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Character amenability of real Banach algebras

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Abstract

Let $(A, \|\cdot\|)$ be a real Banach algebra. In this paper we first introduce left and right φ-amenability of A and discuss the relation between left (right, respectively) φ-amenability and $\overline{\varphi}$-amenability of A for $\varphi \in \triangle(A) \cup \{0\}$ where $\overline{\varphi} \in \triangle(A)$ is the conjugate of φ. Next we show that A is left (right, respectively) φ-amenable if and only if $A_{\mathbb{C}}$ is left (right, respectively) $\varphi_{\mathbb{C}}$-amenable, where $A_{\mathbb{C}}$ is a suitable complexification of A and $\varphi_{\mathbb{C}} \in \triangle(A_{\mathbb{C}})$ is the induced character by φ on $A_{\mathbb{C}}$. In continue, we give a hereditary property for 0-amenability of A. We also study relations between the injectivity of Banach left A-modules and right φ-amenability of A. Finally, we characterize the left character amenability of certain real Banach algebras.

Keywords: Banach algebra, Character amenable, Complexification, Banach left module, injectivity.
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1. Introduction and preliminaries

The symbol $\mathbb{F}$ denotes a field that can be $\mathbb{R}$ or $\mathbb{C}$. For a Banach space $(\mathfrak{X}, \|\cdot\|)$ over $\mathbb{F}$, we denote by $\mathfrak{X}^*$ the dual space of $\mathfrak{X}$. Let A be an algebra and $\mathfrak{X}$ be an A-bimodule over $\mathbb{F}$ with the left module action $(a, x) \longmapsto a \cdot x : A \times \mathfrak{X} \longrightarrow \mathfrak{X}$ and the right module action $(a, x) \longmapsto x \cdot a : A \times \mathfrak{X} \longrightarrow \mathfrak{X}$. A linear map $D : A \longrightarrow \mathfrak{X}$ over $\mathbb{F}$ is called an $\mathfrak{X}$-*derivation* on A if $D(ab) = D(a) \cdot b + a \cdot D(b)$ for all $a, b \in A$. For each $x \in \mathfrak{X}$, the map $d_{A,\mathfrak{X},x} : A \longrightarrow \mathfrak{X}$ defined by $d_{A,\mathfrak{X},x}(a) = a \cdot x - x \cdot a \quad (a \in A)$, is an $\mathfrak{X}$-derivation on A over $\mathbb{F}$. An $\mathfrak{X}$-derivation D on A over $\mathbb{F}$ is called *inner* if $D = d_{A,\mathfrak{X},x}$ for some $x \in \mathfrak{X}$.

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Let $(A, \|\cdot\|)$ be a Banach algebra over $\mathbb{F}$. An A-bimodule $\mathfrak{X}$ over $\mathbb{F}$ is called a *Banach A-bimodule* if $\mathfrak{X}$ is a Banach space with a norm $\|\cdot\|$ and there exists a positive constant k such that

$$\|a\cdot x\| \leq k\|a\|\|x\|, \quad \|x\cdot a\| \leq k\|a\|\|x\|,$$

for all $a \in A$ and $x \in \mathfrak{X}$. Let $\mathfrak{X}$ be a Banach A-bimodule over $\mathbb{F}$ with the module operations $(a, x) \mapsto a\cdot x$, $(a, x) \mapsto x\cdot a : A\times\mathfrak{X} \longrightarrow \mathfrak{X}$. Then $\mathfrak{X}^*$ is a Banach A-module over $\mathbb{F}$ with the natural module operations $(\lambda, a) \longmapsto a\cdot\lambda$, $(\lambda, a) \longmapsto \lambda\cdot a : A\times\mathfrak{X}^* \longrightarrow \mathfrak{X}^*$ given by

$$(a\cdot\lambda)(x) = \lambda(x\cdot a), \quad (\lambda\cdot a)(x) = \lambda(a\cdot x) \qquad (a \in A, \lambda \in \mathfrak{X}^*, x \in \mathfrak{X}),$$

and with the operator norm $\|\cdot\|_{op}$. We denote by $Z^1_{\mathbb{F}}(A, \mathfrak{X})$ the set of all continuous $\mathfrak{X}$-derivations on A over $\mathbb{F}$. It is known that $Z^1_{\mathbb{F}}(A, \mathfrak{X})$ is a linear subspace of $\mathcal{B}_{\mathbb{F}}(A, \mathfrak{X})$, the linear space of all bounded linear operators from A to $\mathfrak{X}$ over $\mathbb{F}$. We denote by $N^1_{\mathbb{F}}(A, \mathfrak{X})$ the set of all inner $\mathfrak{X}$-derivations on A over $\mathbb{F}$. Clearly, $N^1_{\mathbb{F}}(A, \mathfrak{X})$ is a linear subspace of $Z^1_{\mathbb{F}}(A, \mathfrak{X})$ over $\mathbb{F}$. We denote by $H^1_{\mathbb{F}}(A, \mathfrak{X})$ the quotient space $Z^1_{\mathbb{F}}(A, \mathfrak{X})/N^1_{\mathbb{F}}(A, \mathfrak{X})$ which is called the *first cohomology group* of A over $\mathbb{F}$ with the coefficients in $\mathfrak{X}$.

A Banach algebra A over $\mathbb{F}$ is called *amenable* if $H^1_{\mathbb{F}}(A, \mathfrak{X}^*) = \{0\}$ for all Banach A-bimodule $\mathfrak{X}$ over $\mathbb{F}$.

Let A be a Banach algebra over $\mathbb{F}$ and let $\varphi : A \longrightarrow \mathbb{C}$ be an algebra homomorphism from A to $\mathbb{C}$ over $\mathbb{F}$. We say that φ is a *character* of A (the zero homomorphism from A to $\mathbb{C}$, respectively) if $\varphi(a_0) \neq 0$ for some $a_0 \in A$ ($\varphi(a) = 0$ for all $a \in A$, respectively). The zero homomorphism from A to $\mathbb{C}$ is denoted by 0. We denote by $\triangle(A)$ the set of all characters of A. It is known that $\triangle(A)$ is a subset of $\mathcal{B}_{\mathbb{F}}(A, \mathbb{C})$. If A is a commutative Banach algebra with identity over $\mathbb{F}$, then $\triangle(A)$ is nonempty. It is not true whenever A is noncommutative. For example $\mathcal{H}$, the set of all quaternion numbers, is a real noncommutative Banach algebra with identity but $\triangle(\mathcal{H}) = \emptyset$ (see [16, Page 20]). Note that it is possible $\triangle(A) = \emptyset$ wherever A has not the identity (see [14, Examples 2.1.6 and 2.1.7]). If A is a real Banach algebra, then $\varphi \in \triangle(A)$ if and only if $\overline{\varphi} \in \triangle(A)$, where $\overline{\varphi} : A \longrightarrow \mathbb{C}$ is defined by $\overline{\varphi}(a) = \overline{\varphi(a)}$ $(a \in A)$.

Let A be a Banach algebra over $\mathbb{F}$ and $\varphi \in \triangle(A) \cup \{0\}$. We denote by $\mathcal{M}^r_{\mathbb{F}}(A, \varphi)$ ($\mathcal{M}^l_{\mathbb{F}}(A, \varphi)$, respectively) the collection of all complex Banach space $\mathfrak{X}$ for which $\mathfrak{X}$ is a Banach A-bimodule over $\mathbb{F}$ with the right (left, respectively) module action defined by $x\cdot a = \varphi(a)x$ ($a\cdot x = \varphi(a)x$, respectively) for all $(a, x) \in A\times\mathfrak{X}$.

Definition 1.1. Let $(B, \|\cdot\|)$ be a complex Banach algebra and let $\varphi \in \triangle(B) \cup \{0\}$. Then B is called *left* (*right*, respectively) *φ-amenable* if $H^1_{\mathbb{C}}(B, \mathfrak{X}^*) = \{0\}$ for all $\mathfrak{X} \in \mathcal{M}^l_{\mathbb{C}}(B, \varphi)$ ($\mathfrak{X} \in \mathcal{M}^r_{\mathbb{C}}(B, \varphi)$, respectively).

The concepts of left and right φ-amenability of complex Banach algebras were first introduced by Hu, Sangani Monfared and Traynor in [11] which is modified by Nasr-Isfahani and Soltani in [19] as the definition above.

Let $(A, \|\cdot\|)$ be a real Banach algebra and $\varphi \in \triangle(A)\cup\{0\}$. It is easy to see that if $\mathfrak{X} \in \mathcal{M}^r_{\mathbb{R}}(A, \varphi)$ satisfying $i(a\cdot x) = a\cdot(ix)$ for all $(a, x) \in A\times\mathfrak{X}$ ($\mathfrak{X} \in \mathcal{M}^l_{\mathbb{R}}(A, \varphi)$ satisfying $i(x\cdot a) = (ix)\cdot a$ for all $(a, x) \in A\times\mathfrak{X}$, respectively), then $\mathfrak{X}^* \in \mathcal{M}^l_{\mathbb{R}}(A, \varphi)$ and $i(f\cdot a) = (if)\cdot a$ holds for all $(f, a) \in \mathfrak{X}^*\times A$ ($\mathfrak{X}^* \in \mathcal{M}^r_{\mathbb{R}}(A, \varphi)$and $i(a\cdot f) = a\cdot(if)$ holds for all $(a, f) \in A\times\mathfrak{X}^*$, respectively), where $i = \sqrt{-1}$. We now introduce the left and right φ-amenability for real Banach algebras A as the following.

Definition 1.2. Let A be a real Banach algebra and let $\varphi \in \triangle(A)\cup\{0\}$. We say that A is *left* (*right*, respectively) *φ-amenable* if $H^1_{\mathbb{R}}(A, \mathfrak{X}^*) = \{0\}$ for all $\mathfrak{X} \in \mathcal{M}^l_{\mathbb{R}}(A, \varphi)$ ($\mathfrak{X} \in \mathcal{M}^r_{\mathbb{R}}(A, \varphi)$, respectively) satisfying

$$i(x\cdot a) = (ix)\cdot a \;\; (i(a\cdot x) = a\cdot(ix), \text{respectively}),$$

for all $(a, x) \in A \times \mathfrak{X}$.

Definition 1.3. Let A be a Banach algebra over $\mathbb{F}$.

(i) For $\varphi \in \triangle(A) \cup \{0\}$, we say that A is *φ-amenable* if A is left and right φ-amenable.
(ii) A is called *left* (*right*, respectively) *character amenable* if A is left (right, respectively) φ-amenable for all $\varphi \in \triangle(A) \cup \{0\}$.
(iii) A is called *character amenable* if A is left and right character amenable.

Let E be a real linear space (real algebra, respectively). A complex linear space (complex algebra, respectively) $E_{\mathbb{C}}$ is called a *complexification* of E if there exists an injective real linear mapping (a real algebra homomorphism, respectively) $J : E \longrightarrow E_{\mathbb{C}}$ such that $E_{\mathbb{C}} = J(E) \oplus iJ(E)$.

If $\mathfrak{X}$ is a real linear space, then $\mathfrak{X} \times \mathfrak{X}$ with the additive operation and scalar multiplication defined by

$$\begin{aligned} &(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \qquad (x_1, x_2, y_1, y_2 \in \mathfrak{X}), \\ &(\alpha + i\beta)(x, y) = (\alpha x - \beta y, \alpha y + \beta x) \qquad (\alpha, \beta \in \mathbb{R}, x, y \in \mathfrak{X}), \end{aligned} \tag{1.1}$$

is a complexification of $\mathfrak{X}$ with respect to the injective linear map $J : \mathfrak{X} \longrightarrow \mathfrak{X} \times \mathfrak{X}$ defined by $J(x) = (x, 0)$, $x \in \mathfrak{X}$.

If A is a real algebra, then $A \times A$ with the algebra operations

$$\begin{aligned} &(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2) \qquad (a_1, a_2, b_1, b_2 \in A), \\ &(\alpha + i\beta)(a, b) = (\alpha a - \beta b, \alpha b + \beta a) \qquad (\alpha, \beta \in \mathbb{R}, a, b \in A), \\ &(a_1, b_1)(a_2, b_2) = (a_1 a_2 - b_1 b_2, a_1 b_2 + b_1 a_2) \qquad (a_1, b_1, a_2, b_2 \in A), \end{aligned} \tag{1.2}$$

is a complexification of A with the algebra homomorphism $J : A \longrightarrow A \times A$ defined by $J(a) = (a, 0)$, $a \in A$.

Let $(E, \|\cdot\|)$ be a real normed linear space (algebra, respectively), $E_{\mathbb{C}}$ be a complexification of E with respect to an injective real linear mapping (algebra homomorphism, respectively) $J : E \longrightarrow E_{\mathbb{C}}$ and $|||\cdot|||$ be a norm (an algebra norm, respectively) on $E_{\mathbb{C}}$. We say that $|||\cdot|||$ satisfies in the $(*)$ condition if there exist positive constants k_1 and k_2 such that

$$\max\{\|a\|, \|b\|\} \leq k_1 |||J(a) + iJ(b)||| \leq k_2 \max\{\|a\|, \|b\|\},$$

for all $a, b \in E$. By [5, Proposition I.1.13], there exists a norm (an algebra norm) $|||\cdot|||$ on $E_{\mathbb{C}}$ satisfying in the $(*)$ condition with $k_1 = 1$ and $k_2 = 2$ where $E_{\mathbb{C}} = E \times E$ and $J : E \longrightarrow E_{\mathbb{C}}$ is defined by $J(a) = (a, 0)$, $a \in E$. Note that the $(*)$ condition implies that $(E, \|\cdot\|)$ is a real Banach space (a real Banach algebra, respectively) if and only if $(E_{\mathbb{C}}, |||\cdot|||)$ is a complex Banach space (a complex Banach algebra, respectively).

Let $(A, \|\cdot\|)$ be a real Banach algebra, $A_{\mathbb{C}}$ be a complexification of A with respect to an injective real algebra homomorphism $J : A \longrightarrow A_{\mathbb{C}}$ and $|||\cdot|||$ be an algebra norm on $A_{\mathbb{C}}$ satisfying in the $(*)$ condition. It is known [3, Theorem 2.4] that A is amenable if and only if $A_{\mathbb{C}}$ is amenable. In Section 2, we prove that A is left (right, respectively) φ-amenable if and only if A is $\overline{\varphi}$-amenable, whenever $\varphi \in \triangle(A)$. Moreover, we give a characterization of left and right φ-amenability of A whenever $\varphi \in \triangle(A)$ with $\overline{\varphi} = \varphi$. In Section 3, we show that A is right character (right character, character) amenable if and only if $A_{\mathbb{C}}$ is left character (right character, character) amenable, respectively. In Section 4, we give a characterization of the left (right, respectively) 0-amenability of A. In Section

5, we show that if $\varphi \in \triangle(A)$ and $\mathfrak{X}$ is a complex Banach space, then A is left φ-amenable if and only if the real left Banach A-module $\mathfrak{X}$, with the left module action $a \cdot x = \varphi(a)x \quad ((a, x) \in A \times \mathfrak{X})$, is injective. In Section 6, for a complex Banach algebra B we assume that $B_{\mathbb{R}}$ is B regarded as a real Banach algebra and show that $B_{\mathbb{R}}$ is right character amenable if and only if B is right character amenable. In Section 7, applying certain known results for left and right character amenability of complex Banach algebras and some obtained results in Sections 2-6, we give some results for the left and right character amenability of certain real Banach algebras.

2. φ-amenability and $\overline{\varphi}$-amenability

We first investigate the relation between φ-amenability and $\overline{\varphi}$-amenability for a real Banach algebra A, where $\varphi \in \triangle(A)$.

