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Inertial approximation method for split variational inclusion problem in Banach spaces

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Abstract

In this paper, we introduce a new iterative algorithm of inertial form for approximating the solution of Split Variational Inclusion Problem (SVIP) involving accrective operators in Banach space. Motivated by the inertial technique, we incorporate the inertial term to accelerate the convergence of the proposed method. Under standard and mild assumption of monotonicity of the SVIP associated mappings, we establish the weak convergence of the sequence generated by our algorithm. Some applications and numerical example are presented to illustrate the performance of our method as well as comparing it with the non-inertial version.

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1. Introduction

Let H be a real Hilbert space, $\langle \cdot, \cdot \rangle$ an inner product and $||\cdot||$ the corresponding norm on H.. An operator $Q : H \to 2^H$ is said to be monotone if $\langle u - v, x - y\rangle \geq 0$ for all $u \in Q(x)$, $v \in Q(y)$. It is said to be maximal monotone if, in addition, the graph $G(Q)$ of Q is not properly contained in the graph of any other monotone mapping i.e $(x, u) \in H \times H, \langle x - y, u - v\rangle \geq 0$ for every $(y, v) \in G(Q)$ implies $u \in Q(x)$. A single-valued operator $Q : H \to H$ is called α inverse-strongly monotone if there exists a positive real number α such that $\langle Qx - Qy, x - y\rangle \geq \alpha||x - y||^2$ for all $x, y \in H$, see

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[21, 22, 23]. It is well known that for each $x \in H$ and $\lambda > 0$, there is a unique $z \in H$ such that $x \in (I + \lambda Q)z$, where Q is a maximal monotone operator and I is an identity operator on H. The single-valued operator $J_\lambda^Q = (I + \lambda Q)^{-1}$ is called the resolvent of Q of parameter λ. It is a firmly nonexpansive and nonexpansive mapping which is everywhere defined and satisfies $z = J_\lambda^Q z$ if and only if $0 \in Qz$.

We recall that a real-valued mapping h on H is lower semi-continous if $h(x) \le \liminf_{n\to\infty} h(x_n)$ for all sequence $\{x_n\} \subset X$ such that $x_n \to x$ (strongly). Similarly h is weakly sequentially lower semi-continuous (weakly lsc) if $h(x) \le \liminf_{n\to\infty} h(x_n)$ for all sequence $\{x_n\} \subset X$ such that $x_n \rightharpoonup x$ (weakly).

1.1. Splitting method for sum of accretive mappings

Splitting method have received more attention recently due to the fact that many nonlinear problems arising in applied areas such as image recovery, machine learning and signal processing can be mathematically modelled as a nonlinear operator equation, which in turn can be further decomposed into the sum of possibly simpler nonlinear operators. Splitting method for linear equations were introduced by Peaceman and Rachford [39] and Douglas and Rachford [16]. Extension to Hilbert spaces were carried out by Kellog [26], Lions and Mercier [27]. The defining problem is to iteratively find a zero of the sum of two monotone operators T_1 and T_2 in Hilbert space H, that is the solution to the inclusion problem

$$0 \in (T_1 + T_2)x. \tag{1.1}$$

Many problems in real life can be formulated as (1.1). A prominent example is the stationary solution to the initial value problem of the evolution

$$\frac{\partial u}{\partial t} + Fu \ni 0, \quad u(0) = u_0,$$

where the governing maximal monotone F is of the form $T_1 + T_2$. This problem models the optimization problem

$$\min_{x\in H}\{f(x) + gT(x)\}, \tag{1.2}$$

where f, g are proper lower semicontinuous functions from H to the extended real line $\mathbb{R} = \{-\infty, +\infty\}$ and T is a bounded linear operator on H. The minimization problem (1.2) is widely used in image recovery, machine learning and signal processing. A splitting method for solving (1.1) involves an iterative algorithm for which each iteration involves only with the individual operators T_1 and T_2, but not the sum $T_1 + T_2$ concurrently. To solve (1.1), Lions and Mercier [27] introduced the nonlinear Peaceman-Rachford and Douglas-Rachford which generate a sequence $\{x_n\}$ by the recursion formula $x_{n+1} = (2J_\lambda^{T_1} - I)(2J_\lambda^{T_2} - I)x_n$ and a sequence $\{x_n\}$ generated by $x_{n+1} = J_\lambda^{T_1}(2J_\lambda^{T_2} - I)x_n + (I - J_\lambda^{T_2})x_n$, where $J_\lambda^{T_1}$ denotes the resolvent of the monotone operator T_1. Of the two recursion formula, the Douglas-Rachford algorithm always converges in the weak topology to a point y^* and $y^* = J_\lambda^{T_2}x$ is a solution of (1.1), since the generating operator $J_\lambda^{T_1}(2J_\lambda^{T_2} - I) + (I - J_\lambda^{T_2})$ for this algorithm is firmly nonexpansive. The Peaceman-Rachford algorithm however fails to converge even in the weak topology in the infinite dimensional settings.

1.2. Split monotone variational inclusion

In 2011, Moudafi [31] introduced the following Split Monotone Variational Inclusion Problem (SMVIP): Find $x^* \in H_1$, such that

$$\begin{cases} 0 \in (T_1(x^*) + S_1(x^*)) \\ y^* = Ax^* \in H_2 : 0 \in (T_2(y^*) + S_2(y^*)), \end{cases} \tag{1.3}$$

where $T_1 : H_1 \to 2^{H_1}$ and $T_2 : H_2 \to 2^{H_2}$ are set-valued maximal monotone mappings, $S_1 : H_1 \to H_1$ and $S_2 : H_2 \to H_2$ are single valued monotone operators and $A : H_1 \to H_2$ is a bounded linear operator. In [31], Moudafi obtained a weak convergence theorem for approximating the solution SMVIP in Hilbert spaces.

The SMVIP includes as special cases, split common fixed points problem, split variational inequality problem, the split feasibility problem and split zero problem. All of which have been studied by several authors (see [6, 15, 19, 30, 35, 36] and the references therein).

