit as $n \to \infty$ in (2.23) and by (2.21) and (2.22) we obtain

$$g(x) = \lim_{n\to\infty} g(g(x_{n+1})) = \lim_{n\to\infty} F(g(x_n), g(y_n)) = F(x, y),$$
$$g(y) = \lim_{n\to\infty} g(g(y_{n+1})) = \lim_{n\to\infty} F(g(y_n), g(x_n)) = F(y, x).$$

Now suppose that (ii) holds. Since $\{g(x_n)\}$ and $\{g(y_n)\}$ are non-decreasing and $\lim_{n\to\infty} g(x_n) = x$ and $\lim_{n\to\infty} g(y_n) = y$, from (2.2) and (2.3) we have $g(x_n) \leq x$ and $g(y_n) \geq y$, for all n. Therefore by (2.23) and (2.1) we get

$$\begin{aligned}
\rho\Big(\beta(g(x) - F(x, y))\Big) &= \rho\Big(\frac{\beta}{\alpha}\Big[\alpha(g(x) - g(g(x_{n+1}))\Big] + \frac{1}{\alpha_0}\Big[\alpha_0\beta(g(g(x_{n+1}) - F(x, y))\Big]\Big) \\
&\leq \rho\Big(\alpha(g(x) - g(g(x_{n+1}))\Big) + \rho\Big(\alpha_0\beta(g(g(x_{n+1}) - F(x, y))\Big) \\
&= \rho\Big(\alpha(g(x) - g(g(x_{n+1}))\Big) + \rho\Big(\alpha_0\beta(F(g(x_n), g(y_n)) - F(x, y))\Big) \\
&\leq \rho\Big(\alpha(g(x) - g(g(x_{n+1}))\Big) + \kappa^r\rho\Big(\alpha(F(g(x_n), g(y_n)) - F(x, y))\Big) \\
&\leq \rho\Big(\alpha(g(x) - g(g(x_{n+1}))\Big) \\
&\quad + \kappa^r\psi\Big(\frac{\rho\Big(\beta(g(g(x_n)) - g(x))\Big) + \rho\Big(\beta(g(g(y_n)) - g(y))\Big)}{\kappa^r}\Big).
\end{aligned}$$

Letting $n \to \infty$ implies that $\rho\Big(\beta(g(x) - F(x, y))\Big) = 0$. Therefore $g(x) = F(x, y)$. Similarly one can prove that $g(y) = F(y, x)$. That is F and g have a coupled coincidence point. □

Now, we give an example in which satisfies in the hypothesis of Theorem 2.2.

Example 2.3. *Let $\mathcal{X} = \mathbb{R}$ and $\rho(x) = |x|$ for each $x \in \mathbb{R}$, then ρ is a modular which satisfies in Δ_2-condition with $\kappa = 2$. Consider $\alpha = 2$ and $\beta = 1$, then we get $\alpha_0 = 2$ and $r = 1$. Define mappings $F : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ and $g : \mathcal{X} \to \mathcal{X}$ by $F(x,y) = \frac{1}{8}\ln(1+|x|) - \frac{1}{8}\ln(1+|y|)$ and $g(x) = x$ for all $x, y \in \mathcal{X}$. Then F and g satisfies in the requirement of Theorem 2.2 and we get*

$$\begin{aligned}
\rho\Big(\alpha(F(x,y) - F(z,w))\Big) &= 2\Big|\frac{1}{8}\ln(1+|x|) - \frac{1}{8}\ln(1+|y|) - \frac{1}{8}\ln(1+|z|) + \frac{1}{8}\ln(1+|w|)\Big| \\
&\le \frac{1}{8}\Big|\ln\frac{1+|x|}{1+|z|}\Big| + \frac{1}{8}\Big|\ln\frac{1+|y|}{1+|w|}\Big| \\
&\le \frac{1}{2}\Big[\frac{1}{4}\ln(1+|x-z|)\Big| + \frac{1}{2}\Big[\frac{1}{4}\ln(1+|y-w|)\Big]\Big] \\
&\le \frac{1}{2}\ln\Big(\frac{4+|x-z|+|y-w|}{4}\Big) \\
&\le \frac{1}{2}\ln\Big(1+\frac{|x-z|+|y-w|}{4}\Big) \\
&= \frac{1}{2}\ln\Big(1+\frac{\rho(x-z)+\rho(y-w)}{4}\Big).
\end{aligned}$$

Therefore (2.25) holds for $\psi(t) = \frac{1}{2}\ln(1+t)$ for all $t > 0$, and also the hypothesis of Theorem 2.2 is fulfilled. Hence F and g have a coupled coincidence point that, where $(0,0)$ is a coupled coincidence point of F and g.