Theorem 2.1. *Let $(A, \|\cdot\|)$ be a real Banach algebra with $\triangle(A) \neq \emptyset$ and let $\varphi \in \triangle(A)$. Then the following assertions hold.*

(i) *A is left φ-amenable if and only if A is left $\overline{\varphi}$-amenable.*
(ii) *A is right φ-amenable if and only if A is right $\overline{\varphi}$-amenable.*
(iii) *A is φ-amenable if and only if A is $\overline{\varphi}$-amenable.*

Proof . (i) We first assume that A is left φ-amenable. Let $\mathfrak{X} \in \mathcal{M}^l_{\mathbb{R}}(A, \overline{\varphi})$ with the norm $\|\cdot\|$ such that $i(x \cdot a) = (ix) \cdot a$ for all $(a, x) \in A \times \mathfrak{X}$. Let $\underline{\mathfrak{X}}$ denote $\mathfrak{X}$ with the scalar multiplication $(\alpha, x) \longmapsto \alpha * x : \underline{\mathfrak{X}} \times \mathbb{C} \longrightarrow \underline{\mathfrak{X}}$ defined by

$$\alpha * x = \bar{\alpha} x \quad (\alpha \in \mathbb{C}, x \in \underline{\mathfrak{X}}).$$

It is easy to see that $\underline{\mathfrak{X}}$ is a complex Banach space with the norm $\|\cdot\|$ and a real Banach A-bimodule with the module actions $(a, x) \longmapsto a \odot x : A \times \underline{\mathfrak{X}} \longrightarrow \underline{\mathfrak{X}}$ and $(a, x) \longmapsto x \odot a : A \times \underline{\mathfrak{X}} \longrightarrow \underline{\mathfrak{X}}$ defined by

$$\begin{aligned} a \odot x &= \varphi(a) * x = \overline{\varphi}(a)x \quad (a \in A, x \in \underline{\mathfrak{X}}), \\ x \odot a &= x \cdot a \quad (x \in \underline{\mathfrak{X}}, a \in A). \end{aligned}$$

Hence, $\underline{\mathfrak{X}} \in \mathcal{M}^l_{\mathbb{R}}(A, \varphi)$. Moreover, for each $(a, x) \in (A \times \underline{\mathfrak{X}})$ we have

$$\begin{aligned} i * (x \odot a) &= \bar{i}(x \odot a) = -i(x \odot a) = -((ix) \cdot a) \\ &= (-ix) \cdot a = (\bar{i}x) \cdot a = (i * x) \odot a. \end{aligned}$$

It is easy to see that $(\underline{\mathfrak{X}})^* = \{\overline{f} : \quad f \in \mathfrak{X}^*\}$. Moreover, one can show that

$$\overline{f \cdot a} = \overline{f} \odot a, \quad \overline{a \cdot f} = a \odot \overline{f} \quad (a \in A, f \in \mathfrak{X}^*). \tag{2.1}$$

Since A is left φ-amenable, we deduce that

$$H^1_{\mathbb{R}}(A, (\underline{\mathfrak{X}})^*) = \{0\}. \tag{2.2}$$

Assume that $d \in Z^1_{\mathbb{R}}(A, \mathfrak{X}^*)$. Define the map $\underline{d} : A \longrightarrow (\underline{\mathfrak{X}})^*$ by

$$\underline{d}(a) = \overline{d(a)} \quad (a \in A). \tag{2.3}$$

It is easy to see that $\underline{d}$ is a bounded real linear operator from A to $(\underline{\mathfrak{X}})^*$ and $\|\underline{d}\| = \|d\|$. Moreover, by (2.3) and (2.1) we have

$$\begin{aligned}\underline{d}(ab) &= \overline{d(ab)} = \overline{d(a)\cdot b + a\cdot d(b)} = \overline{d(a)\cdot b} + \overline{a\cdot d(b)} \\ &= \overline{d(a)} \odot b + a \odot \overline{d(b)} = \underline{d}(a)\odot b + a\odot \underline{d}(b),\end{aligned}$$

for all $a, b \in A$. Hence, $\underline{d} \in Z^1_{\mathbb{R}}(A, (\underline{\mathfrak{X}})^*)$ and so, by (2.2), there exists $g \in (\underline{\mathfrak{X}})^*$ such that

$$\underline{d} = d_{A,(\underline{\mathfrak{X}})^*,g}. \tag{2.4}$$

Applying (2.4), for each $a \in A$ we get

$$\begin{aligned}d(a) &= \overline{\underline{d}(a)} = \overline{d_{A,(\underline{\mathfrak{X}})^*,g}}(a) = \overline{a\odot g - g\odot a} \\ &= \overline{a\odot g} - \overline{g\odot a} = a\cdot \overline{g} - \overline{g}\cdot a = d_{A,\mathfrak{X}^*,\overline{g}}(a).\end{aligned}$$

Hence, $d = d_{A,\mathfrak{X}^*,\overline{g}}$. Therefore, $H^1_{\mathbb{R}}(A, \mathfrak{X}^*) = \{0\}$ and so A is left $\overline{\varphi}$-amenable.

We now assume that A is left $\overline{\varphi}$-amenable. By the necessity part, A is left $\overline{\overline{\varphi}}$-amenable, that is, A is left φ-amenable. Hence, (i) holds.

(ii) It follows similar to (i).

(iii) This follows from (i) and (ii). □

We now characterize the φ-amenability of a real Banach algebra A, where $\varphi \in \triangle(A)$ with $\overline{\varphi} = \varphi$.

Theorem 2.2. *Let $(A, \|\cdot\|)$ be a real Banach algebra with $\triangle(A) \neq \emptyset$ and let $\varphi \in \triangle(A)$ with $\overline{\varphi} = \varphi$. Then the following assertions are equivalent.*

(i) *A is left φ-amenable.*

(ii) *$H^1_{\mathbb{R}}(A, \mathfrak{X}^*) = \{0\}$ for each real Banach A-bimodule $\mathfrak{X}$ with the left module action $a \cdot x = \varphi(a)x, (a, x) \in A \times \mathfrak{X}$.*

(iii) *There is an element $m \in A^{**}$ such that $m(\varphi) = 1$ and $m(f.a) = \varphi(a)m(f)$ for all $a \in A$ and $f \in A^*$.*

Proof . (i)⇒(ii) Let $(\mathfrak{X}, \|\cdot\|)$ be a real Banach A-bimodule with the left module actions defined by $a \cdot x = \varphi(a)x \quad (x \in \mathfrak{X}, \quad a \in A)$. Set $\mathfrak{X}_{\mathbb{C}} = \mathfrak{X} \times \mathfrak{X}$. Then $\mathfrak{X}_{\mathbb{C}}$ is a complex linear space with the additive and scalar multiplication defined by (1.1). Moreover, $\mathfrak{X}_{\mathbb{C}}$ is a complexification of $\mathfrak{X}$ with the injective real linear mapping $J : \mathfrak{X} \longrightarrow \mathfrak{X}_{\mathbb{C}}$ defined by $J(x) = (x, 0)$, It is known that there exists a norm $|||\cdot|||$ on $\mathfrak{X}_{\mathbb{C}}$ satisfying in the $(*)$ condition with the positive constant $k_1 = 1$ and $k_2 = 2$. Hence, $(\mathfrak{X}_{\mathbb{C}}, |||\cdot|||)$ is a complex Banach space. It is easy to see that $\mathfrak{X}_{\mathbb{C}}$ is a real Banach A-bimodule with the module actions $(a, (x, y)) \longmapsto a(x, y) : A \times \mathfrak{X}_{\mathbb{C}} \longrightarrow \mathfrak{X}_{\mathbb{C}}$ and $(a, (x, y)) \longmapsto (x, y)a : A \times \mathfrak{X}_{\mathbb{C}} \longrightarrow \mathfrak{X}_{\mathbb{C}}$ defined by

$$\begin{aligned}a(x, y) &= (a\cdot x, a\cdot y) \quad (a \in A, \quad (x, y) \in \mathfrak{X}_{\mathbb{C}}),\\ (x, y)a &= (x\cdot a, y\cdot a) \quad ((x, y) \in \mathfrak{X}_{\mathbb{C}}, \quad a \in A).\end{aligned}$$

On the other hand, for all $(a, (x, y)) \in A \times \mathfrak{X}_{\mathbb{C}}$ we have

$$\begin{aligned}i((x, y)a) &= i(x\cdot a, y\cdot a) = (-(y\cdot a), x\cdot a)\\ &= (-y\cdot a, x\cdot a) = (-y, x)a\\ &= i(x, y)a.\end{aligned}$$

Since φ is real-valued, for each $(a,(x,y)) \in A \times \mathfrak{X}_{\mathbb{C}}$ we have

$$a \odot (x,y) = (a \cdot x, a \cdot y) = (\varphi(a)x, \varphi(a)y) = \varphi(a)(x,y).$$

Therefore, $\mathfrak{X}_{\mathbb{C}} \in \mathcal{M}^l_{\mathbb{R}}(A,\varphi)$ and so by (i) we have

$$H^1_{\mathbb{R}}(A,(\mathfrak{X}_{\mathbb{C}})^*) = \{0\}. \tag{2.5}$$

Assume that $d \in Z^1_{\mathbb{R}}(A,\mathfrak{X}^*)$. Define the map $D : A \longrightarrow (\mathfrak{X}_{\mathbb{C}})^*$ by

$$D(a)(x,y) = d(a)(x) + id(a)(y) \quad (a \in A, \quad (x,y) \in \mathfrak{X}_{\mathbb{C}}).$$

It is easy to see that D is a real linear mapping from A to $(\mathfrak{X}_{\mathbb{C}})^*$. Let $a,b \in A$, since for each $(x,y) \in \mathfrak{X}_{\mathbb{C}}$ we have

$$\begin{aligned} D(ab)(x,y) &= d(ab)(x) + id(ab)(y) \\ &= (d(a)\cdot b + a \cdot d(b))(x) + i(d(a)\cdot b + a\cdot d(b))(y) \\ &= [d(a)(b\cdot x) + id(a)(b \cdot y)] + [d(b)(x\cdot a) + id(b)(y \cdot a)] \\ &= D(a)(b\cdot x, b\cdot y) + D(b)(x\cdot a, y \cdot a) \\ &= D(a)(b(x,y)) + D(b)((x,y)a) \\ &= (D(a)b)(x,y) + (aD(b))(x,y) \\ &= (D(a)b + aD(b))(x,y), \end{aligned}$$

we deduce that $D(ab) = D(a)b + aD(b)$. Therefore, D is an $(\mathfrak{X}_{\mathbb{C}})^*$-derivation on A over $\mathbb{R}$. On the other hand, $\|D(a)\| \le 2\|d\|\|a\|$ for all $a \in A$. Hence, D is bounded and $\|D\| \le 2\|d\|$. Therefore, $D \in Z^1_{\mathbb{R}}(A,(\mathfrak{X}_{\mathbb{C}})^*)$ and so, by (2.5), there exists $f \in (\mathfrak{X}_{\mathbb{C}})^*$ such that

$$D = d_{A,(\mathfrak{X}_{\mathbb{C}})^*,f}. \tag{2.6}$$

Define the function $\lambda : \mathfrak{X} \longrightarrow \mathbb{R}$ by

$$\lambda(x) = \mathrm{Re}\ f(x,0) \quad (x \in \mathfrak{X}).$$

Clearly, $\lambda \in \mathfrak{X}^*$. Let $a \in A$. Since $d(a)(x) \in \mathbb{R}$ for all $x \in \mathfrak{X}$, we have

$$\begin{aligned} d(a)(x) &= \mathrm{Re}\ d(a)(x) = \mathrm{Re}\ d_{A,(\mathfrak{X}_{\mathbb{C}})^*,f}(a)(x,0) \\ &= \mathrm{Re}\ (af(x,0) - fa(x,0) = \mathrm{Re}\ (f((x,0)a) - f(a(x,0)) \\ &= \mathrm{Re}\ (f(x\cdot a,0) - f(a\cdot x,0)) = \mathrm{Re}\ f(x\cdot a,0) - \mathrm{Re}\ f(a\cdot x,0) \\ &= \lambda(x\cdot a) - \lambda(a\cdot x) = a\cdot\lambda(x) - \lambda\cdot a(x) \\ &= (a\cdot\lambda - \lambda\cdot a)(x) = d_{A,\mathfrak{X}^*,\lambda}(a)(x), \end{aligned}$$

for all $x \in \mathfrak{X}$. Hence, $d(a) = d_{A,\mathfrak{X}^*,\lambda}(a)$. Since this equality holds for all $a \in A$, we deduce that $d = d_{A,\mathfrak{X}^*,\lambda}$. Hence, $H^1_{\mathbb{R}}(A,\mathfrak{X}^*) = \{0\}$ and so (ii) holds.

(ii)⇒(iii) Clearly, A^* is a real Banach A-bimodule with the module actions defined by

$$\begin{aligned} &f\cdot a(b) = f(ab) \quad (f \in A^*, \quad a,b \in A), \\ &a\cdot f(b) = \varphi(a)f(b) \quad (a \in A, \quad f \in A^*, \quad b \in A). \end{aligned}$$

Since φ is real-valued, we deduce that $\varphi \in A^*$. Set $M = \{r\varphi : \quad r \in \mathbb{R}\}$. Then M is a closed real subspace of A^*. Let $a \in A$. Then $a \cdot \varphi = \varphi(a)\varphi \in M$. Since for each $b \in A$ we have

$$\varphi \cdot a(b) = \varphi(ab) = \varphi(a)\varphi(b) = (\varphi(a)\varphi)(b),$$

we deduce that $\varphi \cdot a = \varphi(a)\varphi$ and so $\varphi \cdot a \in M$. Therefore, M is a closed A-submodule of A^*. Set $\mathfrak{X} = A^*/M$. It is easy to see that $\mathfrak{X}$ is a real Banach A-bimodule with the module actions

$$\begin{aligned} a \cdot (f + M) &= (a \cdot f) + M \quad (a \in A, \quad f \in A^*) \\ (f + M) \cdot a &= (f \cdot a) + M \quad (a \in A, \quad f \in A^*). \end{aligned}$$

Moreover, for each $a \in A$ and $f \in A^*$ we have

$$a \cdot (f + M) = (a \cdot f) + M = \varphi(a)f + M = \varphi(a)(f + M). \tag{2.7}$$

Define the map $\pi : A^* \longrightarrow \mathfrak{X}$ by

$$\pi(f) = f + M \quad (f \in A^*).$$

Then π is a surjective continuous linear mapping. Moreover, π is module homomorphism. Hence, $\pi^* : \mathfrak{X}^* \longrightarrow A^{**}$, the adjoint of π, is a injective linear operator. Moreover, π^* is module homomorphism. Since $\varphi \in A^* \setminus \{0\}$, there exist $\nu \in A^{**}$ with $\nu(\varphi) = 1$. Define the map $d : A \longrightarrow A^{**}$ with $d = d_{A,A^{**},\nu}$. We claim that for each $a \in A$ there exists a unique $\Lambda_a \in \mathfrak{X}^*$ such that $\pi^*(\Lambda_a) = d(a)$. Let $a \in A$. Then

$$\begin{aligned} d(a)(\varphi) &= (a \cdot \nu - \nu \cdot a)(\varphi) = (a \cdot \nu)(\varphi) - (\nu \cdot a)(\varphi) \\ &= \nu(\varphi \cdot a) - \nu(a \cdot \varphi) = \nu(\varphi \cdot a - a \cdot \varphi) \\ &= \nu(\varphi(a)\varphi - \varphi(a)\varphi) = \nu(0) = 0. \end{aligned}$$

This implies that $d(a)(M) = \{0\}$ and so $M \subseteq \ker(d(a))$. Define the function $\Lambda_a : \mathfrak{X} \longrightarrow \mathbb{R}$ by

$$\Lambda_a(f + M) = d(a)(f) \quad (f \in A^*).$$

Then, Λ_a is well-defined since $M \subseteq \ker(d(a))$. It is easy to see that $\Lambda_a \in \mathfrak{X}^*$.On the other hand, for each $f \in A^*$ we have

$$\pi^*(\Lambda_a)(f) = \Lambda_a \circ \pi(f) = \Lambda_a(\pi(f)) = \Lambda_a(f + M) = d(a)(f).$$

Hence, $\pi^*(\Lambda_a) = d(a)$. The injectivity of π^* implies that Λ_a is unique. Hence, our claim is justified. Now define the map $D : A \longrightarrow \mathfrak{X}^*$ by $D(a) = \Lambda_a$ for all $a \in A$. It is easy to see that D is a real linear operator. The surjectivity of π implies that there exist a $\delta > 0$ such that $\|\pi^*(x^*)\| \geq \delta\|x^*\|$ for all $x^* \in \mathfrak{X}^*$. Hence, for each $a \in A$, we have

$$\|D(a)\| = \|\Lambda_a\| \leq \frac{1}{\delta}\|\pi^*(\Lambda_a)\| = \frac{1}{\delta}\|d(a)\| \leq \frac{1}{\delta}\|d\|\|a\|.$$

Therefore, D is continuous. Since π^* is a module homomorphism from $\mathfrak{X}^*$ to A^{**}, we deduce that

$$\begin{aligned} \pi^*(D(ab)) &= d(ab) = d(a) \cdot b + a \cdot d(b) \\ &= \pi^*(\Lambda_a) \cdot b + a \cdot \pi^*(\Lambda_b) = \pi^*(\Lambda_a \cdot b) + \pi^*(a \cdot \Lambda_b) \\ &= \pi^*(\Lambda_a \cdot b + a \cdot \Lambda_b) = \pi^*(D(a) \cdot b + a \cdot D(b)), \end{aligned}$$

for all $a, b \in A$. The injectivity of π^* implies that $D(ab) = D(a) \cdot b + a \cdot D(b)$ for all $a, b \in A$. Therefore, $D \in Z^1_{\mathbb{R}}(A, \mathfrak{X}^*)$. Since the left module action of A on $\mathfrak{X}^*$ is given by (2.7), we deduce that $H^1_{\mathbb{R}}(A, \mathfrak{X}^*) = \{0\}$. Thus, there exists $\lambda \in \mathfrak{X}^*$ such that $D = d_{A,\mathfrak{X}^*,\lambda}$. This implies that

$$a \cdot \pi^*(\lambda) - \pi^*(\lambda) \cdot a = \pi^*(a \cdot \lambda - \lambda \cdot a) = \pi^*(D(a)) = d(a) = a \cdot \nu - \nu \cdot a, \tag{2.8}$$

for all $a \in A$. Take $m = \nu - \pi^*(\lambda)$. Then $m \in A^{**}$ and

$$\begin{aligned} m(\varphi) &= \nu(\varphi) - \pi^*(\lambda)(\varphi) = 1 - \lambda \circ \pi(\varphi) = 1 - \lambda(\pi(\varphi)) \\ &= 1 - \lambda(\varphi + M) = 1 - \lambda(M) = 1 - \lambda(0_{\mathfrak{X}}) = 1. \end{aligned}$$

On the other hand, by (2.8) for each $a \in A$ we have

$$\begin{aligned} a \cdot m &= a \cdot (\nu - \pi^*(\lambda)) = a \cdot \nu - a \cdot \pi^*(\lambda) \\ &= a \cdot \nu - (a \cdot \nu - \nu \cdot a) - \pi^*(\lambda) \cdot a = \nu \cdot a - \pi^*(\lambda) \cdot a \\ &= (\nu - \pi^*(\lambda)) \cdot a = m \cdot a. \end{aligned}$$

Therefore,

$$m(f \cdot a) = a \cdot m(f) = m \cdot a(f) = m(a \cdot f) = m(\varphi(a)f) = \varphi(a)m(f),$$

for all $a \in A$ and $f \in A^*$. Hence, (iii) holds.