Very recently, Zhang and Wang [47] proposed an iterative algorithm and proved that the algorithm converges weakly and strongly to a split common fixed point problem for nonexpansive semigroups in Banach spaces under some suitable conditions. Precisely, they proved the following theorem:

Theorem 1.1. *Let X_1 be a real uniformly convex and 2-uniformly smooth Banach space satisfying Opial's condition and with the best smoothness constant k satisfying $0 < k < \frac{1}{\sqrt{2}}$, X_2 be a real Banach space, $A : X_1 \to X_2$ be a bounded linear operator, and A^* be the adjoint of A. Let $\{S(t) : t \geq 0\} : X_1 \to X_1$ be a uniformly asymptotically regular nonexpansive semigroup with $\mathcal{C} := \cap_{t\geq 0} F(S(t)) \neq \emptyset$ and $\{T(t) : t \geq 0\} : X_2 \to X_2$ be a uniformly asymptotically regular nonexpansive semigroup with $\mathcal{Q} := \cap_{t\geq 0} F(T(t)) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by: $x_1 \in X_1$*

$$\begin{cases} z_n = x_n + \gamma J_1^{-1} A^* J_2 (T(t_n) - I) A x_n, \\ x_{n+1} = (1-\alpha_n) z_n + \alpha_n S(t_n) z_n, \quad \forall \ \ n \geq 1, \end{cases} \tag{1.4}$$

where $\{t_n\}$ is sequence of real numbers, $\{\alpha_n\}$ a sequence in $(0,1)$ and γ is a positive constant satisfying

(1) $t_n > 0$ and $\lim\limits_{n\to\infty} t_n = \infty$;

(2) $\liminf\limits_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $0 < \gamma < \frac{1-2k^2}{||A||^2}$.

(I) If $\Gamma = \{p \in C : Ap \in Q\} \neq \emptyset$, then $\{x_n\}$ converges weakly to a split common fixed point $x^* \in \Gamma$.

(II) In addition, if $\Gamma = \{p \in C : Ap \in Q\} \neq \emptyset$, and there is at least one $S(t) \in \{S(t) : t \geq 0\}$ which is semi-compact, then $\{x_n\}$ converges strongly to a split common fixed point $x^* \in \Gamma$.

1.3. Inertial technique

Polyak [40] proposed an inertial extrapolation as an acceleration process for solving the smooth convex minimization problem. It is based on the heavy ball method of the second order dynamical system with friction:

$$\ddot{x}(t) + \alpha_1 \dot{x}(t) + \alpha_2 \nabla(x(t)) = 0,$$

where $\alpha_1, \alpha_2 > 0$ are friction parameters. The inertial algorithm is given as a two step iterative method which is written as

$$\begin{cases} y_n = x_n + \alpha_n(x_n - x_{n-1}) \\ x_{n+1} = y_n - \lambda_n \nabla f(x_n), \quad n \geq 1, \end{cases} \tag{1.5}$$

where $f : H \to \mathbb{R}$ is a smooth convex function, $\alpha_n \in (0,1)$ is an extrapolation factor and $\lambda_n > 0$ is a stepwise positive parameter which has to be sufficiently small. The inertial term in (1.5) is introduced as a means of speeding up the rate of convergence properties of the scheme. This is due to the fact that the new iterate is given by taking a step which is a combination of the direction $x_n - x_{n-1}$ and the current anti-gradient $-\nabla f(x_n)$. Because of this increase in the speed of convergence rates of iterative algorithms, there have been an increasing interest in the study of inertial type iterative schemes (see e.g [1, 10, 13, 23, 29]). Moreover the acceleration scheme developed by Nestrov [34] improves the theoretical rate of convergence of forward-backward method from the standard $\mathcal{O}(k^{-1})$ down to $\mathcal{O}(k^{-2})$ and the inertial extrapolation scheme of Nestorov's accelerated forward-backward method which is actually $o(k^{-2})$ rather than the $\mathcal{O}(k^{-2})$ see [7].
Alvarez and Attouch [8], applied the idea of the heavy ball method to the setting of a general maximal monotone operator using the proximal point algorithm. They came up with the following inertial proximal point algorithm.

$$\begin{cases} y_n = x_n + \alpha_n(x_n - x_{n-1}), \\ x_{n+1} = (I + r_n T)^{-1} y_n, \quad n \geq 1. \end{cases} \tag{1.6}$$

They proved a weak convergence theorem using (1.6) to a zero of the maximal monotone operator T with conditions that $\{r_n\}$ is nondecreasing and $\alpha_n \in (0,1)$ is such that $\sum_{n\geq 0} \alpha_n ||x_n - x_{n-1}||^2 < \infty$. Moudafi and Oliny [33] improved on Algorithm (1.6) by introducing an additional single-valued co-coercive, and Lipschitz continuous operator S into the inertial proximal point algorithm as:

$$\begin{cases} y_n = x_n + \alpha_n(x_n - x_{n-1}), \\ x_{n+1} = (I + r_n T)^{-1}(I - r_n S) y_n, \quad n \geq 1. \end{cases} \tag{1.7}$$

They obtained a weak convergence theorem for finding the zero of the sum $(S + T)$ provided that the conditions given on the parameters in (1.6) are satisfied.
It is worthy of mention that most of the works involving the inertial extrapolation were carried out in Hilbert spaces, in which weak convergence results were obtained in most cases.
Inspired by the ongoing research interest in inertial extrapolation, we consider the following split variational inclusion problem involving accretive operators: Let X_1 and X_2 be Banach spaces. The split variational inclusion problem for accretive operators is given as: Find $x_1 \in X_1$ such that

$$\begin{cases} 0 \in X_1 : x^* \in (T_1 + S_1), \\ y^* = Ax^* \in X_2 : y^* \in (T_2 + S_2), \end{cases} \tag{1.8}$$

where $T_1 : X_1 \to 2^{X_1}, T_2 : X_2 \to 2^{X_2}$ are set-valued accretive operators, $S_1 : X_1 \to X_1$, $S_2 : X_2 \to X_2$ are inverse strongly accretive operators and $A : X_1 \to X_2$ is a bounded linear operator.
Furthermore, we introduce an inertial-type iterative scheme and prove a weak convergence theorem of the scheme to the solution of (1.8).