In the following theorem we will prove the existence and uniquness of the coupled fixed point for mixed g-monotone contractive mappings in partially ordered modular spaces. Let $(\mathcal{X}, \le)$ be a partially ordered set. We endow the product $\mathcal{X} \times \mathcal{X}$ with the following partial order relation:

$$(x,y) \le (z,w) \leftrightarrow x \le z, y \ge w$$

for all $(x,y), (z,w) \in \mathcal{X} \times \mathcal{X}$.

Theorem 2.4. *In addition to the hypothesis of Theorem 2.2, suppose that for each $(x,y), (z,w) \in \mathcal{X}\times\mathcal{X}$ there exists an element $(s,t) \in \mathcal{X}\times\mathcal{X}$ such that $(F(s,t), F(t,s))$ is comparable to $(F(x,y), F(y,x))$ and $(F(z,w), F(w,z))$. Then F and g have a unique coupled fixed point.*

Proof . *From Theorem 2.2, F and g have a coupled fixed point. Suppose (x,y) and (z,w) are coupled fixed points of the mappings F and g, that is, $g(x) = F(x,y), g(y) = F(y,x)$ and $g(z) = F(z,w), g(w) = F(w,z)$. By assumption there exists $(s,t) \in \mathcal{X} \times \mathcal{X}$ such that $(F(s,t), F(t,s))$ is comparable to $(F(x,y), F(y,x))$ and $(F(z,w), F(w,z))$. Put $s_0 = s, t_0 = t$ and choose $s_1, t_1 \in \mathcal{X}$ so that $g(s_1) = F(s_0,t_0), g(t_1) = F(t_0,s_0)$. As in the proof of Theorem 2.2, we can define sequences $\{g(s_n)\}$ and $\{g(t_n)\}$ as*

$$g(s_{n+1}) = F(s_n,t_n), g(t_{n+1}) = F(t_n,s_n), g(s_n) \le g(s_{n+1}) \quad \text{and} \quad g(t_n) \ge g(t_{n+1}).$$

Also, put $x_0 = x, y_0 = y, z_0 = z, w_0 = w$ and on the same way, define the sequences $\{g(x_n)\}$, $\{g(y_n)\}$ and $\{g(z_n)\}$ $\{g(w_n)\}$. Then $g(x) = g(x_0) = F(x_0,y_0) = g(x_1)$ and $g(y) = g(y_0) = F(y_0,x_0) = g(y_1)$.

Furthermore we have

$$\begin{aligned}\rho\Big(\beta(g(x)-g(x_2))\Big) &= \rho\Big(\beta(g(x)-F(x_1,y_1))\Big)\\ &= \rho\Big(\frac{\beta}{\alpha}\Big[\alpha(g(x)-g(g(x_{n+1}))\Big]+\frac{1}{\alpha_0}\Big[\alpha_0\beta(g(g(x_{n+1})-F(x_1,y_1))\Big]\Big)\\ &\le \rho\Big(\alpha(g(x)-g(g(x_{n+1}))\Big)+\rho\Big(\alpha_0\beta(g(g(x_{n+1})-F(x_1,y_1))\Big)\\ &= \rho\Big(\alpha(g(x)-g(g(x_{n+1}))\Big)+\rho\Big(\alpha_0\beta(F(g(x_n),g(y_n))-F(x_1,y_1))\Big)\\ &\le \rho\Big(\alpha(g(x)-g(g(x_{n+1}))\Big)+\kappa^r\rho\Big(\alpha(F(g(x_n),g(y_n))-F(x_1,y_1))\Big)\\ &\le \rho\Big(\alpha(g(x)-g(g(x_{n+1}))\Big)\\ &\quad+\kappa^r\psi\Big(\frac{\rho\Big(\beta(g(g(x_n))-g(x_1))\Big)+\rho\Big(\beta(g(g(y_n))-g(y_1))\Big)}{2\kappa^r}\Big)\\ &\le \rho\Big(\alpha(g(x)-g(g(x_{n+1}))\Big)\\ &\quad+\kappa^r\psi\Big(\frac{\rho\Big(\beta(g(g(x_n))-g(x))\Big)+\rho\Big(\beta(g(g(y_n))-g(y))\Big)}{2\kappa^r}\Big).\end{aligned}$$