(iii) $\Rightarrow$ (i) Let $\mathfrak{X} \in \mathcal{M}^l_{\mathbb{R}}(A, \varphi)$ and $d \in Z^1_{\mathbb{R}}(A, \mathfrak{X}^*)$. Let $x \in \mathfrak{X}$. Define the map $d_x : A \longrightarrow \mathbb{R}$ by

$$d_x(a) = \text{Re } d(a)(x) \quad (a \in A).$$

Clearly, d_x is a real linear functional on A and

$$|d_x(a)| = |\text{Re } d(a)(x)| \leq |d(a)(x)| \leq \|d(a)\|\|x\| \leq \|d\|\|a\|\|x\|,$$

for all $a \in A$. Therefore, $d_x \in A^*$ and $\|d_x\| \leq \|d\|\|x\|$. We now define the map $D : \mathfrak{X} \longrightarrow A^*$ by

$$D(x) = d_x \quad (x \in \mathfrak{X}).$$

Suppose that $x, y \in \mathfrak{X}$ with $d_x \neq d_y$. Then there exist $a \in A$ such that $d_x(a) \neq d_y(a)$, i.e, Re $d(a)(x) \neq$ Re $d(a)(y)$. This implies that $x \neq y$. Therefore, D is well-defined. It is easy to see that D is a real linear mapping. On the other hand,

$$\|D(x)\| = \|d_x\| \leq \|d\|\|x\|,$$

for all $x \in \mathfrak{X}$. Thus, D is bounded and $\|D\| \leq \|d\|$. According to $\varphi = \overline{\varphi}$ and $a \cdot x = \varphi(a)x$ for all $(a, x) \in A \times \mathfrak{X}$, we deduce that

$$D(a \cdot x) = \varphi(a)D(x) \quad (a \in A, \quad x \in \mathfrak{X}). \tag{2.9}$$

Since $\mathfrak{X} \in \mathcal{M}^l_{\mathbb{R}}(A, \varphi)$, $\varphi = \overline{\varphi}$ and $d \in Z^1_{\mathbb{R}}(A, \mathfrak{X}^*)$, for each $(a, x) \in A \times \mathfrak{X}$ and every $b \in A$ we have

$$\begin{aligned} D(x \cdot a)(b) &= d_{x \cdot a}(b) = \text{Re } (a \cdot d(b))(x) \\ &= \text{Re } (d(ab) - d(a) \cdot b)(x) = \text{Re } d(ab)(x) - \text{Re } (d(a) \cdot b)(x) \\ &= d_x(ab) - \text{Re } d(a)(b \cdot x) = D(x)(ab) - \text{Re } d(a)(\varphi(b)(x) \\ &= D(x) \cdot a(b) - \varphi(b)\text{Re } d(a)(x) = (D(x) \cdot a)(b) - \varphi(b)d_x(a) \\ &= (D(x) \cdot a)(b) - D(x)(a)\varphi(b) = (D(x) \cdot a)(b) - (D(x)(a)\varphi)(b) \\ &= (D(x) \cdot a - D(x)(a)\varphi)(b). \end{aligned}$$

This implies that

$$D(x \cdot a) = D(x) \cdot a - D(x)(a)\varphi, \tag{2.10}$$

for all $(a, x) \in A \times \mathfrak{X}$. Assume that $\mathfrak{X}_{\mathbb{R}}$ denotes $\mathfrak{X}$ regarded as a real Banach space. Let $D^* : A^{**} \longrightarrow (\mathfrak{X}_{\mathbb{R}})^*$ be adjoint operator of D. Take $\lambda = D^*(m)$. Then $\lambda \in (\mathfrak{X}_{\mathbb{R}})^*$. Let $a \in A$ be given. By (2.9), we have

$$\begin{aligned}(\lambda \cdot a)(x) &= \lambda(a \cdot x) = D^*(m)(a \cdot x) = m(D(a \cdot x))\\ &= m(\varphi(a)D(x)) = \varphi(a)m(D(x)) = \varphi(a)D^*(m)(x)\\ &= \varphi(a)\lambda(x) = (\varphi(a)\lambda)(x)\end{aligned}$$

for all $x \in \mathfrak{X}$. This implies that

$$\lambda \cdot a = \varphi(a)\lambda. \tag{2.11}$$

By the definition of λ, (2.10), (iii) and (2.11) for each $x \in \mathfrak{X}$ we have

$$\begin{aligned}(a \cdot \lambda)(x) &= D^*(m)(x \cdot a) = m(D(x) \cdot a - D(x)(a)\varphi)\\ &= m(D(x) \cdot a) - D(x)(a)m(\varphi) = \varphi(a)m(D(x)) - \text{Re } d(a)(x)\\ &= \varphi(a)D^*(m)(x) - (\text{Re } d(a))(x) = \varphi(a)\lambda(x) - (\text{Re } d(a))(x)\\ &= (\varphi(a)\lambda - \text{Re } d(a))(x) = ((\lambda \cdot a) - \text{Re } d(a))(x).\end{aligned}$$

Therefore,

$$a \cdot \lambda = (\lambda \cdot a) - \text{Re } d(a). \tag{2.12}$$

Define the map $\Psi : \mathfrak{X}^* \longrightarrow (\mathfrak{X}_{\mathbb{R}})^*$ by

$$\Psi(\Gamma) = \text{Re } \Gamma \quad (\Gamma \in \mathfrak{X}^*).$$

It is known that Ψ is a surjective real linear isometry. The surjectivity of Ψ implies that there exist $\Lambda \in \mathfrak{X}^*$ such that

$$\lambda = \Psi(\Lambda). \tag{2.13}$$

By the definition of Ψ and (2.13), for each $x \in \mathfrak{X}$ we have

$$\begin{aligned}\Psi(a \cdot \Lambda)(x) &= (\text{Re } (a \cdot \Lambda))(x) = \text{Re } (a \cdot \Lambda)(x) = \text{Re } (\Lambda)(x \cdot a)\\ &= (\text{Re } (\Lambda))(x \cdot a) = \Psi(\Lambda)(x \cdot a) = \lambda(x \cdot a) = (a \cdot \lambda)(x).\end{aligned}$$

Therefore, $\Psi(a \cdot \Lambda) = a \cdot \lambda$. One can similary show that $\Psi(\Lambda \cdot a) = \lambda \cdot a$. Hence, by (2.12) we get

$$\begin{aligned}\Psi(d(a)) &= \text{Re } d(a) = \lambda \cdot a - a \cdot \lambda\\ &= \Psi(\Lambda \cdot a - a \cdot \Lambda) = \Psi(d_{A,\mathfrak{X}^*,-\Lambda}(a)).\end{aligned}$$

This implies that $d(a) = d_{A,\mathfrak{X}^*,-\Lambda}(a)$. Since a was arbitrary chosen, we deduce that $d = d_{A,\mathfrak{X}^*,-\Lambda}$. Therefore, $H^1_{\mathbb{R}}(A, \mathfrak{X}^*) = \{0\}$ and so A is left φ-amenable. Hence, (i) holds. □

Similarly, we obtain the following result.

Theorem 2.3. *Let $(A, \|\cdot\|)$ be a real Banach algebra with $\triangle(A) \neq \emptyset$ and let $\varphi \in \triangle(A)$ with $\overline{\varphi} = \varphi$. Then the following assertions are equivalent.*

(i) *A is right φ-amenable.*

(ii) *$H^1_{\mathbb{R}}(A, \mathfrak{X}^*) = \{0\}$ for each real Banach A-bimodule $\mathfrak{X}$ with the left module action $x \cdot a = \varphi(a)x, (a, x) \in A \times \mathfrak{X}$.*

(iii) *There is an element $m \in A^{**}$ such that $m(\varphi) = 1$ and $m(a \cdot f) = \varphi(a)m(f)$ for all $a \in A$ and $f \in A^*$.*

3. Character amenability of A and $A_{\mathbb{C}}$

Let $(A, \|\cdot\|)$ be a real Banach algebra, let $A_{\mathbb{C}}$ be a complexification of A with respect to an injective real algebra homomorphism $J : A \longrightarrow A_{\mathbb{C}}$ and let $|||\cdot|||$ be an algebra norm on $A_{\mathbb{C}}$ satisfying in the $(*)$ condition. For $\varphi \in \triangle(A) \cup \{0\}$, we define the map $\varphi_C : A_{\mathbb{C}} \longrightarrow \mathbb{C}$ by

$$\varphi_C(J(a) + iJ(b)) = \varphi(a) + i\varphi(b) \qquad (a, b \in A). \tag{3.1}$$

Clearly, $\varphi_C \in \triangle(A_{\mathbb{C}})$ if $\varphi \in \triangle(A)$ and $\varphi_C = 0$ if $\varphi = 0$. Moreover, the map $\Phi : \triangle(A) \cup \{0\} \longrightarrow \triangle(A_{\mathbb{C}}) \cup \{0\}$ defined by

$$\Phi(\varphi) = \varphi_C \quad (\varphi \in \triangle(A) \cup \{0\}), \tag{3.2}$$

is bijection and $\Phi(0) = 0$. For $\varphi \in \triangle(A)$, φ_C is called the character of $A_{\mathbb{C}}$ induced φ. Here, we show that character amenability of real Banach algebra $(A, \|\cdot\|)$ is equivalent to character amenability of complex Banach algebra $(A_{\mathbb{C}}, |||\cdot|||)$.

Theorem 3.1. *Let $(A, \|\cdot\|)$ be a real Banach algebra, let $A_{\mathbb{C}}$ be a complexification of A with respect to an injective real algebra homomorphism $J : A \longrightarrow A_{\mathbb{C}}$, and let $|||\cdot|||$ be an algebra norm on $A_{\mathbb{C}}$ satisfying in the $(*)$ condition. Then the followings hold.*

(i) *For $\varphi \in \triangle(A) \cup \{0\}$, A is left (right, respectively) φ-amenable if and only if $A_{\mathbb{C}}$ is left (right, respectively) φ_C-amenable.*

(ii) *A is left (right, respectively) character amenable if and only if $A_{\mathbb{C}}$ is left (right, respectively) character amenable.*

(iii) *A is character amenable if and only if $A_{\mathbb{C}}$ is character amenable.*

Proof . (i) Since the algebra norm $|||\cdot|||$ satisfies in the $(*)$ condition, there exist positive constants k_1 and k_2 such that

$$\max\{\|a\|, \|b\|\} \leq k_1 |||J(a) + iJ(b)||| \leq k_2 \max\{\|a\|, \|b\|\}, \tag{3.3}$$

for all $a, b \in A$. We first assume that $\varphi \in \triangle(A) \cup \{0\}$ and A is right φ-amenable. Let $\mathfrak{X} \in \mathcal{M}^r_{\mathbb{C}}(A_{\mathbb{C}}, \varphi_C)$ with the norm $\|\cdot\|$ and the module actions $(c, x) \longmapsto c \cdot x$ and $(c, x) \longmapsto x \cdot c$. It is easy to see that $\mathfrak{X}$ is a real A-bimodule with the module actions $(a, x) \longmapsto a \odot x$ and $(a, x) \longmapsto x \odot a$ defined by

$$a \odot x = J(a) \cdot x \quad (a \in A, x \in \mathfrak{X}), \tag{3.4}$$
$$x \odot a = x \cdot J(a) \quad (a \in A, x \in \mathfrak{X}). \tag{3.5}$$

Since $\mathfrak{X}$ is a Banach $A_{\mathbb{C}}$-module with the norm $\|\cdot\|$, there exists a positive constant k such that

$$\|(J(a) + iJ(b)) \cdot x\| \leq k |||J(a) + iJ(b)||| \;\; \|x\| \quad (a, b \in A, x \in \mathfrak{X}), \tag{3.6}$$
$$\|x \cdot (J(a) + iJ(b))\| \leq k |||J(a) + iJ(b)||| \;\; \|x\| \quad (a, b \in A, x \in \mathfrak{X}). \tag{3.7}$$

Applying (3.3), (3.4), (3.5), (3.6) and (3.7), we have

$$\|a \odot x\| = \|J(a) \cdot x\| \leq k |||J(a)||| \;\; \|x\| \leq \frac{kk_2}{k_1} \|a\| \;\; \|x\| \quad (a, b \in A, x \in \mathfrak{X}),$$
$$\|x \odot a\| = \|x \cdot J(a)\| \leq k |||J(a)||| \;\; \|x\| \leq \frac{kk_2}{k_1} \|a\| \;\; \|x\| \quad (a, b \in A, x \in \mathfrak{X}).$$

Thus $\mathfrak{X}$ is a real Banach A-bimodule. Since $\mathfrak{X} \in \mathcal{M}^r_{\mathbb{C}}(A_{\mathbb{C}}, \varphi_C)$, we have $x \cdot J(a) = \varphi_C(J(a))x$ for all $(x, a) \in \mathfrak{X} \times A$. This implies that $x \odot a = \varphi(a)x$ for all $(x, a) \in \mathfrak{X} \times A$. Hence, $\mathfrak{X} \in \mathcal{M}^r_{\mathbb{R}}(A, \varphi)$. On the other hand, for each $(a, x) \in A \times \mathfrak{X}$ we have

$$i(a \odot x) = i(J(a) \cdot x) = J(a) \cdot (ix) = a \odot (ix).$$

Since A is right φ-amenable, we have

$$H^1_{\mathbb{R}}(A, \mathfrak{X}^*) = \{0\}. \tag{3.8}$$

Let $D \in Z^1_{\mathbb{C}}(A_{\mathbb{C}}, \mathfrak{X}^*)$. Define the map $d : A \longrightarrow \mathfrak{X}^*$ by

$$d(a) = D(J(a)) \quad (a \in A).$$

It is easy to see that d is a real $\mathfrak{X}^*$-derivation on A. Since

$$\|d(a)\| = \|D(J(a)\| \leq \|D\| \ \ |||J(a)||| \leq \frac{k_2}{k_1}\|D\| \ \ \|a\|,$$

for all $a \in A$, we deduce that d is bounded and $\|d\| \leq \frac{k_2}{k_1}\|D\|$. Thus $d \in Z^1_{\mathbb{R}}(A, \mathfrak{X}^*)$. According to (3.8), there exists $\Lambda \in \mathfrak{X}^*$ such that

$$d = d_{A,\mathfrak{X}^*,\Lambda}. \tag{3.9}$$

It is easy to see that

$$a \odot \Lambda = J(a) \cdot \Lambda \quad (a \in A), \quad \Lambda \odot a = \Lambda \cdot J(a) \quad (a \in A). \tag{3.10}$$

By the definition of d and applying (3.9) and (3.10) we get

$$\begin{aligned} D(J(a) + iJ(b)) &= D(J(a)) + iD(J(b)) \\ &= d(a) + id(b) \\ &= d_{A,\mathfrak{X}^*,\Lambda}(a) + id_{A,\mathfrak{X}^*,\Lambda}(b) \\ &= a \odot \Lambda - \Lambda \odot a + i(b \odot \Lambda - \Lambda \odot b) \\ &= J(a) \cdot \Lambda - \Lambda \cdot J(a) + i(J(b) \cdot \Lambda - \Lambda \cdot J(b)) \\ &= (J(a) \cdot \Lambda - \Lambda \cdot J(a)) + ((iJ(b)) \cdot \Lambda - \Lambda \cdot (iJ(b)) \\ &= (J(a) + iJ(b)) \cdot \Lambda - \Lambda \cdot (J(a) + iJ(b)) \\ &= d_{A_{\mathbb{C}},\mathfrak{X}^*,\Lambda}(J(a) + iJ(b)) \end{aligned}$$

for all $a, b \in A$. This implies that $D = d_{A_{\mathbb{C}},\mathfrak{X}^*,\Lambda}$ and so

$$H^1_{\mathbb{C}}(A_{\mathbb{C}}, \mathfrak{X}^*) = \{0\}.$$

Therefore, $A_{\mathbb{C}}$ is right φ_C-amenable.