2. Preliminaries

Let X be a Banach space with the dual X^*, and let $\langle \cdot, \cdot \rangle$ denotes the duality pairing between the elements of X and X^*. Let $B = \{x \in X : ||x|| = 1\}$ be the unit sphere in X. Then, X is said to be strictly convex, if for any $x, y \in B$,

$$x \neq y \text{ implies } \frac{||x + y||}{2} < 1. \tag{2.1}$$

Define a function $\delta : [0,2] \to [0,1]$ called the modulus of convexity as follows:

$$\delta(\epsilon) = \inf \big\{ 1 - \frac{||x+y||}{2} : x, y \in X, ||x|| = ||y||, ||x-y|| \geq \epsilon \big\}, \tag{2.2}$$

for all $\epsilon \in [0,2]$. X is said to be uniformly convex if and only if $\delta(\epsilon) > 0$ for all $\epsilon \in [0,2]$. The Banach space X is said to be smooth if the limit

$$\lim_{t \to 0} \frac{||x+ty|| - ||x||}{t} \tag{2.3}$$

exists for every $x, y \in B$. It is said to be uniformly smooth if the limit (2.3) is attained uniformly for $x, y \in X$, (see [44]). The norm of X is said to be Fréchet differentiable if, for each $x \in X$, the limit (2.3) exists and is attained uniformly for all y such that $||y|| = 1$. It is therefore trivial that a uniformly smooth Banach space is Fréchet differentiable. The normalized duality mapping J from X to 2^{X^*} is defined by

$$Jx = \{x^* \in X^* : \langle x, x^* \rangle = ||x||^2, ||x|| = ||x^*||\}, \quad \forall \;\; x \in X. \tag{2.4}$$

It is widely known that J is single-valued and norm-to-norm uniformly continuous on each bounded subsets of X if X is a real smooth and uniformly convex Banach space, (see [37, 42]). Let X be a Banach space and C a nonempty, closed and convex subset of X. An operator $T : C \to C$ is said to be L-Lipschitz if there exists a constant $L > 0$ such that

$$||Tx - Ty|| \leq L||x - y|| \;\; \forall \;\; x, y \in C. \tag{2.5}$$

In particular, if $L = 1$ then the operator T is nonexpansive. We denote by $F(T)$ the set of fixed points of T, that is $F(T) = \{x \in C : Tx = x\}$. It is known that if T is nonexpansive then $F(T) \neq \emptyset$. A set-valued mapping $G : X \to 2^X$, with domain $D(G)$ and range $R(G)$, is said to be accretive if, for all $t > 0$ and every $x, y \in D(G) \subset X$,

$$||x - y|| \leq ||x - y + t(u - v)||, \;\; u \;\; \in Gx, \;\; v \;\; \in Gy. \tag{2.6}$$

An equivalent definition for the accretive operator was derived by Kato [25], that is G is said to be accretive if and only if, for each $x, y \in D(G)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle u - v, j(x - y) \rangle \geq 0, \;\; u \in Gx, \;\; v \in Gy. \tag{2.7}$$

In addition, an operator G is said to be m-accretive if it is accretive and $R(I + rG)$ is X for all $r > 0$. Given $\alpha > 0$ and $q \in (0, \infty)$, we say that an accretive operator G is α-inversely strongly accretive (α-isa) of order q, if for each $x, y \in D(G)$, there exists $j_q(x - y) \in J_q(x - y)$ such that

$$\langle u - v, j_q(x - y) \rangle \geq \alpha ||u - v||^q, \;\; u \in Gx, \;\; v \in Gy. \tag{2.8}$$

For $q = 2$, we simply say G is α-isa, that is G is α-isa if,

$$\langle u - v, j(x - y) \rangle \geq \alpha ||u - v||^2, \;\; u \in Gx, \;\; v \in Gy. \tag{2.9}$$

We remark that every α- inverse strongly accretive operator T is accretive and $\frac{1}{\alpha}$-Lipschitz. For more on accretive operators see [2] and the references contained therein. If G is accretive, then we can define a nonexpansive single-valued mapping $J_{\lambda_n} : R(I + \lambda_n G) \to D(G)$ for each nondecreasing $\lambda_n > 0$ by $J_{\lambda_n} := (I + \lambda_n G)^{-1}$, which is called the resolvent of G of parameter λ_n. Denote by $G^{-1}(0)$ the set of zero of G; that is $G^{-1}(0) := \{x \in D(G) : 0 \in Gx\}$. It is well known that $F(J^G_{\lambda_n}) = G^{-1}(0)$. The following results will be useful in this sequel:

Lemma 2.1. *[29] Let* $\{\phi_n\} \subset [0,\infty)$ *and* $\{\delta_n\} \subset [0,\infty)$ *satisfying:*

(1) $\phi_{n+1} - \phi_n \leq \theta_n(\phi_n - \phi_{n-1}) + \delta_n$,

(2) $\sum \delta_n < \infty$,

(3) $\theta_n \subset [0,\theta]$, *where* $\theta \in [0,1)$.

Then ϕ_n *is a convergent sequence and* $\sum[\phi_{n+1} - \phi_n]_+ < \infty$, *where* $[t]_+ := \max\{t, 0\}$ *for any* $t \in \mathbb{R}$.

Lemma 2.2. *[46] Given a number* $r > 0$. *A real Banach space* X *is uniformly convex if and only if there exists a continuous strictly increasing function* $g : [0,\infty) \to [0,\infty)$ *with* $g(0) = 0$ *such that*

$$||\lambda x + (1-\lambda)y||^2 \leq \lambda||x||^2 + (1-\lambda)||y||^2 - \lambda(1-\lambda)g(||x-y||),$$

for all $x, y \in X$, $\lambda \in [0,1]$, *with* $||x|| < r$ *and* $||y|| < r$.