Letting $n\to\infty$ *implies that* $\rho\Big(\beta(g(x)-g(x_2))\Big)=0$. *Hence* $F(x,y)=g(x)=g(x_2)$. *Similarly we can prove that* $F(y,x)=g(y)=g(y_2)$ *and continuing this process we have*

$$g(x_n)=F(x,y),\quad g(y_n)=F(y,x),\quad g(z_n)=F(z,w),\quad g(w_n)=F(w,z)\quad(n\ge 1). \tag{2.25}$$

Moreover $(F(x,y),F(y,x))=(g(x_1),g(y_1))=(g(x),g(y))$ *and* $(F(s,t),F(t,s))=(g(s_1),g(t_1))$ *are comparable, so* $g(x)\le g(s_1)$ *and* $g(y)\ge g(t_1)$. *Since* $g(s_n)\le g(s_{n+1})$ *and* $g(t_n)\ge g(t_{n+1})$, *hence* $(g(x),g(y))$ *and* $(g(s_n),g(t_n))$ *are comparable, that is* $g(x)\le g(s_n)$ *and* $g(y)\ge g(t_n)$ *for all* $n\ge 1$. *Therefore from (2.1) we get*

$$\begin{aligned}\rho\Big(\alpha(g(x)-g(s_{n+1}))\Big) &= \rho\Big(\beta(F(x,y)-F(s_n,t_n))\Big)\\ &\le \psi\Big(\frac{\rho\Big(\beta(g(x)-g(s_n))\Big)+\rho\Big(\beta(g(y)-g(t_n))\Big)}{2\kappa^r},\end{aligned} \tag{2.26}$$

and

$$\begin{aligned}\rho\Big(\alpha(g(y)-g(t_{n+1}))\Big) &= \rho\Big(\beta(F(y,x)-F(t_n,s_n))\Big)\\ &\le \psi\Big(\frac{\rho\Big(\beta(g(y)-g(t_n))\Big)+\rho\Big(\beta(g(x)-g(s_n))\Big)}{2\kappa^r}.\end{aligned} \tag{2.27}$$

Adding (2.26) and (2.27) we get

$$\begin{aligned}\frac{\rho\Big(\alpha(g(x)-g(s_{n+1}))\Big)+\rho\Big(\alpha(g(y)-g(t_{n+1}))\Big)}{2} &\le \psi\Big(\frac{\rho\Big(\beta(g(x)-g(s_n))\Big)+\rho\Big(\beta(g(y)-g(t_n))\Big)}{2\kappa^r}\\ &\le \psi\Big(\frac{\rho\Big(\beta(g(x)-g(s_n))\Big)+\rho\Big(\beta(g(y)-g(t_n))\Big)}{2}.\end{aligned}$$

Repeating the above process, for $n \geq 1$ we get

$$\frac{\rho\Big(\alpha(g(x)-g(s_{n+1}))\Big)+\rho\Big(\alpha(g(y)-g(t_{n+1}))\Big)}{2} \leq \psi^n\Big(\frac{\rho\Big(\beta(g(x)-g(s_1))\Big)+\rho\Big(\beta(g(y)-g(t_1))\Big)}{2\kappa^r}.$$

This implies that

$$\lim_{n\to\infty}\rho\Big(\alpha(g(x)-g(s_{n+1}))\Big)=0 \quad and \quad \lim_{n\to\infty}\rho\Big(\alpha(g(y)-g(t_{n+1}))\Big)=0, \tag{2.28}$$

and similarly we can prove that

$$\lim_{n\to\infty}\rho\Big(\alpha(g(z)-g(s_{n+1}))\Big)=0 \quad and \quad \lim_{n\to\infty}\rho\Big(\alpha(g(w)-g(t_{n+1}))\Big)=0. \tag{2.29}$$

Now, using the Δ_2-condition, (2.28) and (2.29) leads to

$$\rho\Big(\beta(g(x)-g(z))\Big) \leq \rho\Big(\alpha(g(x)-g(s_{n+1}))\Big)+\rho\Big(\alpha_0\beta(g(z)-g(s_{n+1}))\Big) \to 0 \quad as \quad n\to\infty,$$
$$\rho\Big(\beta(g(x)-g(z))\Big) \leq \rho\Big(\alpha(g(x)-g(s_{n+1}))\Big)+\rho\Big(\alpha_0\beta(g(z)-g(s_{n+1}))\Big) \to 0 \quad as \quad n\to\infty.$$

Consequently

$$g(x)=g(z) \quad and \quad g(y)=g(w). \tag{2.30}$$

Since $g(x)=F(x,y)$ and $g(y)=F(y,x)$, by commutativity of F and g we have

$$g(g(x))=g(F(x,y))=F(g(x),g(y)) \quad and \quad g(g(y))=g(F(y,x))=F(g(y),g(x)).$$