We now assume that $\varphi \in \triangle(A) \cup \{0\}$ and $A_{\mathbb{C}}$ is right φ_C-amenable. We show that A is right φ-amenable. Let $\mathfrak{X} \in M^r_{\mathbb{R}}(A, \varphi)$ with the norm $\|\cdot\|$ and with the module actions $(a, x) \longmapsto a \cdot x$ and $(a, x) \longmapsto x \cdot a$ such that

$$i(a \cdot x) = a \cdot (ix), \tag{3.11}$$

for all $(a, x) \in A \times \mathfrak{X}$. Define the map $(J(a) + iJ(b), x) \longmapsto (J(a) + iJ(b))x : A_{\mathbb{C}} \times \mathfrak{X} \longrightarrow \mathfrak{X}$ by

$$(J(a) + iJ(b))x = (a \cdot x) + i(b \cdot x) \quad (a, b \in A, x \in \mathfrak{X}), \tag{3.12}$$

and the map $(J(a)+iJ(b),x)\longmapsto x(J(a)+iJ(b)):A_{\mathbb{C}}\times\mathfrak{X}\longrightarrow\mathfrak{X}$ by

$$x(J(a)+iJ(b))=(x\cdot a)+i(x\cdot b)\quad (x\in\mathfrak{X},a,b\in A). \tag{3.13}$$

Applying (3.11) and (3.12), we can show that

$$\begin{aligned}(\alpha+i\beta)((J(a)+iJ(b))x)&=((\alpha+i\beta)(J(a)+iJ(b)))x\\&=(J(a)+iJ(b))((\alpha+i\beta)x)\end{aligned}$$

for all $(\alpha,\beta,a,b,x)\in\mathbb{R}\times\mathbb{R}\times A\times A\times\mathfrak{X}$. Since $\varphi\in\triangle(A)\cup\{0\}$ and $\mathfrak{X}\in\mathcal{M}^r_{\mathbb{R}}(A,\varphi)$, we have $x\cdot a=\varphi(a)x$ for all $(x,a)\in\mathfrak{X}\times A$. This implies that

$$x(J(a)+iJ(b))=\varphi_C(J(a)+iJ(b))x, \tag{3.14}$$

for all $x\in\mathfrak{X}$ and $a,b\in A$. Applying (3.14) and (3.13), we get

$$\begin{aligned}(\alpha+i\beta)(x(J(a)+iJ(b)))&=((\alpha+i\beta)(xJ(a)+iJ(b)))\\&=x((\alpha+i\beta)(J(a)+iJ(b)))\end{aligned}$$

for all $(\alpha,\beta,a,b,x)\in\mathbb{R}\times\mathbb{R}\times A\times A\times\mathfrak{X}$. Hence, $\mathfrak{X}$ is a complex $A_{\mathbb{C}}$-bimodule. Since $\mathfrak{X}$ is a real Banach A-bimodule, there exists a positive constant k such that

$$\|a\cdot x\|\le k\|a\|\ \ \|x\|\quad (a\in A,x\in\mathfrak{X}), \tag{3.15}$$
$$\|x\cdot a\|\le k\|a\|\ \ \|x\|\quad (a\in A,x\in\mathfrak{X}). \tag{3.16}$$

Applying (3.3), (3.12) and (3.15), we get

$$\|(J(a)+iJ(b))x\|\le 2kk_1|||J(a)+iJ(b)|||\ \ \|x\|,$$

for all $(a,b,x)\in A\times A\times\mathfrak{X}$ and applying (3.4), (3.13) and (3.16), we get

$$\|x(J(a)+iJ(b))\|\le 2kk_1|||J(a)+iJ(b)|||\ \ \|x\|,$$

for all $(a,b,x)\in A\times A\times\mathfrak{X}$. Hence, $\mathfrak{X}$ is a complex Banach $A_{\mathbb{C}}$-bimodule and so, by (3.14), $\mathfrak{X}\in\mathcal{M}^r_{\mathbb{C}}(A_{\mathbb{C}},\varphi_C)$. Therefore,

$$H^1_{\mathbb{C}}(A_{\mathbb{C}},\mathfrak{X}^*)=\{0\}. \tag{3.17}$$

Let $d\in Z^1_{\mathbb{R}}(A,\mathfrak{X}^*)$. Define the map $D:A_{\mathbb{C}}\longrightarrow\mathfrak{X}^*$ by

$$D(J(a)+iJ(b))=d(a)+id(b)\quad (a,b\in A). \tag{3.18}$$

It is easy to show that D is a complex linear operator. According to $d\in Z^1_{\mathbb{R}}(A,\mathfrak{X}^*)$ and applying (3.12), (3.13) and (3.18), one can show that

$$\begin{aligned}D((J(a)+iJ(b))(J(a')+iJ(b')))&=D((J(a)+iJ(b)))(J(a')+iJ(b'))\\&\quad+(J(a)+iJ(b))D(J(a')+iJ(b')),\end{aligned}$$

for all $a,b,a',b'\in A$. Hence, D is a complex $\mathfrak{X}^*$-derivation on $A_{\mathbb{C}}$. By (3.18) and (3.3), we have

$$\begin{aligned}\|D(J(a)+iJ(b))\|&=\|d(a)+id(b)\|\le||d(a)||+\|d(b)\|\\&\le\|d\|\ \ \|a\|+\|d\|\ \ \|b\|\le 2\|d\|\max\{\|a\|,\|b\|\}\\&\le 2k_1\|d\|\ \ |||J(a)+iJ(b)|||,\end{aligned}$$

for all $a, b \in A$. This implies that D is bounded and $\|D\| \leq 2k_1\|d\|$. Hence, $D \in Z^1_{\mathbb{C}}(A_{\mathbb{C}}, \mathfrak{X}^*)$. By (3.17), there exists $\Lambda \in \mathfrak{X}^*$ such that

$$D = d_{A_{\mathbb{C}}, \mathfrak{X}^*, \Lambda}. \tag{3.19}$$

It is easy to see that

$$J(a)\Lambda = a \cdot \Lambda, \quad \Lambda J(a) = \Lambda \cdot a, \tag{3.20}$$

for all $a \in A$. Applying the definition of D, (3.19) and (3.20), we have

$$\begin{aligned} d(a) &= D(J(a)) = d_{A_{\mathbb{C}}, \mathfrak{X}^*, \Lambda}(J(a)) = J(a)\Lambda - \Lambda J(a) \\ &= a \cdot \Lambda - \Lambda \cdot a = d_{A, \mathfrak{X}, x}(a), \end{aligned}$$

for all $a \in A$. Hence, $d = d_{A, \mathfrak{X}^*, \Lambda}$ and so $H^1_{\mathbb{R}}(A, \mathfrak{X}^*) = \{0\}$. Therefore, A is right φ-amenable.

Similarly, we can show that if $\varphi \in \triangle(A) \cup \{0\}$ then A is left φ-amenable if and only if $A_{\mathbb{C}}$ is left φ_C-amenable. Hence, (i) holds.

(ii) Since the map $\Phi : \triangle(A) \cup \{0\} \longrightarrow \triangle(A_{\mathbb{C}}) \cup \{0\}$ defined by (3.2) is bijection, (ii) follows from (i).

(iii) Clearly, (ii) implies that (iii) holds. □

4. A hereditary property of left and right 0-amenability

A hereditary property of the left 0-amenability of complex Banach algebras studied by Nasr-Isfahani and Soltani [19, Proposition 3.4(i)] which is modified as the following.

Proposition 4.1. *Let $(B, \|\cdot\|)$ be a complex Banach algebra. Then B is left (right, respectively) 0-amenable if and only if B has a bounded right (left, respectively) approximate identity.*

Applying Proposition 4.1 and part (i) of Theorem 3.1 for $\varphi = 0$, we obtain a hereditary property of left and right 0-amenability for real Banach algebras as the following.

Proposition 4.2. *Let $(A, \|\cdot\|)$ be a real Banach algebra. Then A is left (right, respectively) 0-amenable if and only if A has a bounded right (left, respectively) approximate identity.*

Proof . Take $A_{\mathbb{C}} = A \times A$. Recall that $A_{\mathbb{C}}$ is a complex algebra with the algebra operations defined by (1.2) and so it is a complexification of A with respect to the injective real algebra homomorphism $J : A \longrightarrow A_{\mathbb{C}}$ defined by $J(a) = (a, 0)$, $(a \in A)$. By [5, Proposition, I.1.13], there exists an algebra norm $|||\cdot|||$ on $A_{\mathbb{C}}$ satisfy the $(*)$ condition with $k_1 = 1$ and $k_2 = 2$.

We first assume that A is left (right, respectively) 0-amenable. By part (i) of Theorem 3.1 for $\varphi = 0$, the complex Banach algebra $(A_{\mathbb{C}}, |||\cdot|||)$ is left (right, respectively) 0-amenable. Hence, $A_{\mathbb{C}}$ has a bounded right (left, respectively) approximate identity $\{(u_\gamma, v_\gamma)\}_{\gamma \in \Gamma}$ by Proposition 4.1. It is easy to see that $\{u_\gamma\}_{\gamma \in \Gamma}$ is a bounded right (left, respectively) approximate identity for A.

We now assume that A has a bounded right (left, respectively) approximate identity $\{u_\gamma\}_{\gamma \in \Gamma}$. It is easy to see that $\{(u_\gamma, 0)\}_{\gamma \in \Gamma}$ is a bounded right (left, respectively) approximate identity for $A_{\mathbb{C}}$. Hence, $A_{\mathbb{C}}$ is left (right, respectively) 0-amenable by Proposition 4.1. Therefore, A is left (right, respectively) 0-amenable by part (i) of Theorem 3.1 for $\varphi = 0$. □ As consequences of Propositions 4.1 and 4.2, we obtain the following results.

Corollary 4.3. *Let $(A, \|\cdot\|)$ be a commutative Banach algebra over $\mathbb{F}$. Then A is left 0-amenable if and only if A is right 0-amenable.*

Corollary 4.4. *Let $(A, \|\cdot\|)$ be a Banach algebra over $\mathbb{F}$. Then A is 0-amenable if and only if A has a bounded approximate identity.*

Corollary 4.5. *Let $(B, \|\cdot\|)$ be a complex Banach algebra and let $B_{\mathbb{R}}$ denote B regarded as a real algebra. Then B is left (right, respectively) 0-amenable if and only if $B_{\mathbb{R}}$ is left (right, respectively) 0-amenable.*

5. Right φ-amenability and injectivity

In this section, we assume that A is a real Banach algebra with $\triangle(A) \neq \emptyset$ and $\varphi \in \triangle(A)$. We discuss the relation between left φ-amenability of A and injectivity of real Banach left A-modules.

Let A be a Banach algebra and $\mathfrak{X}$ be a left Banach A-module over $\mathbb{F}$. We say that $\mathfrak{X}$ is *faithful* if $A \cdot x \neq \{0\}$ for all $x \in \mathfrak{X} \setminus \{0\}$, where $A \cdot x = \{a \cdot x : \ a \in A\}$ for $x \in \mathfrak{X}$.

The following result is a modification of [19, Proposition 4.1] which is useful in the sequel.

Proposition 5.1. *Let A be a Banach algebra over $\mathbb{F}$ with $\triangle(A) \neq \emptyset$, let $\varphi \in \triangle(A)$ and let $\mathfrak{X}$ be a complex Banach space. Then $\mathfrak{X}$ is a faithful Banach left A-module over $\mathbb{F}$ with the left module action $(a, x) \longmapsto a \cdot x : A \times \mathfrak{X} \longrightarrow \mathfrak{X}$ defined by $a \cdot x = \varphi(a)x, \quad (a, x) \in A \times \mathfrak{X}$.*

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces over $\mathbb{F}$. We denote by $\mathcal{B}_{\mathbb{F}}(\mathfrak{X}, \mathfrak{Y})$ the Banach space of all bounded linear operators from $\mathfrak{X}$ to $\mathfrak{Y}$ over $\mathbb{F}$ with the operator norm. We say that $T \in \mathcal{B}_{\mathbb{F}}(\mathfrak{X}, \mathfrak{Y})$ is *admissible* if $T \circ S \circ T = T$ for some $S \in \mathcal{B}_{\mathbb{F}}(\mathfrak{Y}, \mathfrak{X})$.

Let A be a Banach algebra over $\mathbb{F}$ and let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach left A-modules over $\mathbb{F}$. We denote by ${}_A\mathcal{B}_{\mathbb{F}}(\mathfrak{X}, \mathfrak{Y})$ the set of all $T \in \mathcal{B}_{\mathbb{F}}(\mathfrak{X}, \mathfrak{Y})$ for which T is an A-module morphism. Clearly, ${}_A\mathcal{B}_{\mathbb{F}}(\mathfrak{X}, \mathfrak{Y})$ is a closed subspace of $\mathcal{B}_{\mathbb{F}}(\mathfrak{X}, \mathfrak{Y})$ over $\mathbb{F}$. An operator $T \in {}_A\mathcal{B}_{\mathbb{F}}(\mathfrak{X}, \mathfrak{Y})$ is called a *coretraction* if there exists $S \in {}_A\mathcal{B}(\mathfrak{Y}, \mathfrak{X})$ with $S \circ T = I_{\mathfrak{X}}$, the identity self-map on $\mathfrak{X}$.

Let A be a Banach algebra and let $\mathcal{J}$ be a Banach left A-module over $\mathbb{F}$. We say that $\mathcal{J}$ is *injective* if for any Banach left A-modules $\mathfrak{X}$ and $\mathfrak{Y}$ over $\mathbb{F}$, each admissible monomorphism $T \in {}_A\mathcal{B}_{\mathbb{F}}(\mathfrak{X}, \mathfrak{Y})$ and each $S \in {}_A\mathcal{B}_{\mathbb{F}}(\mathfrak{X}, \mathcal{J})$, there exists $R \in {}_A\mathcal{B}_{\mathbb{F}}(\mathfrak{Y}, \mathcal{J})$ such that $R \circ T = S$.

Let A be a Banach algebra and let $\mathfrak{X}$ be a Banach space over $\mathbb{F}$. It is known [6, Example 2.6.2(viii)] that $\mathcal{B}_{\mathbb{F}}(A, \mathfrak{X})$ is a Banach A-bimodule with the module actions $(a, T) \longrightarrow a \cdot T$ and $(a, T) \longrightarrow T \cdot a$ defined by

$$(a \cdot T)(b) = T(ba) \quad (a \in A, \quad T \in \mathcal{B}_{\mathbb{F}}(A, \mathfrak{X}), \quad b \in A),$$
$$(T \cdot a)(b) = T(ab) \quad (a \in A, \quad T \in \mathcal{B}_{\mathbb{F}}(A, \mathfrak{X}), \quad b \in A).$$

Let A be a Banach algebra and let $\mathfrak{X}$ be a Banach A-bimodule over $\mathbb{F}$. For each $x \in \mathfrak{X}$, define the map $T_x : A \longrightarrow \mathfrak{X}$ by

$$T_x(a) = a \cdot x \quad (a \in A).$$

It is easy to see that $T_x \in \mathcal{B}_{\mathbb{F}}(A, \mathfrak{X})$ for all $x \in \mathfrak{X}$. Define the map $\Pi_{\mathbb{F}} : \mathfrak{X} \longrightarrow \mathcal{B}_{\mathbb{F}}(A, \mathfrak{X})$ by

$$\Pi_{\mathbb{F}}(x) = T_x \quad (x \in \mathfrak{X}).$$

It is easy that $\Pi_{\mathbb{F}} \in {}_A\mathcal{B}_{\mathbb{F}}(\mathfrak{X}, \mathcal{B}_{\mathbb{F}}(A, \mathfrak{X}))$. $\Pi_{\mathbb{F}}$ is called the canonical embedding from $\mathfrak{X}$ to $\mathcal{B}_{\mathbb{F}}(A, \mathfrak{X})$. The following result is due to Helemskii which is useful in the sequel.

Proposition 5.2. [9, Proposition III.1.31]. *Let A be a Banach algebra and let $\mathfrak{X}$ be a faithful left A-module over $\mathbb{F}$. Then $\mathfrak{X}$ is injective if and only if the canonical embedding $\Pi_{\mathbb{F}} \in {}_A\mathcal{B}_{\mathbb{F}}(\mathfrak{X}, \mathcal{B}_{\mathbb{F}}(A, \mathfrak{X}))$ is a coretraction.*

For a real Banach algebra $(A, \|\cdot\|)$ and a complex Banach space $(\mathfrak{X}, \|\cdot\|)$, we show that $\mathfrak{X}$ is an injective real Banach left A-module with a suitable left module action if and only if $\mathfrak{X}$ is an injective complex Banach $A_{\mathbb{C}}$-module with a suitable left module action. For this purpose we need the following lemma which its proof is straightforward.