Recall that a Banach space X is said to satisfy the Opial's condition, if whenever $\{x_n\}$ is a sequence in X which converges weakly to x as $n \to \infty$, then

$$\limsup_{n\to\infty} ||x_n - x|| < \limsup_{n\to\infty} ||x_n - y||, \quad \forall \ \ y \in X, \quad y \neq x. \tag{2.10}$$

Lemma 2.3. *[17] Let* X *be a uniformly convex Banach space, let* C *be a nonempty closed convex subset of* X *and let* $T : X \to X$ *be a nonexpansive mapping. Then* $(I - T)$ *is demiclosed at zero.*

Lemma 2.4. *[14] Let* X *be a real Banach space with Fréchet differentiable norm. For* $x \in X$, *let* $\beta^*(t)$ *be defined for* $t \in (0,\infty)$ *by*

$$\beta^*(t) = \sup\left\{\left|\frac{||x+ty||^2 - ||x||^2}{t} - 2\langle y, j(x)\rangle\right| : ||y|| = 1\right\}.$$

Then, $\lim_{t\to 0^+} \beta^*(t) = 0$ *and*

$$||x+h||^2 \leq ||x||^2 + 2\langle h, j(x)\rangle + ||h||\beta^*||h|| \tag{2.11}$$

for all $h \in X\{0\}$.

Remark 2.5. *In Lemma 2.4 we will assume* $\beta^*(t) \leq ct$, $t > 0$ *for some* $c > 1$. *It is easy therefore to obtain the following estimate*

$$2\langle h, j(x)\rangle \leq ||x||^2 + c||h||^2 - ||x-h||^2,$$

(2.12)

by replacing h *in* (2.11) *by* $-h$.

Lemma 2.6. *[28] Let* X *be a real Banach space. Let* $T_1 : X \to 2^X$ *be an m-accretive operator and* $S_1 : X \to X$ *be an* α*-inverse strongly accretive mapping on* X. *Then we have*

(i) *for* $\lambda > 0, F(Q_\lambda) = (T_1 + S_1)^{-1}(0)$,

(ii) *for* $0 < \lambda < \mu$ *and* $x \in X$, $||x - Q_\lambda x|| \leq 2||x - Q_\mu x||$,

where $Q_\lambda = J_\lambda^{T_1}(I - \lambda S_1) = (I + \lambda T_1)^{-1}(I - \lambda S_1)$.

Lemma 2.7. *[46] Let X be a 2-uniformly smooth Banach space with the best of smoothness constants $k > 0$. Then the following inequality holds:*

$$||x + y||^2 \leq ||x||^2 + 2\langle j(x), y\rangle + 2k^2||y||^2. \;\; \forall \;\; x, y \in X. \tag{2.13}$$

In this sequel we shall use the following notations $P_{\lambda_n} := J_{\lambda_n}^{T_2}(I - \lambda_n S_2) = (I + \lambda_n T_2)^{-1}(I - \lambda_n S_2)$ and $Q_{\lambda_n} := J_{\lambda_n}^{T_1}(I - \lambda_n S_1) = (I + \lambda_n T_1)^{-1}(I - \lambda_n S_1)$, where T_1, S_1, T_2 and S_2 are as defined in (1.8).

3. Main results

In this section, we give our main results.

Lemma 3.1. *Let X_1 be real uniformly convex and 2-uniformly smooth Banach spaces, X_2 a real Banach space with Féchet differentiable norm, $A : X_1 \to X_2$ a bounded linear operator and A^* the adjoint of A. Let $T_1 : X_1 \to 2^{X_1}, T_2 : X_2 \to 2^{X_2}$ be set-valued accretive operators and $S_1 : X_1 \to X_1, S_2 : X_2 \to X_2$ be α-inverse strongly accretive operators. Assume $\Gamma := \{q \in (T_1 + S_1)^{-1}(0) : Aq \in (T_2 + S_2)^{-1}(0)\} \neq \emptyset$. Let $\{\lambda_n\}$ be a sequence of non-negative real numbers, for $x_1 \in X_1$, let $\{x_n\}$ be a sequence given by*

$$\begin{cases} u_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = u_n + \gamma J_1^{-1} A^* J_2(P_{\lambda_n} - I)Au_n, \\ x_{n+1} = \alpha_n y_n + (1 - \alpha_n)Q_{\lambda_n} y_n, \;\; \forall \;\; n \geq 1, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$, γ is a positive constant and $\theta_n \subset [0, \theta]$ where $\theta \in [0, 1)$ satisfying the following conditions:

(1) $\sum_{n\geq 0} \theta_n ||x_n - x_{n-1}||^2 < \infty$;

(2) $0 < \gamma < \frac{1-2k^2}{||A||^2}$, *where k is the smoothness constant satisfying* $0 < k^2 < \frac{1}{2}$;

(3) $\lambda_n \in (0, \frac{2\alpha}{c}), \;\; \forall \;\; n \geq 1, \quad c > 1$.

Then $\{x_n\}$ is bounded.

Proof . For each $n \geq 1$, let $Q_{\lambda_n}^{T_1} := J_{\lambda_n}^{T_1}(I - \lambda_n S_1)$ and fix $q \in \Gamma$, then $q \in (T_1 + S_1)$ and $Aq \in (T_2 + S_2)$. For all $x, y \in X_1$, using the nonexpansivity of J_{λ_n} and Lemma 2.4, we have

$$\begin{aligned} ||Q_{\lambda_n}x - Q_{\lambda_n}y||^2 &= ||J_{\lambda_n}^{T_1}(I - \lambda_n S_1)x - J_{\lambda_n}^{T_1}(I - \lambda_n S_1)y||^2 \\ &\leq ||x - y - \lambda_n(s_1 x - S_1 y)||^2 \\ &\leq ||x - y|| - 2\lambda_n\langle S_1 x - S_1 y, j(x - y)\rangle + c\lambda_n^2||S_1 x - S_1 y||^2 \\ &\leq ||x - y||^2 - \lambda_n(2\alpha - c\lambda_n)||S_1 x - S_2 y||^2 \\ &\leq ||x - y||^2. \end{aligned} \tag{3.2}$$

Thus, Q_{λ_n} is nonexpansive for all $n \geq 1$. Similary, P_{λ_n} is nonexpansive.