Set $u=g(x), v=g(y)$. Then

$$g(u)=F(u,v) \quad and \quad g(v)=F(v,u). \tag{2.31}$$

Hence (u,v) is a coupled coincidence point. Then from (2.30) with $z=u$ and $w=v$ it follows that $g(u)=g(x)$ and $g(v)=g(y)$, thus

$$g(u)=u \quad and \quad g(v)=v. \tag{2.32}$$

From (2.31) and (2.32), we get

$$u=g(u)=F(u,v) \quad and \quad v=g(v)=F(v,u).$$

Therefore (u,v) is a coupled common fixed point of F and g. Now, if (u',v') is another coupled common fixed point, then (2.30) implies that $u'=g(u')=g(u)=u$ and $v'=g(v')=g(v)=v$. □

If we put $\psi(t)=mt$ for $m\in[0,1)$ in Theorem 2.2, we obtain the following corollary.

Corollary 2.5. *Let $(\mathcal{X},\preceq,\rho)$ be a complete ordered modular function space. Suppose that $F:\mathcal{X}\times\mathcal{X}\to\mathcal{X}$ and $g:\mathcal{X}\to\mathcal{X}$ be mappings such that F has the mixed g-monotone property and there exist $\alpha,\beta\in\mathbb{R}^+$ with $\alpha>\beta$ such that*

$$\rho\Big(\alpha(F(x,y)-F(z,w))\Big) \leq \frac{m}{2\kappa^r}\Big[\rho\Big(\beta(g(x)-g(z))\Big)+\rho\Big(\beta(g(y)-g(w))\Big)\Big] \tag{2.33}$$

for all $x, y, z, w \in \mathcal{X}$ for which $g(x) \leq g(z)$ and $g(y) \geq g(w)$. Let $F(\mathcal{X} \times \mathcal{X}) \subseteq g(\mathcal{X})$, g is continuous and commutes with F. Also assume either
$(i)F$ is continuous or
$(ii)\mathcal{X}$ has the following properties:

$$(a) \text{if } \{x_n\} \text{is a non-decreasing sequence such that } x_n \to x, \text{ then } x_n \leq x, \text{for all } n, \tag{2.34}$$

$$(b) \text{if } \{y_n\} \text{is a non-decreasing sequence such that } y_n \to y, \text{ then } y \leq y_n, \text{for all } n. \tag{2.35}$$

If there exist $x_0, y_0 \in \mathcal{X}$ such that

$$g(x_0) \leq F(x_0, y_0) \quad \text{and} \quad g(y_0) \geq F(y_0, x_0), \tag{2.36}$$

then F and g have a coupled coincidence point.

Taking $g = I$(the identity mapping) in Theorem 2.2 we have the folloing result.

Corollary 2.6. *Let $(\mathcal{X}, \preceq, \rho)$ be a complete ordered modular function space and $\psi \in \Psi$. Also suppose $F : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ be a mapping having the mixed monotone property and there exist $\alpha, \beta \in \mathbb{R}^+$ with $\alpha > \beta$ such that and there exists a $k \in [0, 1)$ such that*

$$\rho\Big(\alpha(F(x, y) - F(z, w))\Big) \leq \psi\Big[\frac{\rho\Big(\beta(x - z)\Big) + \rho\Big(\beta(y - w)\Big)}{2\kappa^r}\Big] \tag{2.37}$$

for all $x, y, z, w \in \mathcal{X}$ for which $x \leq z$ and $y \geq w$. Also suppose either
$(i)F$ is continuous or
$(ii)\mathcal{X}$ has the following properties:

$$(a) \text{ if } \{x_n\} \text{is a non-decreasing sequence such that } x_n \to x, \text{ then } x_n \leq x, \text{for all } n, \tag{2.38}$$

$$(b) \text{ if } \{y_n\} \text{is a non-decreasing sequence such that } y_n \to y, \text{ then } y \leq y_n, \text{for all } n. \tag{2.39}$$