Lemma 5.3. *Let $(A, \|\cdot\|)$ be a real Banach algebra and let $(\mathfrak{X}, \|\cdot\|)$ be a complex Banach space. Then the followings hold.*

(i) $\mathcal{B}_{\mathbb{R}}(A, \mathfrak{X})$ *with the operator norm is a complex Banach space whenever the scalar multiplication is determined by*

$$(\alpha S)(a) = \alpha S(a) \quad (\alpha \in \mathbb{C}, \quad S \in \mathcal{B}_{\mathbb{R}}(A, \mathfrak{X}), \quad a \in A). \tag{5.1}$$

(ii) *Real Banach left A-module $\mathcal{B}_{\mathbb{R}}(A, \mathfrak{X})$ satisfies*

$$i(a \cdot T) = a \cdot (iT) \quad (a \in A, \quad T \in \mathcal{B}_{\mathbb{R}}(A, \mathfrak{X})). \tag{5.2}$$

Theorem 5.4. *Let $(A, \|\cdot\|)$ be a real Banach algebra and let $\varphi \in \triangle(A)$ and let $A_{\mathbb{C}}$ be a complexification of A with respect to an injective real algebra homomorphism $J : A \longrightarrow A_{\mathbb{C}}$, let $|||\cdot|||$ be an algebra norm on $A_{\mathbb{C}}$ satisfying in the $(*)$ condition. Suppose that $\mathfrak{X}$ is a complex Banach space. Then the following assertions are equivalent.*

(i) $\mathfrak{X}$ *is an injective real Banach left A-module with the left module action $(a, x) \longmapsto a \cdot x : A \times \mathfrak{X} \longrightarrow \mathfrak{X}$ defined by $a \cdot x = \varphi(a)x, \quad (a, x) \in A \times \mathfrak{X}$.*

(ii) $\mathfrak{X}$ *is an injective complex Banach left $A_{\mathbb{C}}$-module with the left module action $(J(a)+iJ(b), x) \longmapsto (J(a)+iJ(b)) \cdot x : A_{\mathbb{C}} \times \mathfrak{X} \longrightarrow \mathfrak{X}$ defined by $(J(a)+iJ(b)) \cdot x = \varphi_C(J(a)+iJ(b))x, \quad (a, b, x) \in A \times A \times \mathfrak{X}$.*

Proof . Clearly, $\mathfrak{X}$ is a real Banach left A-module (a complex Banach left $A_{\mathbb{C}}$-module, respectively) with the left module action defined in (i) (in (ii), respectively). Hence, A ($A_{\mathbb{C}}$, respectively) is a faithful real (complex, respectively) Banach left $A_{\mathbb{C}}$-module by Proposition 5.1. Let $\Pi_{\mathbb{R}} : \mathfrak{X} \longrightarrow \mathcal{B}_{\mathbb{R}}(A, \mathfrak{X})$ be the canonical embedding from $\mathfrak{X}$ to $\mathcal{B}_{\mathbb{R}}(A, \mathfrak{X})$ and $\Pi_{\mathbb{C}} : \mathfrak{X} \longrightarrow \mathcal{B}_{\mathbb{C}}(A_{\mathbb{C}}, \mathfrak{X})$ be the canonical embedding from $\mathfrak{X}$ to $\mathcal{B}_{\mathbb{C}}(A_{\mathbb{C}}, \mathfrak{X})$. Then

$$\Pi_{\mathbb{R}}(x)(a) = a \cdot x = \varphi(a)x \quad (x \in \mathfrak{X}, \quad a \in A),$$

and

$$\Pi_{\mathbb{C}}(x)(J(a) + iJ(b)) = J(a) + iJ(b) \cdot x = \varphi_C(J(a) + iJ(b))x \quad (x \in \mathfrak{X}, \quad a, b \in A).$$

Applying (5.2), one can show $\mathcal{B}_{\mathbb{R}}(A, \mathfrak{X})$ is a complex Banach left $A_{\mathbb{C}}$-module with the left module action

$$J(a) + iJ(b)S = a \cdot S + i(b \cdot S) \quad (a, b \in A, \quad S \in \mathcal{B}_{\mathbb{R}}(A, \mathfrak{X}).$$

Moreover, we can easily show that for each $T \in \mathcal{B}_{\mathbb{C}}(A_{\mathbb{C}}, \mathfrak{X})$, $T \circ J \in \mathcal{B}_{\mathbb{R}}(A, \mathfrak{X})$ and $\|T \circ J\| \leq \frac{k_2}{k_1}\|T\|$. We now define the map $\Theta : \mathcal{B}_{\mathbb{C}}(A_{\mathbb{C}}, \mathfrak{X}) \longrightarrow \mathcal{B}_{\mathbb{R}}(A, \mathfrak{X})$ by

$$\Theta(T) = T \circ J \quad (T \in \mathcal{B}_{\mathbb{C}}(A_{\mathbb{C}}, \mathfrak{X})).$$

Clearly, Θ is a real linear mapping from the complex Banach space $\mathcal{B}_{\mathbb{C}}(A_{\mathbb{C}}, \mathfrak{X})$ to the complex Banach space $\mathcal{B}_{\mathbb{R}}(A, \mathfrak{X})$. Since for each $T \in \mathcal{B}_{\mathbb{C}}(A_{\mathbb{C}}, \mathfrak{X})$ we have

$$\Theta(iT)(a) = ((iT) \circ J)(a) = (iT)(J(a)) = iT(J(a)) = (i\Theta(T))(a),$$

for all $a \in A$, we deduce that $\Theta(iT) = i\Theta(T)$ for all $T \in \mathcal{B}_{\mathbb{C}}(A_{\mathbb{C}}, \mathfrak{X})$. Hence, Θ is a complex linear mapping. Since $\|T \circ J\| \leq \frac{k_2}{k_1}\|T\|$ for all $T \in \mathcal{B}_{\mathbb{C}}(A_{\mathbb{C}}, \mathfrak{X})$, we deduce that Θ is bounded and $\|\Theta\| \leq \frac{k_2}{k_1}$. Let $a, b \in A$ and $T \in \mathcal{B}_{\mathbb{C}}(A_{\mathbb{C}}, \mathfrak{X})$. Then

$$\begin{aligned}
\Theta((J(a) + iJ(b)) \cdot T)(c) &= ((J(a) + iJ(b)) \cdot T)(J(c)) \\
&= T(J(c)((J(a) + iJ(b)) \\
&= T(J(c)J(a)) + iT(J(c)J(b)) \\
&= T(J(ca)) + iT(J(cb)) \\
&= \Theta(T)(ca) + i\Theta(T)(cb) \\
&= (a \cdot \Theta(T))(c) + i((b \cdot \Theta(T))(c)) \\
&= (a \cdot \Theta(T))(c) + (i(b \cdot \Theta(T))(c)) \\
&= (J(a)\Theta(T))(c) + (i(J(b)\Theta(T))(c)) \\
&= (J(a)\Theta(T)) + (iJ(b))\Theta(T))(c) \\
&= ((J(a) + iJ(b))\Theta(T))(c),
\end{aligned}$$

for all $c \in A$.Hence,

$$\Theta((J(a) + iJ(b)) \cdot T) = (J(a) + iJ(b))\Theta(T).$$

Therefore, $\Theta \in {}_{A_{\mathbb{C}}}\mathcal{B}_{\mathbb{C}}(\mathcal{B}_{\mathbb{C}}(A_{\mathbb{C}}, \mathfrak{X}), \mathcal{B}_{\mathbb{R}}(A, \mathfrak{X}))$.

For each $S \in \mathcal{B}_{\mathbb{R}}(A, \mathfrak{X}))$, define the map $\Lambda_S : A_{\mathbb{C}} \longrightarrow \mathfrak{X}$ by

$$\Lambda_S(J(a) + iJ(b)) = S(a) + iS(b) \quad (a, b \in A).$$

It is easy to see that $\Lambda_S \in \mathcal{B}_{\mathbb{C}}(A_{\mathbb{C}}, \mathfrak{X})$. Define the map $\Gamma : \mathcal{B}_{\mathbb{R}}(A, \mathfrak{X}) \longrightarrow \mathcal{B}_{\mathbb{C}}(A_{\mathbb{C}}, \mathfrak{X})$ by

$$\Gamma(S) = \Lambda_S \quad (S \in \mathcal{B}_{\mathbb{R}}(A, \mathfrak{X})).$$

It is easy to see that $\Theta \circ \Gamma = I_{\mathcal{B}_{\mathbb{R}}(A,\mathfrak{X})}$ and $\Gamma \circ \Theta = I_{\mathcal{B}_{\mathbb{C}}(A_{\mathbb{C}},\mathfrak{X})}$. Therefore, $\Gamma = \Theta^{-1}$ and $\Gamma \in \mathcal{B}_{\mathbb{C}}(\mathcal{B}_{\mathbb{R}}(A, \mathfrak{X}), \mathcal{B}_{\mathbb{C}}(A_{\mathbb{C}}, \mathfrak{X}))$ by open mapping theorem for complex Banach spaces.

Clearly, $\mathcal{B}_{\mathbb{C}}(A_{\mathbb{C}}, \mathfrak{X})$ is a real Banach left A-module with the left module action

$$a \odot T = J(a) \cdot T \quad (a \in A, \quad T \in \mathcal{B}_{\mathbb{C}}(A_{\mathbb{C}}, \mathfrak{X})).$$

Let $c \in A$ and $S \in \mathcal{B}_{\mathbb{R}}(A, \mathfrak{X})$. Then for each $a, b \in A$ we have

$$\begin{aligned}
\Gamma(c \cdot S)((J(a) + iJ(b)) &= \Lambda_{c \cdot S}((J(a) + iJ(b)) = c \cdot S(a) + i(c \cdot S)(b) \\
&= S(ac) + iS(bc) = \Lambda_S(J(ac) + iJ(bc)) \\
&= \Lambda_S((J(a) + iJ(b))J(c)) = (J(c) \cdot \Lambda_S)(J(a) + iJ(b)) \\
&= (c \odot \Lambda_S)(J(a) + iJ(b)) = (c \odot \Gamma(S))(J(a) + iJ(b)).
\end{aligned}$$

Therefore, $\Gamma(c \cdot S) = c \odot \Gamma(S)$ and so $\Gamma \in {}_A\mathcal{B}_{\mathbb{C}}(\mathcal{B}_{\mathbb{R}}(A, \mathfrak{X}), \mathcal{B}_{\mathbb{C}}(A_{\mathbb{C}}, \mathfrak{X}))$. Let $x \in \mathfrak{X}$. Since

$$\begin{aligned}
((\Theta \circ \Pi_{\mathbb{C}})(x))(a) &= (\Theta(\Pi_{\mathbb{C}}(x))(a) = (\Pi_{\mathbb{C}}(x) \circ J)(a) \\
&= \varphi_C(J(a))x = \varphi(a)x = \Pi_{\mathbb{R}}(x)(a),
\end{aligned}$$

for all $a \in A$, we deduce that $(\Theta \circ \Pi_{\mathbb{C}})(x) = \Pi_{\mathbb{R}}(x)$. Therefore,

$$\Theta \circ \Pi_{\mathbb{C}} = \Pi_{\mathbb{R}}. \tag{5.3}$$

Since $\Gamma = \Theta^{-1}$, we have
$$\Pi_{\mathbb{C}} = \Gamma \circ \Pi_{\mathbb{R}}. \tag{5.4}$$
By (5.3) and the complex linearity of $\Pi_{\mathbb{C}}$ and Θ, we deduce that $\Pi_{\mathbb{R}}$ is complex linear. To prove $(i) \Rightarrow (ii)$, assume that $\mathfrak{X}$ is a injective real Banach left A-module with the left module action defined by
$$a \cdot x = \varphi(a)x \quad (a \in A, x \in \mathfrak{X}).$$
By Proposition 5.1 for $\mathbb{F} = \mathbb{R}$, $\mathfrak{X}$ is faithful real Banach left A-module. Therefore, by Proposition 5.2 for $\mathbb{F} = \mathbb{R}$, we deduce that $\Pi_{\mathbb{R}}$ is a coretraction. Hence, there exists $Q \in {}_A\mathcal{B}_{_\mathbb{R}}(\mathcal{B}_{\mathbb{R}}(A, \mathfrak{X}), \mathfrak{X})$ such that
$$Q \circ \Pi_{\mathbb{R}} = I_{\mathfrak{X}}. \tag{5.5}$$
Define the map $Q_C : \mathcal{B}_{\mathbb{R}}(A, \mathfrak{X}) \longrightarrow \mathfrak{X}$ by
$$Q_C(S) = Q(S) - iQ(iS) \quad (S \in \mathcal{B}_{\mathbb{R}}(A, \mathfrak{X})).$$
It is easy to see that $Q_C \in \mathcal{B}_{_\mathbb{C}}(\mathcal{B}_{\mathbb{R}}(A, \mathfrak{X}), \mathfrak{X})$. Applying (5.2), we get
$$\begin{aligned}Q_C(J(a)S) &= Q_C(a \cdot S) = Q(a \cdot S) - iQ(i(a \cdot S)) \\ &= Q(a \cdot S) - iQ(a \cdot (iS)) = a \cdot Q(S) - i(a \cdot Q(iS)) \\ &= \varphi(a)Q(S) - i(\varphi(a)Q(iS)) = \varphi(a)(Q(S) - i(Q(iS)) \\ &= \varphi_C(J(a))(Q(S) - iQ(iS)) = J(a) \cdot Q_C(S),\end{aligned}$$
for all $a \in A$ and $S \in \mathcal{B}_{\mathbb{R}}(A, \mathfrak{X})$. This implies that
$$Q_C((J(a) + iJ(b))S) = (J(a) + iJ(b)) \cdot Q_C(S),$$
for all $a, b \in A$ and $S \in \mathcal{B}_{\mathbb{R}}(A, \mathfrak{X})$. Hence, $Q_C \in {}_{A_{\mathbb{C}}}\mathcal{B}_{_\mathbb{C}}(\mathcal{B}_{\mathbb{R}}(A, \mathfrak{X}), \mathfrak{X})$. Therefore, $\frac{1}{2}Q_C \circ \Theta \in {}_{A_{\mathbb{C}}}\mathcal{B}_{\mathbb{C}}(\mathcal{B}_{\mathbb{C}}(A_{\mathbb{C}}, \mathfrak{X}), \mathfrak{X})$ since $\Theta \in {}_{A_{\mathbb{C}}}\mathcal{B}_{\mathbb{C}}(\mathcal{B}_{\mathbb{C}}(A_{\mathbb{C}}, \mathfrak{X}), \mathcal{B}_{\mathbb{R}}(A, \mathfrak{X}))$. Applying (5.3) and (5.5) and complex linearity of $\Pi_{\mathbb{R}}$, we have
$$\begin{aligned}\frac{1}{2}(Q_C \circ \Theta \circ \Pi_{\mathbb{C}})(x) &= \frac{1}{2}(Q_C \circ \Pi_{\mathbb{R}})(x) = \frac{1}{2}(Q_C)(\Pi_{\mathbb{R}}(x)) \\ &= \frac{1}{2}(Q(\Pi_{\mathbb{R}}(x)) - iQ(i\Pi_{\mathbb{R}}(x)) = \frac{1}{2}(Q(\Pi_{\mathbb{R}}(x)) - iQ(\Pi_{\mathbb{R}}(ix)) \\ &= \frac{1}{2}((Q \circ \Pi_{\mathbb{R}}(x)) - i(Q \circ \Pi_{\mathbb{R}}(ix)) = \frac{1}{2}(I_{\mathfrak{X}}(x) - iI_{\mathfrak{X}}(ix)) \\ &= \frac{1}{2}(x + x) = x\end{aligned}$$
for all $x \in \mathfrak{X}$ and so $\frac{1}{2}(Q_C \circ \Theta) \circ \Pi_{\mathbb{C}} = I_{\mathfrak{X}}$. Therefore, $\Pi_{\mathbb{C}}$ is a coretraction. Since $\mathfrak{X}$ is a faithful complex Banach left $A_{\mathbb{C}}$-module, by proposition 5.2 for $\mathbb{F} = \mathbb{C}$, we deduce that $\mathfrak{X}$ is an injective complex Banach left $A_{\mathbb{C}}$-module with the module action defined in (ii). Hence, (i) implies (ii).

To prove $(ii) \Rightarrow (i)$, assume that $\mathfrak{X}$ is an injective complex Banach left $A_{\mathbb{C}}$-module with the left module action $(J(a) + iJ(b), x) \longmapsto (J(a) + iJ(b)) \cdot x : A_{\mathbb{C}} \times \mathfrak{X} \longrightarrow \mathfrak{X}$ defined by
$$(J(a) + iJ(b)) \cdot x = \varphi_C(J(a) + iJ(b))x \quad (a, b \in A, x \in \mathfrak{X}).$$
By Proposition 5.1 for $\mathbb{F} = \mathbb{C}$, $\mathfrak{X}$ is a faithful complex Banach left $A_{\mathbb{C}}$-module. Therefore, by Proposition 5.2 for $\mathbb{F} = \mathbb{C}$, the complex canonical embedding $\Pi_{\mathbb{C}} : \mathfrak{X} \longrightarrow \mathcal{B}_{\mathbb{C}}(A_{\mathbb{C}}, \mathfrak{X})$ is a coretraction. Thus, there exists $P \in {}_{A_{\mathbb{C}}}\mathcal{B}_{_\mathbb{C}}(\mathcal{B}_{\mathbb{C}}(A_{\mathbb{C}}, \mathfrak{X}), \mathfrak{X})$ such that
$$P \circ \Pi_{\mathbb{C}} = I_{\mathfrak{X}}. \tag{5.6}$$

Define tha map $P' : \mathcal{B}_{\mathbb{R}}(A, \mathfrak{X}) \longrightarrow \mathfrak{X}$ by

$$P' = P \circ \Gamma. \tag{5.7}$$

Applying (5.6), (5.7) and (5.4), we get

$$P' \circ \Pi_{\mathbb{R}} = (P \circ \Gamma) \circ \Pi_{\mathbb{R}} = P \circ (\Gamma \circ \Pi_{\mathbb{R}}) = P \circ \Pi_{\mathbb{C}} = I_{\mathfrak{X}}$$

Hence, $\Pi_{\mathbb{R}}$ is a coretraction. Since $\mathfrak{X}$ is a faithful real Banach left A-module, we deduce that $\mathfrak{X}$ is an injective real Banach left $A_{\mathbb{C}}$-module by Proposition 5.2 for $\mathbb{F} = \mathbb{R}$. Hence, (ii) implies (i). □

A relation between φ-amenability of a complex Banach algebra B and the injectivity of certain Banach left B-modules is given in [19, Theorem 5.2]. We obtain similar result for real Banach algebras as the following.

Theorem 5.5. *Let $(A, \|\cdot\|)$ be a real Banach algebra and let $\varphi \in \triangle(A)$. Then the following assertions are equivalent.*

(i) *If $\mathfrak{X}$ is a complex dual Banach space, then $\mathfrak{X}$ is an injective real Banach left A-module with the left module action $(a, x) \longmapsto a \cdot x : A \times \mathfrak{X} \longrightarrow \mathfrak{X}$ defined by $a \cdot x = \varphi(a)x, \quad (a \in A, x \in \mathfrak{X})$.*

(ii) *$\mathbb{C}$ is an injective real Banach left A-module with the left module action $(a, z) \longmapsto a \cdot z : A \times \mathbb{C} \longrightarrow \mathbb{C}$ defined by $a \cdot z = \varphi(a)z, \quad (a \in A, x \in \mathbb{C})$.*

(iii) *There is a complex Banach space $\mathfrak{X}$ such that $\mathfrak{X}$ is an injective real Banach left A-module with the left module action $(a, x) \longmapsto a \cdot x : A \times \mathfrak{X} \longrightarrow \mathfrak{X}$ defined by $a \cdot x = \varphi(a)x, \quad (a \in A, x \in \mathfrak{X})$.*

(iv) *A is right φ-amenable.*

Proof . (i)⇒ (ii) Since $\mathbb{C}$ is a complex dual Banach space, we deduce that $\mathbb{C}$ is an injective real Banach left A-module with the left module action $(a, z) \longmapsto a \cdot z : A \times \mathbb{C} \longrightarrow \mathbb{C}$ defined by $a \cdot z = \varphi(a)z, \quad (a \in A, x \in \mathbb{C})$, by (i).Hence (ii) holds.