So, it follows from (3.1) and Lemma 2.2, that

$$\begin{aligned} ||x_{n+1} - q||^2 &= ||\alpha_n(y_n - p) + (1-\alpha_n)(Q_{\lambda_n} y_n - q)||^2 \\ &\leq \alpha_n ||y_n - q||^2 + (1-\alpha_n)||Q_{\lambda_n} y_n - q||^2 - \alpha_n(1-\alpha_n) g(||y_n - Q_{\lambda_n} y_n||) \\ &\leq \alpha_n ||y_n - q||^2 + (1-\alpha_n)||y_n - q||^2 - \alpha_n(1-\alpha_n) g(||y_n - Q_{\lambda_n} y_n||) \\ &\leq ||y_n - q||^2 - \alpha_n(1-\alpha_n) g(||y_n - Q_{\lambda_n} y_n||). \end{aligned} \tag{3.3}$$

Again, from (3.1) and Lemma 2.7, we have

$$\begin{aligned} ||y_n - q||^2 &= ||(u_n - q) + \gamma J_1^{-1} A^* J_2(P_{\lambda_n} - I)Au_n||^2 \\ &\leq ||\gamma J_1^{-1} A^* J_2(P_{\lambda_n} - I)Au_n||^2 + 2\gamma\langle u_n - q, A^* J_2(P_{\lambda_n} - I)Au_n\rangle + 2k^2||u_n - q||^2 \\ &\leq \gamma^2 ||A||^2 ||(P_{\lambda_n} - I)Au_n||^2 + 2\gamma\langle u_n - q, A^* J_2(P_{\lambda_n} - I)Au_n\rangle + 2k^2||u_n - q||^2 \\ &\leq \gamma^2 ||A||^2 ||(P_{\lambda_n} - I)Au_n||^2 + 2\gamma\langle Au_n - Ap, J_2(P_{\lambda_n} - I)Au_n\rangle + 2k^2||u_n - q||^2 \\ &= \gamma^2 ||A||^2 ||(P_{\lambda_n} - I)||^2 + 2\gamma\langle Au_n - P_{\lambda_n} Au_n + P_{\lambda_n} Au_n - Ap, J_2(P_{\lambda_n} - I)Au_n\rangle + 2k^2||u_n - q||^2 \\ &= \gamma^2 ||A||^2 ||(P_{\lambda_n} - I)Au_n||^2 \\ &+ 2\gamma\langle P_{\lambda_n} Au_n - P_{\lambda_n} Ap, J_2(P_{\lambda_n} - I)Au_n\rangle - 2\gamma||Au_n - P_{\lambda_n} Au_n||^2 + 2k^2||u_n - q||^2 \\ &\leq \gamma^2 ||A||^2 ||(P_{\lambda_n} - I)Au_n||^2 - 2\gamma||Au_n - P_{\lambda_n} Au_n||^2 \\ &+ \gamma[||P_n Au_n - P_{\lambda_n} Ap||^2 + ||(P_{\lambda_n} - I)Au_n||^2] + \\ &\quad 2k^2||u_n - q||^2 \\ &\leq \gamma^2 ||A||^2 ||(P_{\lambda_n} - I)Au_n||^2 - \gamma||Au_n - P_{\lambda_n} Au_n||^2 + 2k^2||u_n - q||^2 + \gamma||P_{\lambda_n} Au_n - P_{\lambda_n} Ap||^2 \\ &\leq \gamma^2 ||A||^2 ||(P_{\lambda_n} - I)Au_n||^2 - \gamma||Au_n - P_{\lambda_n} Au_n||^2 + 2k^2||u_n - q|| + \gamma||A||^2||u_n - q||^2 \\ &\leq \gamma(\gamma||A||^2 - 1)||(P_{\lambda_n} - I)Au_n||^2 + (\gamma||A||^2 + 2k^2)||u_n - q||^2 \\ &\leq (\gamma||A||^2 + 2k^2)||u_n - q||^2 - \gamma(1 - \gamma||A||^2)||(P_{\lambda_n} - I)Au_n||^2. \end{aligned} \tag{3.4}$$

Furthermore, from (3.1), Lemma 2.4 and Remark 2.5, we have

$$\begin{aligned} ||u_n - q||^2 &= ||(x_n - q) + \theta_n(x_n - x_n - 1)||^2 \\ &\leq ||x_n - q||^2 + 2\theta_n\langle x_n - x_{n-1}, j(x_n - q)\rangle + c\theta_n^2||x_n - x_{n-1}||^2 \\ &\leq ||x_n - q||^2 + \theta_n[||x_n - q||^2 + c||x_n - x_{n-1}||^2 - ||x_{n-1} - q||^2] + c\theta_n^2||x_n - x_{n-1}||^2 \\ &\leq ||x_n - q||^2 + \theta_n[||x_n - q||^2 - ||x_{n-1} - q||^2] + 2c\theta_n||x_n - x_{n-1}||^2, \end{aligned} \tag{3.5}$$

which together with (3.5), implies

$$\begin{aligned} ||y_n - q||^2 \leq\ & (\gamma||A||^2 + 2k^2)[||x_n - q||^2 + \theta_n(||x_n - q||^2 - ||x_{n-1} - q||^2) + 2c\theta_n||x_n - x_{n-1}||^2] - \\ & \gamma(1 - \gamma||A||^2)||(P_{\lambda_n} - I)Au_n||^2. \end{aligned} \tag{3.6}$$

Thus, from (3.3), we obtain

$$\begin{aligned} ||x_{n+1} - q||^2 \leq\ & (\gamma||A||^2 + 2k^2)[||x_n - q||^2 + \theta_n(||x_n - q||^2 - ||x_{n-1} - q||^2) + 2c\theta_n||x_n - x_{n-1}||^2] - \\ & \gamma(1 - \gamma||A||^2)||(||(P_{\lambda_n} - I)Au_n||^2) - \alpha_n(1-\alpha_n) g(||y_n - Q_{\lambda_n} y_n||). \end{aligned} \tag{3.7}$$