If there exist $x_0, y_0 \in \mathcal{X}$ such that

$$x_0 \leq F(x_0, y_0) \quad \text{and} \quad y_0 \geq F(y_0, x_0), \tag{2.40}$$

then there exist $x, y \in \mathcal{X}$ such that

$$x = F(x, y) \quad \text{and} \quad y = F(y, x).$$

Moreover, if x_0, y_0 are comparable, then $x = y$, so $x = F(x, x)$.
Proof . *It is enough to show that $x = F(x, x)$. Suppose that $x_0 \leq y_0$. We show that*

$$x_n \leq y_n \quad \text{for all} \quad n \geq 0, \tag{2.41}$$

where $x_n = F(x_{n-1}, y_{n-1}), y_n = F(y_{n-1}, x_{n-1})$. Assume that (2.41) holds for some fixed $n \geq 0$. Then the mixed monotone property of F implies that

$$x_{n+1} = F(x_n, y_n) \leq F(y_n, x_n) = y_{n+1}.$$

Now from (2.41) and (2.37) we get

$$\rho\Big(\alpha(F(x_n, y_n) - F(y_n, x_n))\Big) \leq \psi\Big[\frac{\rho(\beta(x_n - y_n))}{\kappa^r}\Big].$$

Hence we have

$$\begin{aligned}
\rho(\beta(x-y)) &= \rho\Big(\frac{\beta}{\alpha}(\alpha(x-x_{n+1})) + \frac{1}{\alpha_0}(\alpha_0\beta(x_{n+1}-y))\Big)\\
&\le \rho\Big(\alpha(x-x_{n+1})\Big) + \rho\Big(\alpha_0\beta(x_{n+1}-y)\Big)\\
&\le \rho\Big(\alpha(x-x_{n+1})\Big) + \rho\Big(2^r\beta(x_{n+1}-y)\Big)\\
&\le \rho\Big(\alpha(x-x_{n+1})\Big) + \kappa^r\rho\Big(\beta(x_{n+1}-y)\Big)\\
&\le \rho\Big(\alpha(x-x_{n+1})\Big) + \kappa^r\rho\Big(\alpha(x_{n+1}-y_{n+1})\Big) + \kappa^r\rho\Big(\alpha_0\beta(y_{n+1}-y)\Big)\\
&\le \rho\Big(\alpha(x-x_{n+1})\Big) + \kappa^r\rho\Big(\alpha(x_{n+1}-y_{n+1})\Big) + \kappa^{2r}\rho\Big(\beta(y_{n+1}-y)\Big)\\
&= \kappa^r\rho\Big(\alpha(F(x_n,y_n)-F(y_n,x_n))\Big) + \rho\Big(\alpha(x-x_{n+1})\Big) + \kappa^{2r}\rho\Big(\beta(y_{n+1}-y)\Big)\\
&\le \kappa^r\psi\Big[\frac{\rho(\beta(x_n-y_n))}{\kappa^r}\Big] + \rho\Big(\alpha(x-x_{n+1})\Big) + \kappa^{2r}\rho\Big(\beta(y_{n+1}-y)\Big).
\end{aligned}$$

Since $\psi(t) < t$ taking the limit as $n \to \infty$ we get

$$\rho(\beta(x-y)) \le \kappa^r\psi\Big[\frac{\rho(\beta(x-y))}{\kappa^r}\Big] < \rho(\beta(x-y)).$$

Therefore $\rho(\beta(x-y)) = 0$, hence $x = y$ and so $x = F(x,x)$. □

Taking $\psi(t) = mt$ with $m \in [0,1)$ in Corollary 2.6 we obtain the following result.

Corollary 2.7. *Let $(\mathcal{X}, \preceq, \rho)$ be a complete ordered modular function space. Suppose that $F : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ be a mapping having the mixed monotone property and there exist $\alpha, \beta \in \mathbb{R}^+$ with $\alpha > \beta$ and there exists a $k \in [0,1)$ such that*

$$\rho\Big(\alpha(F(x,y)-F(z,w))\Big) \le \frac{m}{2\kappa^r}\Big[\rho\Big(\beta(x-z)\Big) + \rho\Big(\beta(y-w)\Big)\Big]$$

for all $x, y, z, w \in \mathcal{X}$ for which $x \le z$ and $y \ge w$. Also assume either
$(i)F$ is continuous or
$(ii)\mathcal{X}$ has the following properties:

(a)if $\{x_n\}$is a non-decreasing sequence such that $x_n \to x$, then $x_n \le x$, for all n,

(b)if $\{y_n\}$is a non-decreasing sequence such that $y_n \to y$, then $y \le y_n$, for all n.

If there exist $x_0, y_0 \in \mathcal{X}$ such that

$$x_0 \le F(x_0,y_0) \quad \text{and} \quad y_0 \ge F(y_0,x_0),$$

then there exist $x, y \in \mathcal{X}$ such that

$$x = F(x,y) \quad \text{and} \quad y = F(y,x).$$

Moreover, if x_0, y_0 are comparable, then $x = y$, so $x = F(x,x)$.

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