(ii)⇒ (iii) Take $\mathfrak{X} = \mathbb{C}$. Then (iii) holds by (ii).

(iii)⇒ (iv) Set $A_{\mathbb{C}} = A \times A$. Then $A_{\mathbb{C}}$ is a complex algebra with the algebra operations defined in (1.2) and it is a complexification of A with the injective real algebra homomorphism $J : A \longrightarrow A_{\mathbb{C}}$ defined by $J(a) = (a, 0), \quad a \in A$. By [5, Proposition I.1.13], there exists an algebra norm $|||\cdot|||$ on $A_{\mathbb{C}}$ satisfying in the $(*)$ condition with $k_1 = 1$ and $k_2 = 2$. By (iii), there exists a complex Banach space $\mathfrak{X}$ such that $\mathfrak{X}$ is an injective real Banach left A-module with the left module action $(a, x) \longmapsto a \cdot x : A \times \mathfrak{X} \longrightarrow \mathfrak{X}$ defined by $a \cdot x = \varphi(a)x, \quad (a \in A, x \in \mathfrak{X})$. By Theorem 5.4, $\mathfrak{X}$ is an injective complex Banach left $A_{\mathbb{C}}$-module with the left module action $(J(a) + iJ(b), x) \longmapsto (J(a)+iJ(b)) \cdot x : A_{\mathbb{C}} \times \mathfrak{X} \longrightarrow \mathfrak{X}$ defined by $(J(a)+iJ(b)) \cdot x = \varphi_C(J(a)+iJ(b))x, \quad (a, b \in A, x \in \mathfrak{X})$. Therefore, $A_{\mathbb{C}}$ is right φ_C-amenable by [19, Theorem 5.2]. Hence, A is right φ-amenable by part (i) of Theorem 3.1 and so (iv) holds.

(iv)⇒ (i) Let $\mathfrak{X}$ be a complex dual Banach space. Clearly, $\mathfrak{X}$ is a real (complex, respectively) Banach left A-module ($A_{\mathbb{C}}$-module, respectively), with the left module action $a \cdot x = \varphi(a)x$ for all $a \in A, x \in \mathfrak{X}$, $((J(a) + iJ(b)) \cdot x = \varphi_C(J(a) + iJ(b))x$ for all $a, b \in A, x \in \mathfrak{X}$, respectively). By (iv) and part (i) of Theorem 3.1, we deduce that $A_{\mathbb{C}}$ is left φ_C-amenable. Therefore, $\mathfrak{X}$ is an injective complex Banach left $A_{\mathbb{C}}$-module with the mentioned left module action by [19, Theorem 5.2]. Hence, by Theorem 5.4, $\mathfrak{X}$ is an injective real Banach left A-module with the left module action defined by $a \cdot x = \varphi(a)x, \quad (a \in A, x \in \mathfrak{X})$. Thus (i) holds. □

6. Character amenability of B and $B_{\mathbb{R}}$

Let $(B, \|\cdot\|)$ be a complex Banach algebra with $\triangle(B) \neq \emptyset$ and let $B_{\mathbb{R}}$ denote B regarded as a real algebra. Clearly,

$$\triangle(B) \cup \{\overline{\varphi} : \ \varphi \in \triangle(B)\} \subseteq \triangle(B_{\mathbb{R}}).$$

For each $\varphi \in \triangle(B)$, we give a characterization of right φ-amenability of B as the following.

Theorem 6.1. *Let $(B, \|\cdot\|)$ be a complex Banach algebra and let $B_{\mathbb{R}}$ denote B regarded as a real algebra. Then the following assertions are equivalent.*

(i) *B is right φ-amenable.*
(ii) *$\mathbb{C}$ is an injective complex Banach left B-module with the left module action $(b, z) \longmapsto b \cdot z : B \times \mathbb{C} \longrightarrow \mathbb{C}$ defined by $b \cdot z = \varphi(b)z, \quad (b \in B, \quad x \in \mathbb{C})$.*
(iii) *$\mathbb{C}$ is an injective real Banach left $B_{\mathbb{R}}$-module with the left module action $(b, z) \longmapsto b \odot z : B_{\mathbb{R}} \times \mathbb{C} \longrightarrow \mathbb{C}$ defined by $b \odot z = \varphi(b)z, \quad (b \in B, \quad x \in \mathbb{C})$.*
(iv) *$B_{\mathbb{R}}$ is right φ-amenable.*

Proof . (i)$\Rightarrow$ (ii) It follows by [19, Theorem 5.2].

(ii)$\Rightarrow$ (iii) Clearly, $\mathbb{C}$ is a real Banach left $B_{\mathbb{R}}$-module with the left module action $(b, z) \longmapsto b \odot z : B_{\mathbb{R}} \times \mathbb{C} \longrightarrow \mathbb{C}$ defined by

$$b \odot z = \varphi(b)z, \quad (b \in B_{\mathbb{R}}, \quad x \in \mathbb{C}).$$

Take $B' = B_{_{\mathbb{R}}}(B_{\mathbb{R}}, \mathbb{C})$. By part (i) of Lemma 5.3, B' is a complex Banach space. Let $\Pi_{\mathbb{R}} : \mathbb{C} \longrightarrow B'$ be the canonical embedding of $\mathbb{C}$ in B'. Then

$$\Pi_{\mathbb{R}}(z)(b) = \varphi(b)z \quad (z \in \mathbb{C}, \quad b \in B_{\mathbb{R}}).$$

Moreover, for each $z \in \mathbb{C}$ we have

$$\Pi_{\mathbb{R}}(z)(ib) = \varphi(ib)z = i\varphi(b)z = i\Pi_{\mathbb{R}}(z)(b),$$

for all $b \in B$. Therefore, $\Pi_{\mathbb{R}}(z) \in B^*$ for all $z \in \mathbb{C}$.
Let $\Pi_{\mathbb{C}} : \mathbb{C} \longrightarrow B_{_{\mathbb{C}}}(B, \mathbb{C}) = B^*$ be the canonical embedding of $\mathbb{C}$ in B^*. Clearly, $\Pi_{\mathbb{R}} = \Pi_{\mathbb{C}}$. By (ii), there exists $Q \in {}_B\mathcal{B}_{_{\mathbb{C}}}(B^*, \mathbb{C})$ such that

$$Q \circ \Pi_{\mathbb{C}} = I_{\mathbb{C}}. \tag{6.1}$$

It is easy to see that $B^* \times B^*$ is a complex Banach space with the additive operation, scalar multiplication defined by

$$(f_1, g_1) + (f_2, g_2) = (f_1 + f_2, g_1 + g_2) \quad (f_1, f_2, g_1, g_2 \in B^*),$$
$$\alpha(f, g) = (\alpha f, \alpha g) \quad (\alpha \in \mathbb{C}, \quad f, g \in B^*).$$

and with the norm $|||\cdot|||$ defined by

$$|||(f, g)||| = \max\{\|f\|, \|g\|\} \quad (f, g \in B^*).$$

Define the map $\Omega : B^* \times B^* \longrightarrow B'$ by

$$\Omega(f, g) = \text{Re } f + i\text{Im } g \quad (f, g \in B^*).$$

We can easily show that Ω is well-defined and it is a real linear mapping. Let $(f, g) \in B^* \times B^*$ with $\Omega(f, g) = 0$. Then Re $f = 0$ and Im $g = 0$. Therefore, $f = 0$ and $g = 0$ since for each $h \in B^*$ we have

$$h(b) = \text{Re } h(b) - i\text{Re } h(ib),$$

for all $b \in B$. Hence, $(f, g) = (0, 0)$ and so Ω is injective.

Let $\Lambda \in B'$. Thus Re Λ, Im $\Lambda \in (B_{\mathbb{R}})^*$. Define the maps $f, g : B \longrightarrow \mathbb{C}$ by

$$\begin{aligned} f(b) &= \text{Re } \Lambda(b) - i\text{Re } \Lambda(ib) \quad (b \in B), \\ g(b) &= \text{Im } \Lambda(ib) + i\text{Im } \Lambda(b) \quad (b \in B). \end{aligned}$$

It is easy to see that $f, g \in B^*$, Re $f =$ Re Λ and Im $g =$ Im Λ. Therefore, $(f, g) \in B^* \times B^*$ and

$$\Omega(f, g) = \text{Re } f + i\text{Im } g = \text{Re } \Lambda + i\text{Im } \Lambda = \Lambda.$$

Hence, Ω is surjective.
Since

$$\begin{aligned} \|\Omega(f, g)\| &= \|\text{Re } f + i\text{Im } g\| \leq \|\text{Re } f\| + \|\text{Im } g\| \\ &= \|f\| + \|g\| \leq 2\max\{\|f\|, \|g\|\} \\ &= |||(f, g)|||, \end{aligned}$$

for all $f, g \in B^*$, we deduce that Ω is bounded. Therefore, $\Omega \in \mathcal{B}_{\mathbb{R}}(B^* \times B^*, B')$. It is easy to see that B' is a real Banach left B-module with the module action $(b, \Lambda) \longmapsto b \cdot \Lambda : B \times B' \longrightarrow B'$ defined by

$$(b \cdot \Lambda)(c) = \Lambda(cb) \quad (b, c \in B, \quad \Lambda \in B').$$

We can show that $B^* \times B^*$ is a complex Banach left B-module with the module action $(b, (f, g)) \longmapsto b \cdot (f, g) : B \times (B^* \times B^*) \longrightarrow B^* \times B^*$ defined by

$$b \cdot (f, g) = (b \cdot f, b \cdot g) \quad (b \in B, \quad f, g \in B^*).$$

It is easy to see that

$$\text{Re } (b \cdot h) = b \cdot \text{Re } h, \quad \text{Im } (b \cdot h) = b \cdot \text{Im } h \tag{6.2}$$

for all $b \in B$ and $h \in B^*$. We claim that

$$b \cdot i\text{Im } h = i(b \cdot \text{Im } h) \tag{6.3}$$

for all $b \in B$ and $h \in B^*$. Let $b \in B$ and $h \in B^*$. Since for each $b' \in B$ we have

$$\begin{aligned} (b \cdot i\text{Im } h)(b') &= (i\text{Im } h)(b'b) = i(\text{Im } h)(b'b) \\ &= i(b \cdot \text{Im } h)(b') = (i(b \cdot \text{Im } h))(b'), \end{aligned}$$

we deduce that (6.3) holds.
Let $b \in B$ and $f, g \in B^*$. Then, by the definition of Ω, (6.2) and (6.3) we have

$$\begin{aligned} \Omega(b \cdot (f, g)) &= \Omega(b \cdot f, b \cdot g) = \text{Re } (b \cdot f) + i\text{Im } (b \cdot g) \\ &= b \cdot \text{Re } f + i(b \cdot \text{Im } g) = b \cdot \text{Re } f + b \cdot i\text{Im } g \\ &= b \cdot (\text{Re } f + i\text{Im } g) = b \cdot \Omega(f, g). \end{aligned}$$

Therefore, $\Omega \in {}_{B_{\mathbb{R}}}\mathcal{B}_{\mathbb{R}}(B^* \times B^*, B')$. This implies that $\Omega^{-1} \in {}_{B_{\mathbb{R}}}\mathcal{B}_{\mathbb{R}}(B', B^* \times B^*)$. Now define the map $\mu : B^* \times B^* \longrightarrow B^*$ by

$$\mu(f, g) = \frac{1}{2}(f + g) \quad (f, g \in B^*).$$

Clearly, $\mu \in {}_{B_{\mathbb{R}}}\mathcal{B}_{\mathbb{R}}(B^* \times B^*, B^*)$. Thus $Q \circ \mu \circ \Omega^{-1} \in {}_{B_{\mathbb{R}}}\mathcal{B}_{\mathbb{R}}(B', \mathbb{C})$. According to $\Omega(f, f) = \text{Re } f + i\text{Im } f = f$ for all $f \in B^*$, we deduce that $\Omega^{-1}(f) = (f, f)$ for all $f \in B^*$. Let $z \in \mathbb{C}$. Then $\Omega^{-1}(\Pi_{\mathbb{R}}(z)) = (\Pi_{\mathbb{R}}(z), \Pi_{\mathbb{R}}(z))$ and so, by $\Pi_{\mathbb{R}}(z) = \Pi_{\mathbb{C}}(z)$ and (6.1), we get

$$\begin{aligned}(Q \circ \mu \circ \Omega^{-1}) \circ \Pi_{\mathbb{R}}(z) &= (Q \circ \mu)(\Omega^{-1}(\Pi_{\mathbb{R}}(z))) = (Q \circ \mu)(\Pi_{\mathbb{R}}(z), \Pi_{\mathbb{R}}(z)) \\ &= Q(\Pi_{\mathbb{R}}(z)) = Q(\Pi_{\mathbb{C}}(z)) = (Q \circ \Pi_{\mathbb{C}})(z) \\ &= I_{\mathbb{C}}(z).\end{aligned}$$

Therefore, $(Q \circ \mu \circ \Omega^{-1}) \circ \Pi_{\mathbb{R}} = I_{\mathbb{C}}$. This implies that $\Pi_{\mathbb{R}}$ is a coretraction. Since $\varphi \in \triangle(B_{\mathbb{R}})$ and $\mathbb{C}$ is a real Banach left $B_{\mathbb{R}}$-module with the left module action $(b, z) \longmapsto b \odot z : B_{\mathbb{R}} \times \mathbb{C} \longrightarrow \mathbb{C}$ defined by $b \odot z = \varphi(b)z \quad (b \in B_{\mathbb{R}}, z \in \mathbb{C})$, we deduce that $\mathbb{C}$ is an faithful real Banach left $B_{\mathbb{R}}$-module by Proposition 5.1 for $\mathbb{F} = \mathbb{R}$. Hence, $\mathbb{C}$ is an injective real Banach left $B_{\mathbb{R}}$-module with the left module action defined in (iii) and so (iii) holds.

(iii)$\Rightarrow$ (iv) It follows by Theorem 5.5.

(iv)$\Rightarrow$ (i) Let $\mathfrak{X} \in \mathcal{M}^r_{\mathbb{C}}(B, \varphi)$ with the module actions $(b, x) \longmapsto b \cdot x$ and $(b, x) \longmapsto x \cdot b$. Clearly, $\mathfrak{X}$ is a real Banach $B_{\mathbb{R}}$-module with the module actions $(b, x) \longmapsto b \odot x$ and $(b, x) \longmapsto x \odot b$ defined by

$$\begin{aligned}&b \odot x = b \cdot x = \varphi(b)x \quad (b \in B_{\mathbb{R}}, \quad x \in \mathfrak{X}), \\ &x \odot b = x \cdot b \quad (x \in \mathfrak{X}, \quad b \in B_{\mathbb{R}}).\end{aligned}$$

Since $\varphi \in \triangle(B_{\mathbb{R}})$, we deduce that $\mathfrak{X} \in \mathcal{M}^r_{\mathbb{R}}(B_{\mathbb{R}}, \varphi)$. On the other hand

$$i(x \odot b) = i(x \cdot b) = (ix) \cdot b = ix \odot b,$$

for all $x \in \mathfrak{X}$ and $b \in B$. Hence,

$$H^1_{\mathbb{R}}(B_{\mathbb{R}}, \mathfrak{X}^*) = \{0\}, \tag{6.4}$$

by (iv). Let $D \in Z^1_{\mathbb{C}}(B, \mathfrak{X}^*)$. Define the map $d : B_{\mathbb{R}} \longrightarrow \mathfrak{X}^*$ by

$$d(b) = D(b) \quad (b \in B_{\mathbb{R}}).$$

It is easy to see that $d \in Z^1_{\mathbb{R}}(B_{\mathbb{R}}, \mathfrak{X}^*)$. According to (6.4), there exists $f \in \mathfrak{X}^*$ such that

$$d = d_{B_{\mathbb{R}}, \mathfrak{X}^*, f}. \tag{6.5}$$

Since $B = B_{\mathbb{R}}$, by (6.5) we have

$$\begin{aligned}D(b) = d(b) = d_{B_{\mathbb{R}}, \mathfrak{X}^*, f}(b) &= b \odot f - f \odot b \\ &= b \cdot f - f \cdot b = d_{B, \mathfrak{X}^*, f}(b),\end{aligned}$$

for all $b \in B$. Hence, $D = d_{B, \mathfrak{X}^*, f}$ and so $H^1_{\mathbb{C}}(B, \mathfrak{X}^*) = \{0\}$. Therefore, (i) holds. $\square$

By [16, Remark 1.2.8], it is known that if B is a complex commutative Banach algebra with identity, then

$$\triangle(B_{\mathbb{R}}) = \triangle(B) \cup \{\overline{\varphi} : \ \varphi \in \triangle(B)\}. \tag{6.6}$$

Here, we give an extension of the mentioned result as the following.