Since $0 < \gamma||A||^2 + 2k^2 < 1$, we obtain

$$\begin{aligned} ||x_{n+1} - q||^2 \leq\ & ||x_n - q||^2 + \theta_n[||x_n - q||^2 - ||x_{n-1} - q||^2] + 2c\theta_n||x_n - x_{n-1}||^2 - \\ & \gamma(1 - \gamma||A||^2)||(||(P_{\lambda_n} - I)Au_n||^2 - \alpha_n(1-\alpha_n) g(||y_n - Q_{\lambda_n} y_n||). \end{aligned} \tag{3.8}$$

That is,

$$||x_{n+1}-q||^2 \leq ||x_n-q||^2+\theta_n(||x_n-q||^2-||x_{n-1}-q||^2)+2c\theta_n||x_n-x_{n-1}||^2. \tag{3.9}$$

Since $\sum_{n\geq 0}\theta_n||x_n-x_{n-1}||^2 < \infty$, and $\theta_n \subset [0,\theta]$, $[\theta \in (0,1)$, we obtain from Lemma 2.1 that the sequence $\{||x_n-q||\}$ is convergent, hence bounded. Consequently, the sequence $\{||y_n-q||\}$ is bounded. □

Theorem 3.2. *Let X_1 be a real uniformly convex Banach space and 2-uniformly smooth satisfying Opial's condition, X_2 a real Banach space with Fréchet differentiable norm, $A: X_1 \to X_2$ a bounded linear operator and A^* the adjoint of A. Let $T_1: X_1 \to 2^{X_1}, T_2: X_2 \to 2^{X_2}$ be set-valued accretive operators and $S_1: X_1 \to X_1, S_2: X_2 \to X_2$ be α-inverse strongly accretive operators. Assume $\Gamma := \{q \in (T_1+S_1)^{-1}(0) : Aq \in (T_2+S_2)^{-1}(0)\} \neq \emptyset$. Let $\{\lambda_n\}$ be a sequence of non-negative real numbers, for $x_1 \in X_1$, $\{x_n\}$ be the sequence given by (3.1) where α_n is a sequence in $(0,1)$, γ is a positive constant and $\theta_n \subset [0,\theta)$ where $\theta \in [0,1)$ satisfying the following conditions:*

(1) $\sum_{n\geq 0}\theta_n||x_n-x_{n-1}||^2 < \infty$;

(2) $\liminf_{n\to\infty}\alpha_n(1-\alpha_n) > 0$;

(3) $0 < \gamma < \frac{1-2k^2}{||A||^2}$, *where k is the smoothness constant satisfying* $0 < k^2 < \frac{1}{2}$;

(4) $0 < \lambda \leq \lambda_n \leq b < \frac{2\alpha}{c}, \quad \forall\, n \geq 1, \quad c > 1.$

Then $\{x_n\}$ converges weakly to $x^* \in \Gamma$.

Proof . Let $q \in \Gamma$, then by Lemma 2.1 and (3.9), we obtain $\sum_{n\geq 0}[||x_n-q||^2-||x_{n-1}-q||^2]_+ < \infty$, also from (3.8), we have

$$\gamma(1-\gamma||A||^2)||(P_{\lambda_n}-I)Au_n||^2+\alpha_n(1-\alpha_n)g(||y_n-Q_{\lambda_n}y_n||) \leq ||x_{n+1}-q||^2-||x_n-q||^2+\theta_n[||x_n-q||^2-||x_{n-1}-q||^2]_+ +2c\theta_n||x_n-x_{n-1}||^2. \tag{3.10}$$

Hence, we obtain

$$\sum_{n\geq 0}[\gamma(1-\gamma||A||^2)||(P_{\lambda_n}-I)Au_n||^2+\alpha_n(1-\alpha_n)g(||y_n-Q_{\lambda_n}y_n||)] < \infty. \tag{3.11}$$

This implies that

$$\lim_{n\to\infty}||(P_{\lambda_n}-I)Au_n|| = 0. \tag{3.12}$$

Also, condition (2) and the property of the function g in Lemma 2.2, we obtain

$$\lim_{n\to\infty}||y_n-Q_{\lambda_n}y_n|| = 0. \tag{3.13}$$

From Condition (4) we have $\lambda_n > 0, \quad \forall\, n \geq 1$ therefore, there exists $\epsilon > 0$ such that $\lambda_n \geq \epsilon$ for all $n \geq 1$. Then, by Lemma 2.6,

$$\lim_{n\to\infty}||Q_\epsilon x_n-x_n|| \leq 2\lim_{n\to\infty}||Q_{\lambda_n}x_n-x_n|| = 0. \tag{3.14}$$

Since, Q_ϵ is nonexpansive, we have $F(Q_\epsilon) = (T_1+S_1)^{-1}(0) \neq \emptyset$.

Same argument holds for P_ϵ, hence, P_ϵ is nonexpansive and $F(P_\epsilon) = (T_2 + S_2)^{-1}(0) \neq \emptyset$.
From condition (1), we have $\sum_{n\geq 0} \theta_n ||x_n - x_{n-1}||^2 < \infty$, which implies $\theta_n ||x_n - x_{n-1}|| \to 0$.
Observe that

$$||u_n - x_n|| = ||(x_n - x_n) + \theta_n(x_n - x_{n-1})|| \to 0.$$

Hence,

$$\lim_{n\to\infty} ||u_n - x_n|| = 0. \tag{3.15}$$

Also,

$$\begin{aligned} ||y_n - x_n|| &= ||(u_n - x_n) + \gamma J_1^{-1} A^* J_2(P_{\lambda_n} - I)Au_n|| \\ &\leq ||u_n - x_n|| + ||\gamma J_1^{-1} A^* J_2(P_{\lambda_n} - I)Au_n||. \end{aligned} \tag{3.16}$$

Using (3.12) and (3.15), we obtain

$$\lim_{n\to\infty} ||y_n - x_n|| = 0. \tag{3.17}$$

By Lemma 3.1, $\{x_n\}$ is bounded and by the reflexivity of the Banach space X_1, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges weakly to x^*. Using (3.15), we have $\{u_{n_j}\}$ of $\{u_n\}$ converges weakly to x^*. (3.17), also implies that $\{y_{n_j}\}$ of $\{y_n\}$ converges weakly to x^*. From (3.13), we have that $||y_{n_j} - Q_{\lambda_n} y_{n_j}|| \to 0$, as $n \to \infty$. Since Q_{λ_n} is nonexpansive, then by Lemma 2.3 and Lemma 2.6(i) we have that $x^* \in= (T_1 + S_1)^{-1}(0)$.
Furthermore, since the operator A is linear and bounded, we know that $\{Ax_{n_j}\}$ converges weakly to Ax^*. It follows from (3.12) and the fact that P_{λ_n} is demiclosed at zero that $Ax^* \in (T_2 + S_2)^{-1}(0)$. Hence, x^* belongs to Γ.
Now, suppose there exists another subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges to $y^* \in X_1$, we know by (3.16) and previous analysis that $y^* \in (T_2 + S_2)^{-1}(0)$. Applying the Opial's condition on the space X_1, we conclude that $\{x_n\}$ converges weakly to x^*. □

The following results are easily obtained as corollaries to our main result.

Corollary 3.3. *Let H_1 and H_2 be real Hilbert spaces with H_1 satisfying the Opial's condition, and $A : H_1 \to H_2$ a bounded linear operator A^* the adjoint of A. Let $T_1 : H_1 \to 2^{H_1}, T_2 : H_2 \to 2^{H_2}$ be set-valued monotone operators and $S_1 : H_1 \to H_1, S_2 : H_2 \to H_2$ be α-inverse strongly monotone. Assume $\Gamma := \{q \in (T_1 + S_1)^{-1}(0) : Aq \in (T_2 + S_2)^{-1}(0)\} \neq \emptyset$. Let $\{\lambda_n\}$ be a sequence of non-negative real numbers, for $x_1 \in H_1$, let $\{x_n\}$ be a sequence given by*

$$\begin{cases} u_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = u_n + \gamma A^*(P_{\lambda_n} - I)Au_n, \\ x_{n+1} = \alpha_n y_n + (1 - \alpha_n)Q_{\lambda_n} y_n, \quad \forall \ \ n \geq 1, \end{cases} \tag{3.18}$$

where α_n is a sequence in $(0, 1)$, γ is a positive constant and $\theta_n \subset [0, \theta)$, where $\theta \in [0, 1)$ satisfying the following conditions:

(1) $\sum_{n\geq 0} \theta_n ||x_n - x_{n-1}||^2 < \infty$;

(2) $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$;

(3) $0 < \gamma < \frac{1}{||A||^2}$;

(4) $0 < \lambda \leq \lambda_n \leq b < \frac{2\alpha}{c}, \quad \forall\ n \geq 1, \quad c > 1.$

Then $\{x_n\}$ *converges weakly to* $x^* \in \Gamma$.

Suppose in $S_1 \equiv 0$ and $S_2 \equiv 0$ in (1.8), then the split accretive variational inclusion problem (1.8) reduces to split variational inclusion problem: Find $x^* \in X_1$ such that

$$\begin{cases} 0 \in T_1(x^*) \\ y^* = Ax^* \in X_2 : 0 \in T_2(y^*). \end{cases} \tag{3.19}$$

Therefore, we obtain the following corollary.

Corollary 3.4. *Let* X_1 *be a real uniformly convex Banach space and 2-uniformly smooth satisfying Opial's condition,* X_2 *a real Banach space with Féchet differentiable norm,* $A : X_1 \to X_2$ *a bounded linear operator and* A^* *the adjoint of* A*. Let* $T_1 : X_1 \to 2^{X_1}$ *and* $T_2 : X_2 \to 2^{X_2}$ *multi-valued maximal accretive operators. Assume* $\Gamma := \{q \in T_1^{-1}(0) : Aq \in T_2^{-1}(0)\} \neq \emptyset$*. Let* $\{\lambda_n\}$ *be a sequence of non-negative real numbers, for* $x_1 \in X_1$*, let* $\{x_n\}$ *be a sequence given by*

$$\begin{cases} u_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = u_n + \gamma J_1^{-1} A^* J_2(J_{\lambda_n}^{T_2} - I)Au_n, \\ x_{n+1} = \alpha_n y_n + (1 - \alpha_n) J_{\lambda_n}^{T_1} y_n, \quad \forall \quad n \geq 1, \end{cases} \tag{3.20}$$

where α_n *is a sequence in* $(0, 1)$*,* γ *is a positive constant and* $\theta_n \subset [0, \theta)$ *where* $\theta \in [0, 1)$ *satisfying the following conditions:*

(1) $\sum_{n\geq 0} \theta_n ||x_n - x_{n-1}||^2 < \infty$;

(2) $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$;

(3) $0 < \gamma < \frac{1-2k^2}{||A||^2}$*, where* k *is the smoothness constant satisfying* $0 < k^2 < \frac{1}{2}$.

Then $\{x_n\}$ *converges weakly to* $x^* \in \Gamma$.

4. Application and numerical example.

4.1. Convex Minimization Problem

Recall that the concept of accretivity in Banach space coincides with that of monotonicity in Hilbert space. Thus, we apply our result to solve convex minimization problem, which is an important optimization problem.