Proposition 6.2. *Let $(B, \|\cdot\|)$ be a complex Banach algebra and let $B_{\mathbb{R}}$ denote B regarded as a real algebra. Then*

$$\triangle(B_{\mathbb{R}}) = \triangle(B) \cup \{\overline{\varphi} : \ \varphi \in \triangle(B)\}.$$

Proof . Clearly,

$$\triangle(B) \cup \{\overline{\varphi} : \ \varphi \in \triangle(B)\} \subseteq \triangle(B_{\mathbb{R}}). \tag{6.7}$$

Suppose that $\psi \in \triangle(B_{\mathbb{R}})$. Then $\psi(B)$ is real subalgebra of $\mathbb{C}$ and $\{0\}$ is a proper subset of $\psi(B)$. Thus, $\psi(B) = \mathbb{R}$ or $\psi(B) = \mathbb{C}$. Therefore, $1 \in \psi(B)$ and so there exists $b_1 \in B$ with $\psi(b_1) = 1$. It follows that

$$(\psi(ib_1))^2 = (\psi(ib_1)^2) = \psi(-b_1^2) = -\psi(b_1^2) = -(\psi(b_1))^2 = -1.$$

Therefore, either $\psi(ib_1) = i$ or $\psi(ib_1) = -i$. If $\psi(ib_1) = i$, then for each $b \in B$ we have

$$\psi(ib) = \psi(b_1)\psi(ib) = \psi(ib_1b) = \psi(ib_1)\psi(b) = i\psi(b).$$

This implies that $\psi((\alpha + i\beta)b) = (\alpha + i\beta)\psi(b)$ for all $\alpha, \beta \in \mathbb{R}$ and $b \in B$. Hence, $\psi \in \triangle(B)$. If $\psi(ib_1) = -i$, then by a similar calculation we get $\overline{\psi}(b) = i\overline{\psi}(b)$ for all $b \in B$ which implies that $\overline{\psi} \in \triangle(B)$. Therefore, $\psi \in \triangle(B) \cup \{\overline{\varphi} : \ \varphi \in \triangle(B)\}$. Thus,

$$\triangle(B_{\mathbb{R}}) \subseteq \triangle(B) \cup \{\overline{\varphi} : \ \varphi \in \triangle(B)\}. \tag{6.8}$$

From (6.7) and (6.8), we have

$$\triangle(B_{\mathbb{R}}) = \triangle(B) \cup \{\overline{\varphi} : \ \varphi \in \triangle(B)\},$$

and so the proof is complete. □

Theorem 6.3. *Let $(B, \|\cdot\|)$ be a complex Banach algebra and let $B_{\mathbb{R}}$ denote B regarded as a real algebra. Then B is right character amenable if and only if $B_{\mathbb{R}}$ is right character amenable.*

Proof . We firt assume that B is right character amenable . Let $\varphi \in \triangle(B_{\mathbb{R}})$. Then $\varphi \in \triangle(B)$ or $\overline{\varphi} \in \triangle(B)$ by Proposition 6.2. If $\varphi \in \triangle(B)$, then B is right φ-amenable and so $B_{\mathbb{R}}$ is right φ-amenable by Theorem 6.1. If $\overline{\varphi} \in \triangle(B)$, then B is right $\overline{\varphi}$-amenable and so $B_{\mathbb{R}}$ is right $\overline{\varphi}$-amenable by Theorem 6.1. Therefore, $B_{\mathbb{R}}$ is right φ-amenable by part (ii) of Theorem 2.1. Suppose that $\varphi = 0$. Then B is right 0-amenable and so by Corollary 4.5, $B_{\mathbb{R}}$ is right 0-amenable. Therefore, $B_{\mathbb{R}}$ is right character amenable.

Conversely, we assume that $B_{\mathbb{R}}$ is right character amenable. Let $\varphi \in \triangle(B)$. Then $\varphi \in \triangle(B_{\mathbb{R}})$ and so $B_{\mathbb{R}}$ is right φ-amenable. Hence, B is right φ-amenable by Theorem 6.1. Suppose that $\varphi = 0$. Then $B_{\mathbb{R}}$ is right 0-amenable and so B is right 0-amenable by Corollary 4.5. Therefore, B is right character amenable. □

7. Applications and examples

Applying some results in Sections 2-6 and some known results of character amenability for complex commutative Banach algebras, we obtain the following theorems.

Theorem 7.1. *Let $(A, \|\cdot\|)$ be a commutative real Banach algebra. If A is reflexive and character amenable, then A is finite dimensional.*

Proof . Let A be reflexive and character amenable. Set $A_{\mathbb{C}} = A \times A$. Then $A_{\mathbb{C}}$ with the algebra operations defined by (1.2) is complex algebra which is complexification of A with respect to the injective real algebra homomorphism $J : A \longrightarrow A_{\mathbb{C}}$ defined by $J(a) = (a, 0) \quad a \in A$. Moreover, by [5, Proposition I.1.13], there exists an algebra norm $|||\cdot|||$ on $A_{\mathbb{C}}$ satisfying the $(*)$ condition with the positive constants $k_1 = 1$ and $k_2 = 2$. Hence, $A_{\mathbb{C}}$ is character amenable by part (iii) of Theorem 3.1 and also reflexive Banach space by [1, Lemma 2.3(vii)]. Therefore, $A_{\mathbb{C}}$ is finite dimensional by [11, Theorem 3.5] and so there exists a finite subset $\{(a_1, b_1), \cdots, (a_n, b_n)\}$ of $A_{\mathbb{C}}$ which generates $A_{\mathbb{C}}$. It is easy to see that A is generated by the finite set $\{a_1, b_1, \cdots, a_n, b_n\}$. Hence, A is a finite dimensional real linear space. □

Let $(B, \|\cdot\|)$ be a complex Banach algebra with $\triangle(B) \neq \emptyset$. The relative topology on $\triangle(B)$ induced by weak topology (B^{**}-topology) on B^* is called the weak topology on $\triangle(B)$.

Let $(A, \|\cdot\|)$ be a real Banach algebra with $\triangle(A) \neq \emptyset$. set $A' = \mathcal{B}_{\mathbb{R}}(A, \mathbb{C})$. Then $\triangle(A) \subseteq A'$ and A' is a complex Banach space by Lemma 5.3. The relative topology on $\triangle(A)$ induced by $(A')^*$-topology on A' is called the weak topology on $\triangle(A)$.

Theorem 7.2. *Let $(A, \|\cdot\|)$ be a real Banach algebra and let $\varphi \in \triangle(A)$. If A is left or right φ-amenable, then φ is an isolated point in $\triangle(A)$ with the weak topology.*

Proof . Let A be left φ-amenable. Set $A_{\mathbb{C}} = A \times A$. Then $A_{\mathbb{C}}$ with the algebra operations defined by (1.2) is complex algebra which is a complexification of A with respect to the injective real algebra homomorphism $J : A \longrightarrow A_{\mathbb{C}}$ defined by $J(a) = (a, 0) \quad a \in A$. Moreover by [5, Proposition I.1.13], there exists an algebra norm $|||\cdot|||$ on $A_{\mathbb{C}}$ satisfying the $(*)$ condition with the positive constants $k_1 = 1$ and $k_2 = 2$. Hence, $A_{\mathbb{C}}$ is left φ_C-amenable by Theorem 3.1. Thus, there exist $m \in (A_{\mathbb{C}})^{**}$ such that $m(\varphi_C) = 1$ and $m(\eta) = 0$ for all $\eta \in \triangle(A_{\mathbb{C}}) \setminus \{\varphi_C\}$ by [13, Remark 5.1]. Define the map $\sigma : A' \longrightarrow (A_{\mathbb{C}})^*$ by

$$\sigma(\Lambda)(a, b) = \Lambda(a) + i\Lambda(b) \quad (\Lambda \in A', \quad a, b \in A).$$

Clearly, σ is well-defined and $\sigma(\psi) = \psi_C$ for all $\psi \in \triangle(A)$. It is easy to see that σ is a bounded complex linear mapping. Thus, $\sigma^* : (A_{\mathbb{C}})^{**} \longrightarrow (A')^*$, the adjoint operator of σ, is a complex bounded linear mapping. Therefore, $\sigma^*(m) \in (A')^*$ and

$$\sigma^*(m)(\varphi) = m(\sigma(\varphi)) = m(\varphi_C) = 1.$$

Let $\psi \in \triangle(A) \setminus \{\varphi\}$. Then $\psi_C \in \triangle(A_{\mathbb{C}}) \setminus \{\varphi_C\}$ and so $m(\psi_C) = 0$ Thus,

$$\sigma^*(m)(\psi) = m(\sigma(\psi)) = m(\psi_C) = 0.$$

Therefore, $\triangle(A) \cap (\sigma^*(m))^{-1}(\{0\}) = \triangle(A) \setminus \{\varphi\}$. This implies that $\triangle(A) \setminus \{\varphi\}$ is a closed set in $\triangle(A)$ with the weak topology and so $\{\varphi\}$ is an open set in $\triangle(A)$ with the weak topology. Hence, φ is an isolated point of $\triangle(A)$ with the weak topology. □

The following example shows that the converse of Theorem 7.2 is not true in general.

Example 7.3. Let $S = \mathbb{N} \cup \{0\}$ and define the semigroup operation on S by

$$m * n = \begin{cases} m & if\ n = m \\ 0 & if\ n \neq m \end{cases} \qquad (m, n \in S).$$

The semigroup algebra $l^1(S)$ with the convolution product is a complex commutative Banach algebra with the l^1-norm. It is known that $l^1(S)$ generate by $\{e_m : \quad m \in S\}$, where $e_m = \{e_{m,n}\}_{n=0}^{\infty}$ for all $m \in S$ and

$$e_{m,n} = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases} \qquad (n \in \mathbb{N} \cup \{0\}).$$

Moreover, $\triangle(l^1(S)) = \{\varphi_S\} \cup \{\varphi_t : \ t \in \mathbb{N}\}$. where $\varphi_S(e_m) = 1$, $(m \in S)$ and for each $t \in \mathbb{N}$;

$$\varphi_t(e_m) = \begin{cases} 1 & m = t \\ 0 & m \neq t \end{cases} \qquad (m \in S).$$

Let B be the unitisation of $l^1(S)$ with unit e_B. Then $\triangle(B) = \triangle(l^1(S)) \cup \{\varphi_\infty\}$, where

$$\varphi_\infty(e_m) = 0 \quad (m \in S) \quad \text{and} \quad \varphi_\infty(e_B) = 1.$$

Let $B_\mathbb{R}$ be B regarded as a real Banach algebra. Then $B_\mathbb{R}$ is a commutative real Banach algebra and

$$\triangle(B_\mathbb{R}) = \triangle(B) \cup \{\overline{\varphi} : \ \varphi \in \triangle(B)\}.$$

We claim that φ_∞ is an isolated point in $\triangle(B_\mathbb{R})$ with the weak topology. Define the function $f : B' \longrightarrow \mathbb{C}$ by

$$f(\Lambda) = \Lambda(ie_B) \quad (\Lambda \in B').$$

It is easy to see that $f \in (B')^*$. Suppose that φ_∞ is not an isolated point in $\triangle(B_\mathbb{R})$ with the weak topology. Then there exists a net $\{\varphi_\gamma\}_{\gamma\in\Gamma}$ in $\triangle(B_\mathbb{R}) \setminus \{\varphi_\infty\}$ such that

$$\lim_\gamma \varphi_\gamma = \varphi_\infty \quad (\text{in } \ \triangle(B_\mathbb{R}) \ \text{ with the weak topology}).$$

This implies that

$$\lim_\gamma f(\varphi_\gamma) = f(\varphi_\infty) = \varphi_\infty(ie_B) = i\varphi_\infty(e_B) = i \tag{7.1}$$

On the other hand, $f(\varphi_\gamma) \in \{0, -i\}$ for all $\gamma \in \Gamma$. This implies that $\lim_\gamma f(\varphi_\gamma) \neq i$, which contradicts to (7.1). Hence, our claim is justified.
It is known [8, Example 2.2] that B is not φ_∞-amenable. Therefore, $B_\mathbb{R}$ is not φ_∞-amenable by Theorem 6.1.

In continue we study character amenability of certain real Banach algebras.

Let X be a compact Hausdorff space. We denote by $C_\mathbb{F}(X)$ the set of all continuous $\mathbb{F}$-valued functions on X. Then $C_\mathbb{F}(X)$ is a unital commutative Banach algebra over $\mathbb{F}$ with unit 1_X, the constant function on X with value 1, and with the uniform norm $\|\cdot\|_X$ on X defined by

$$\|f\|_X = \sup\{|f(x)| : \ x \in X\} \qquad (f \in C_\mathbb{F}(X)).$$

We write $C(X)$ instead of $C_\mathbb{C}(X)$. A complex subalgebra B of $C(X)$ is called a *Banach function algebra* on X if B separates the points of X, $1_X \in B$ and B is a unital Banach algebra under an algebra norm $\|\cdot\|$. A complex *uniform algebra* on X is a complex Banach function algebra on X with the uniform norm $\|\cdot\|_X$.

Let B be a Banach function algebra on X. For each $x \in X$, the map $e_{B,x} : B \longrightarrow \mathbb{C}$ defined by $e_{B,x}(f) = f(x)$ $(f \in B)$, is a character of B which is called the *evaluation character* on B at x. B is called *natural* if $\triangle(B) = \{e_{B,x} : \ x \in X\}$. The *Choquet boundary* of B is denoted by $Ch(B, X)$ and definded as the set of all $x \in X$ such that δ_x, the point mass measure on X at x, is the unique probability measure μ on X such that μ is a representing measure for $e_{B,x}$, i.e. $e_{B,x}(f) = \int_X f d\mu$ for all $f \in B$. Hu, Sangani Monfared and Traynor studied character amenability of complex Banach function algebra on compact Housdorff space in [11] and obtained the following results which are useful in the sequel.

Theorem 7.4. [11, Theorem 5.1] *Let B be a complex Banach function algebra on a compact Hausdorff space X. If B is character amenable, then $Ch(B, X) = X$.*

Theorem 7.5. [11, Corollary 5.2] *Let B be a complex natural uniform algebra on a compact Hausdorff space X. Then B is character amenable if and only if $Ch(B, X) = X$.*

Let X be a compact Hausdorff space. A self-map $\tau : X \longrightarrow X$ is called a *topological involution* on X if τ is continuous and $\tau(\tau(x)) = x$ for all $x \in X$. Let $\tau : X \longrightarrow X$ be an topological involution on X. Then the map $\tau^* : C(X) \longrightarrow C(X)$ defined by $\tau^*(f) = \bar{f} \circ \tau \quad (f \in C(X))$, is an algebra involution on $C(X)$ which is called the *algebra involution on $C(X)$ induced by τ*. Set

$$C(X, \tau) = \{f \in C(X) : \quad \tau^*(f) = f\}.$$

Then $C(X, \tau)$ is a self-adjoint real uniformly closed subalgebra of $C(X)$ containing 1_X and separating the points of X. Moreover, $C(X) = C(X, \tau) \oplus iC(X, \tau)$ and

$$\max\{\|f\|_X, \|g\|_X\} \leq \|f + ig\|_X \leq 2\max\{\|f\|_X, \|g\|_X\},$$

for all $f, g \in C(X, \tau)$. Furthermore, $C(X, \tau) = C_{\mathbb{R}}(X)$ if and only if τ is the identity map on X. A real subalgebra A of $C(X, \tau)$ is called a *real Banach function algebra* on (X, τ) if A separates the points of X, $1_X \in A$ and A is a unital real Banach algebra with an algebra norm $\| \cdot \|$ on A. If the norm on real Banach function algebra on A is $\| \cdot \|_X$, then A is called a *real uniform algebra* on X.

Let A be a real Banach function algebra on (X, τ). For each $x \in X$, the map $e_{A,x} : A \longrightarrow \mathbb{C}$ defined by $e_{A,x}(f) = f(x) \quad (f \in A)$ is a character of A which is called evaluation character on X. A is called natural if $\triangle(A) = \{e_{A,X} : \quad x \in X\}$. The *Choquet boundary* of A with respect to (X, τ) is denoted by $Ch(A, X, \tau)$ and defined the set of all $x \in X$ such that m_x is the unique real part representing measure μ for $e_{A,x}$, i.e. $e_{A,x}(f) = f(x) = \int_X f d\mu$ for all $f \in A$, where $m_x = \frac{1}{2}(\delta_x + \delta_{\tau(x)})$.

Here, we study character amenability of real Banach function algebras on (X, τ) as the following.

Theorem 7.6. *Let X be a compact Hausdorff space, let $\tau : X \longrightarrow X$ be a topological involution on X and let $(A, \| \cdot \|)$ be a real Banach function algebra on (X, τ). If A is character amenable, then $Ch(A, X, \tau) = X$.*

Proof . Take $B = \{f + ig : \quad f, g \in A\}$. Then B is a complex function algebra on X, $B = A \oplus iA$ and there exists a complex norm algebra $||| \cdot |||$ on B and $C \geq 1$ such that $|||f||| = \|f\|$ for all $f \in A$ and

$$\max\{\|f\|, \|g\|\} \leq C|||f + ig||| \leq 2C\max\{\|f\|, \|g\|\}$$

for all $f, g \in A$. Then B is a complexification of A with the injective real algebra homomorphism $J : A \longrightarrow B$ defined by $J(f) = f \quad (f \in A)$ and $||| \cdot |||$ satisfies in the $(*)$ condition with $k_1 = C$ and $k_2 = 2C$. Thus, $(B, ||| \cdot |||)$ is a complex Banach function algebra on X. Let A be character amenable. Then B is character amenable by part (iii) of Theorem 3.1. Therefore,

$$Ch(B, X) = X, \tag{7.2}$$

by Theorem 7.4. On the other hand,

$$Ch(B, X) = Ch(A, X, \tau), \tag{7.3}$$

by [4, theorem 16]. From (7.2) and (7.3), we get

$$Ch(A, X, \tau) = X,$$

and so the proof is complete. □

Theorem 7.7. *Let X be a compact Hausdorff space, let $\tau : X \longrightarrow X$ be a topological involution on X, and let $(A, \|\cdot\|)$ be a natural real uniform algebra on (X, τ). Then A is character amenable if and only if $Ch(A, X, \tau) = X$.*

Proof . Take $B = \{f + ig : \quad f, g \in A\}$. By [16, Theorem 1.3.20], B is a complex natural uniform algebra on X, $B = A \oplus iA$ and

$$\max\{\|f\|_X, \|g\|_X\} \leq \|f + ig\|_X \leq 2\max\{\|f\|_X, \|g\|_X\}$$

for all $f, g \in A$. Then B is a complexification of A with the injective real algebra homomorphism $J : A \longrightarrow B$ defined by $J(f) = f \quad (f \in A)$ and $\|\cdot\|_X$ satisfies in the $(*)$ condition with $k_1 = 1$ and $k_2 = 2$. By part (iii) of Theorem 3.1, A is character amenable if and only if B is character amenable. By Theorem 7.5, B is character amenable if and only if

$$Ch(B, X) = X.$$

On the other hand,

$$Ch(B, X) = Ch(A, X, \tau)$$

by [16, Theorem 4.3.7]. Therefore, A is character amenable if and only if $Ch(A, X, \tau) = X$ and so the proof is complete. □

The following example show that in sufficient case of Theorem 7.7, we can not omit the naturality condition on A.

Example 7.8. Let $\mathbb{T} = \{z \in \mathbb{C} : \quad |z| = 1\}$ and $P(\mathbb{T})$ be the set of all $f \in C(\mathbb{T})$ for which f is a uniform limit of a sequence of polynomials with coefficients in $\mathbb{C}$ on $\mathbb{T}$. It is known that $P(\mathbb{T})$ is a complex uniform algebra on $\mathbb{T}$, $Ch(P(\mathbb{T}), \mathbb{T}) = \mathbb{T}$ and $P(\mathbb{T}) \neq C(\mathbb{T})$. By [11, Theorem 5.3], $P(\mathbb{T})$ is not character amenable. Define the map $\tau : \mathbb{T} \longrightarrow \mathbb{T}$ by

$$\tau(z) = \overline{z} \quad (z \in \mathbb{C}).$$

Clearly, τ is a topological involution on $\mathbb{T}$. Moreover, it is easy to see that $\tau^*(P(\mathbb{T})) \subseteq P(\mathbb{T})$. Define

$$A = \{f \in P(\mathbb{T}) : \quad \tau^*(f) = f\}.$$

Then A is a real uniform algebra on $(\mathbb{T}, \tau)$, $P(\mathbb{T}) = A \oplus iA$ and

$$\max\{\|f\|_{\mathbb{T}}, \|g\|_{\mathbb{T}}\} \leq \|f + ig\|_{\mathbb{T}} \leq 2\max\{\|f\|_{\mathbb{T}}, \|g\|_{\mathbb{T}}\}$$

for all $f, g \in A$. Moreover, A is not natural and $Ch(A, \mathbb{T}, \tau) = \mathbb{T}$. Thus, $P(\mathbb{T})$ is a complexification of A with the injective real algebra homomorphism $J : A \longrightarrow P(\mathbb{T})$ defined by $J(f) = f \quad (f \in A)$ and $\|\cdot\|_{\mathbb{T}}$ satisfies in the $(*)$ condition with $k_1 = 1$ and $k_2 = 2$. Therefore, A is not character amenable by Theorem 3.1.F

Let (X, d) be a compact metric space and $\alpha \in (0, 1]$. By $\mathrm{Lip}_{\mathbb{F}}(X, d^\alpha)$, we denote the set of all $\mathbb{F}$-valued functions f on X for which

$$p_{(X,d^\alpha)}(f) = \sup\{\frac{|f(x) - f(y)|}{d^\alpha(x, y)} : \quad x, y \in X, x \neq y\} < \infty.$$

Then $\mathrm{Lip}_{\mathbb{F}}(X, d^\alpha)$ is an algebra over $\mathbb{F}$ containing 1_X and separates the points of X. $\mathrm{Lip}_{\mathbb{F}}(X, d^\alpha)$ is called *Lipschitz algebra* of order α over $\mathbb{F}$. For $\alpha \in (0, 1)$, we denote by $\mathrm{lip}_{\mathbb{F}}(X, d^\alpha)$ the set of all $f \in \mathrm{Lip}_{\mathbb{F}}(X, d^\alpha)$ for which

$$\lim_{d(x,y)\longmapsto 0} \frac{|f(x) - f(y)|}{d^\alpha(x, y)} = 0.$$

Then $\mathrm{lip}_{\mathbb{F}}(X, d^\alpha)$ is a subalgebra of $\mathrm{Lip}_{\mathbb{F}}(X, d^\alpha)$ over $\mathbb{F}$. The algebra $\mathrm{lip}_{\mathbb{F}}(X, d^\alpha)$ is called *little Lipschitz algebra* of order α over $\mathbb{F}$. We know that $\mathrm{Lip}_{\mathbb{F}}(X, d^\beta) \subseteq \mathrm{lip}_{\mathbb{F}}(X, d^\alpha) \subseteq \mathrm{Lip}_{\mathbb{F}}(X, d^\alpha) \subseteq C_{\mathbb{F}}(X)$ whenever $0 < \alpha \leq \beta$. The Lipschitz algebra $\mathrm{Lip}_{\mathbb{F}}(X, d^\alpha)$ and the little Lipschitz algebra $\mathrm{Lip}_{\mathbb{F}}(X, d^\alpha)$ were first introduced by Sherbert in [20, 21]. We write $\mathrm{Lip}(X, d^\alpha)$ ($\mathrm{lip}(X, d^\alpha)$, respectively) instead of $\mathrm{Lip}_{\mathbb{C}}(X, d^\alpha)$ ($\mathrm{lip}_{\mathbb{C}}(X, d^\alpha)$, respectively). It is known that $\mathrm{Lip}(X, d^\alpha)$ is a natural complex Banach function algebra on (X, d) under the algebra Lipschitz norm $\| \cdot \|_{\mathrm{Lip}(X,d^\alpha)}$ defined by

$$\|f\|_{\mathrm{Lip}(X,d^\alpha)} = \|f\|_X + p_{(X,d^\alpha)}(f) \quad (f \in \mathrm{Lip}_{\mathbb{F}}(X, d^\alpha)).$$

Moreover, $\mathrm{lip}_{\mathbb{F}}(X, d^\alpha)$ is a closed subalgebra of $(\mathrm{Lip}(X, d^\alpha), \| \cdot \|_{\mathrm{Lip}(X,d^\alpha)})$ whenever $\alpha \in (0, 1)$. Also, $(\mathrm{lip}(X, d^\alpha), \| \cdot \|_{\mathrm{Lip}(X,d^\alpha)})$ is a natural complex Banach function algebra on (X, d). A self-map $\tau : X \longrightarrow X$ is called a *Lipschitz mapping* on (X, d) if

$$p(\tau) = \sup\{\frac{d(\tau(x), \tau(y))}{d(x, y)} : \quad x, y \in X, \quad x \neq y\} < \infty.$$

A Lipschitz mapping τ on (X, d) is called a *Lipschitz involution* on (X, d) if $\tau(\tau(x)) = x$ for all $x \in X$. It is easy to see that if $\tau : X \longrightarrow X$ is a Lipschitz involution on (X, d), then $\tau^*(\mathrm{Lip}(X, d^\alpha)) = \mathrm{Lip}(X, d^\alpha)$ for $\alpha \in (0, 1]$ and $\tau^*(\mathrm{lip}(X, d^\alpha)) = \mathrm{lip}(X, d^\alpha)$ for $\alpha \in (0, 1)$. Define

$$\mathrm{Lip}(X, d^\alpha, \tau) = \{f \in \mathrm{Lip}(X, d^\alpha) : \ \tau^*(f) = f\} \quad (\alpha \in (0,]),$$
$$\mathrm{lip}(X, d^\alpha, \tau) = \{f \in \mathrm{lip}(X, d^\alpha) : \ \tau^*(f) = f\} \quad (\alpha \in (0, 1)).$$

It is known [2, Theorem 2.7] that if $B = \mathrm{Lip}(X, d^\alpha)$ ($B = \mathrm{lip}(X, d^\alpha)$, respectively) and $A = \mathrm{Lip}(X, d^\alpha, \tau)$ ($A = \mathrm{lip}(X, d^\alpha, \tau)$, respectively), then $B = A \oplus iA$,

$$\begin{aligned}\max\{\|f\|_{\mathrm{Lip}(X,d^\alpha)}, \|g\|_{\mathrm{Lip}(X,d^\alpha)}\} &\leq (p(\tau))^\alpha \|f + ig\|_{\mathrm{Lip}(X,d^\alpha)} \\ &\leq 2(p(\tau))^\alpha \max\{\|f\|_{\mathrm{Lip}(X,d^\alpha)}, \|g\|_{\mathrm{Lip}(X,d^\alpha)}\}\end{aligned}$$

for all $f, g \in A$ and $(A, \| \cdot \|_{\mathrm{Lip}(X,d^\alpha)})$ is a natural real Banach function algebra on $((X, d), \tau)$. The real Lipschitz algebras $\mathrm{Lip}(X, d^\alpha, \tau)$ and $\mathrm{lip}(X, d^\alpha, \tau)$ were first introduced in [2].

Theorem 7.9. *Let (X, d) be a compact metric space, let $\tau : X \longrightarrow X$ be a Lipschitz involution on (X, d) and let $A = \mathrm{Lip}(X, d^\alpha, \tau)$ for $\alpha \in (0, 1]$ or $A = \mathrm{lip}(X, d^\alpha, \tau)$ for $\alpha \in (0, 1)$.*

(i) *If $x \in X$, then A is $e_{A,x}$-amenable if and only if x is an isolated point in (X, d).*
(ii) *A is character amenable if and only if X is finite.*

Proof . (i) Let $B = \mathrm{Lip}(X, d^\alpha)$ for $\alpha \in (0, 1]$ ($B = \mathrm{lip}(X, d^\alpha)$ for $\alpha \in (0, 1)$, respectively). Then $(B, \| \cdot \|_{\mathrm{Lip}(X,d^\alpha)})$ is a natural complex Banach function algebra on (X, d) and B is a complexification of A with the injective real algebra homomorphism $J : A \longrightarrow B$ defined by $J(f) = f \quad (f \in A)$ and

$$\begin{aligned}\max\{\|f\|_{\mathrm{Lip}(X,d^\alpha)}, \|g\|_{\mathrm{Lip}(X,d^\alpha)}\} &\leq (p(\tau))^\alpha \|f + ig\|_{\mathrm{Lip}(X,d^\alpha)} \\ &\leq 2(p(\tau))^\alpha \max\{\|f\|_{\mathrm{Lip}(X,d^\alpha)}, \|g\|_{\mathrm{Lip}(X,d^\alpha)}\}\end{aligned}$$

for all $f, g \in A$. According to $e_{B,x} = (e_{A,x})_C$, we deduce that A is $e_{A,x}$-amenable if and only if B is $e_{B,x}$-amenable by part (i) of Theorem 3.1. Since B is a natural complex Banach function algebra on X contained in $\mathrm{Lip}(X, d^\alpha)$, by [8, Theorem 2.6], B is $e_{B,x}$-amenable if and only if x is an isolated point of X. Hence, (i) holds.
(ii) By part (iii) of Theorem 3.1, A is character amenable if and only if B is character amenable. On the other hand, by [8, Corollary 2.7], B is character amenable if and only if X is finite. Hence, (ii) holds. □

Let G be locally compact group. We denote by $M(G)$ the set of all complex Borel measures on G. It is known that $M(G)$ is a complex Banach algebra with the norm

$$\|\mu\| = |\mu|(G) \qquad (\mu \in M(G)).$$

Let λ be a left Haar measure on G and $L^1(G) = L^1(G, \lambda)$, the group algebra on G with respect to measure λ, equipped the $L^1(G)$-norm

$$\|f\|_{L^1(G)} = \int_G |f| \, d\lambda \qquad (f \in L^1(G)).$$

A map $\tau : G \longrightarrow G$ is called a *topological group involution* on G if τ is a continuous group automorphism on G and $\tau(\tau(x)) = x$ for all $x \in G$.

Let G be locally compact group and let $\tau : G \longrightarrow G$ be a topological group involution on G. It is easy to see that $\mu \circ \tau \in M(G)$ for all $\mu \in M(G)$. Define

$$M(G, \tau) = \{\mu \in M(G) : \ \mu \circ \tau = \bar{\mu}\}.$$

It is shown [7, Proposition 2.2] that $M(G, \tau)$ is a closed real subalgebra of $M(G)$, $M(G) = M(G, \tau) \oplus iM(G, \tau)$ and

$$\max\{\|\mu\|, \|\nu\|\} \le \|\mu + i\nu\| \le 2\max\{\|\mu\|, \|\nu\|\},$$

for all $\mu, \nu \in M(G, \tau)$. Let λ be a Haar measure on G. By [7, Theorem 2.4], $\lambda \circ \tau = \lambda$. Define

$$L^1(G, \tau) = \{f \in L^1(G) : \ f \circ \tau = \bar{f}\}.$$

By [7, Theorem 2.5], $L^1(G, \tau)$ is a closed real subalgebra of $L^1(G)$,

$$L^1(G) = L^1(G, \tau) \oplus iL^1(G, \tau),$$

and

$$\max\{\|f\|_{L^1(G)}, \|g\|_{L^1(G)}\} \le \|f + ig\|_{L^1(G)} \le 2\max\{\|f\|_{L^1(G)}, \|g\|_{L^1(G)}\},$$

for all $f, g \in L^1(G, \tau)$.

Theorem 7.10. *Let G be a locally compact group and let $\tau : G \longrightarrow G$ be a topological group involution on G. Then the following assertions are equivalent.*

(i) *$L^1(G, \tau)$ is left character amenable.*
(ii) *$L^1(G, \tau)$ is right character amenable.*
(iii) *G is amenable.*

Proof . Since $L^1(G) = L^1(G,\tau) \oplus iL^1(G,\tau)$,

$$\max\{\|f\|_{L^1(G)}, \|g\|_{L^1(G)}\} \le \|f+ig\|_{L^1(G)} \le 2\max\{\|f\|_{L^1(G)}, \|g\|_{L^1(G)}\},$$

for all $f, g \in L^1(G,\tau)$, we deduce that $L^1(G)$ is left (right, respectively) character amenable if and only if $L^1(G,\tau)$ is left (right, respectively) character amenable by part (ii) of Theorem 3.1. On the other hand, G is amenable if and only if $L^1(G)$ is left (right, respectively) character amenable by [18, Corollary 2.4]. Therefore, the result holds. □

Theorem 7.11. *Let G be a locally compact group and let $\tau : G \longrightarrow G$ be a topological group involution on G. Then $M(G,\tau)$ is character amenable if and only if G is a discrete amenable group.*

Proof . Since $M(G) = M(G,\tau) \oplus iM(G,\tau)$,

$$\max\{\|\mu\|, \|\nu\|\} \le \|\mu + i\nu\| \le 2\max\{\|\mu\|, \|\nu\|\},$$

for all $\mu, \nu \in M(G,\tau)$, we deduce that $M(G,\tau)$ is character amenable if and only if $M(G)$ is character amenable by part (iii) of Theorem 3.1. Therefore, the result holds by [18, Corollary 2.5]. □

Let G be locally compact group, λ be a left Haar measure on G and $L^1(G) = L^1(G,\lambda)$. Let $\tau : G \longrightarrow G$ be a topological group involution on G. Since $L^1(G)$ is a complexification of $L^1(G,\tau)$ with respect to the injective real algebra homomorphism $J : L^1(G,\tau) \longrightarrow L^1(G)$ defined by $J(f) = f \quad (f \in L^1(G,\tau))$ and

$$\max\{\|f\|_{L^1(G)}, \|g\|_{L^1(G)}\} \le \|f+ig\|_{L^1(G)} \le 2\max\{\|f\|_{L^1(G)}, \|g\|_{L^1(G)}\},$$

for all $f, g \in L^1(G,\tau)$, by [1, Lemmas 2.3 and 2.4], $((L^1(G))^{**}, \Box)$ is a complexification of $((L^1(G,\tau))^{**}, \Box)$ with respect to the injective algebra homomorphism $J_2 : (L^1(G,\tau))^{**} \longrightarrow (L^1(G))^{**}$ defined by $J_2(\Phi) = \Phi_C \quad (\Phi \in (L^1(G,\tau))^{**})$ and

$$\max\{\|\Phi\|, \|\Psi\|\} \le 4\|J_2(\Phi) + iJ_2(\Psi)\| \le 32\max\{\|\Phi\|, \|\Psi\|\},$$

for all $\Phi, \Psi \in (L^1(G,\tau))^{**}$.

Applying part (iii) of Theorem 3.1 and [11, Theorem 3.10], we get the following result.

Theorem 7.12. *Let G be locally compact group with a left Haar measure λ, let $L^1(G) = L^1(G,\lambda)$ and let $\tau : G \longrightarrow G$ be a topological group involution on G. Then $(L^1(G,\tau)^{**}, \Box)$ is character amenable if and only if G is finite.*

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