Let H be a Hilbert space, $M : H \to (-\infty, +\infty]$ a proper convex and lower semi-continuous function and $N : H \to \mathbb{R}$ a convex and continuously differentiable function. Then the subdifferential of M denoted ∂M is maximal monotone and the gradient ∇N of N is monotone and continuous (see [41]). Moreover ,

$$M(x^*) + N(x^*) = \min_{x\in X}[M(x) + N(x)] \iff 0 \in \partial(M(x^*) + \nabla N(x^*)). \tag{4.1}$$

We consider the following Split Convex Minimization Problem (SCMP): Find $x^* \in H_1$, such that

$$\begin{cases} M_1(x^*) + N_1(x^*) = \min\limits_{x \in H_1}[M_1(x) + N_1(x)], \\ y^* = Ax^* \in H_2 : M_2(y^*) + N_2(y^*) = \min\limits_{y \in H_2}[M_2(y) + N_2(y)], \end{cases} \tag{4.2}$$

where $A : H_1 \to H_2$ is a bounded linear operator M_1, M_2 are proper convex lower semi-continuous functions and N_1, N_2 are convex and differentiable functions. We denote the solution set of (4.2) by Γ.
By setting $S_1 = \partial N_1, S_2 = \partial N_1, T_1 = \nabla M_1$ and $T_2 = \nabla M_2$ in Corollary 3.3, we obtain the following result for solving SCMP (4.2):

Theorem 4.1. *Let H_1 and H_2 be real Hilbert spaces with H_1 satisfying the Opial's condition, and $A : H_1 \to H_2$ a bounded linear operator A^* the adjoint of A. Let $M_1 : H_1 \to (-\infty, +\infty], M_2 : H_2(-\infty, +\infty]$ be proper convex and continuously differentiable function and $N_1 : H_1 \to \mathbb{R}, N_2 : H_2 \to \mathbb{R}$ be convex and continuously differentiable function such that ∇N_i is $\frac{1}{\alpha}$-Lipschitz for $i = 1, 2$. Assume $\Gamma \neq \emptyset$, for let $\{\lambda_n\}$ be a sequence of non-negative real numbers, for $x_1 \in H_1$, let $\{x_n\}$ be a sequence given by*

$$\begin{cases} u_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = u_n + \gamma A^*((I + \lambda_n \partial M_2)^{-1})(I - \lambda_n \nabla N_2) - I)Au_n, \\ x_{n+1} = \alpha_n y_n + (1 - \alpha_n)(I + \lambda_n \partial M_1)^{-1}(I - \lambda_n \nabla N_1)y_n, \quad \forall \quad n \geq 1, \end{cases} \tag{4.3}$$

where α_n is a sequence in $(0, 1)$, γ is a positive constant and $\theta_n \subset [0, \theta)$ where $\theta \in [0, 1)$ satisfying the following conditions:

(1) $\sum_{n \geq 0} \theta_n ||x_n - x_{n-1}||^2 < \infty$;

(2) $\liminf\limits_{n \to \infty} \alpha_n(1 - \alpha_n) > 0$;

(3) $0 < \gamma < \frac{1}{||A||^2}$;

(4) $0 < \lambda \leq \lambda_n \leq b < \frac{2\alpha}{c}, \quad \forall\ n \geq 1, \quad c > 1.$

Then $\{x_n\}$ converges weakly to $x^* \in \Gamma$.

Let A, M_1 and M_2 be defined as above, we define the Convex Minimization Problem (CMP) as follows: Find $x^* \in H_1$ such that

$$\begin{cases} M_1(x^*) = \min\limits_{x \in H_1} M_1(x), \\ y^* = Ax^* \in H_2 : M_2(y^*) = \min\limits_{y \in H_2} M_2(y). \end{cases} \tag{4.4}$$

We denote the solution set of the CMP (4.4) by denoted by Γ. Several authors have used different iterative algorithms to approximate solutions of SCMP (4.2) and CMP (4.4) and related optimization problems, see [3, 4, 5, 18, 19, 20, 43].
By setting $T_1 = \partial M_1$ and $T_2 = \partial M_2$ in Theorem 3.2, with $S_1 = S_2 \equiv 0$, we obtain the following result:

Corollary 4.2. *Let H_1 and H_2 be real Hilbert spaces with H_1 satisfying the Opial's condition, and $A : H_1 \to H_2$ a bounded linear operator, A^* the adjoint of A. Let $M_1 : H_1 \to (-\infty, +\infty], M_2 : H_2(-\infty, +\infty]$ be proper convex and continuously differentiable function. Assume $\Gamma \neq \emptyset$, let $\{\lambda_n\}$ be a sequence of non-negative real numbers, for $x_1 \in H_1$, let $\{x_n\}$ be a sequence given by*

$$\begin{cases} u_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = u_n + \gamma A^*(J_{\lambda_n}^{\partial M_2} - I)Au_n, \\ x_{n+1} = \alpha_n y_n + (1-\alpha_n)J_{\lambda_n}^{\partial M_1} y_n, \quad \forall \ \ n \geq 1, \end{cases} \tag{4.5}$$

where α_n is a sequence in $(0,1)$, γ is a positive constant and $\theta_n \subset [0, \theta)$, where $\theta \in [0,1)$ satisfying the following conditions:

(1) $\sum_{n\geq 0} \theta_n ||x_n - x_{n-1}||^2 < \infty$;

(2) $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$;

(3) $0 < \gamma < \frac{1}{||A||^2}$.

Then $\{x_n\}$ converges weakly to $x^* \in \Gamma$.

4.2. Numerical example

Here we present a numerical example in $(\mathbb{R}^2, ||.||_2)$ to our result Theorem 3.2.
Let $X_1 = X_2 = \mathbb{R}^2$, we define $A(x) : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$A(x) = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ then, } A^*(x) = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Let $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ and $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T_1(\bar{x}) = (-x_1 - x_2, x_1 + x_2)$ and $T_2(\bar{x}) = (x_1, x_2)$. We obtain the resolvent mappings associated with T_1 and T_2 as follows:

$$\begin{aligned} J_{\lambda_n}^{T_1}(\bar{x}) &= \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -\lambda_n & -\lambda_n \\ \lambda_n & \lambda_n \end{pmatrix} \right]^{-1} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= \begin{pmatrix} 1-\lambda_n & -\lambda \\ \lambda_n & 1+\lambda_n \end{pmatrix}^{-1} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= \begin{pmatrix} 1+\lambda_n & \lambda_n \\ -\lambda_n & 1-\lambda_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= \big((1+\lambda_n)x_1 + \lambda_n x_2, (1-\lambda_n)x_2 - \lambda_n x_1\big). \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} J_{\lambda_n}^{T_2}(\bar{x}) &= \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \lambda_n & 0 \\ 0 & \lambda_n \end{pmatrix} \right]^{-1} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= \left(\frac{1}{1+\lambda_n} x_1, \frac{1}{1+\lambda_n} x_2 \right). \end{aligned}$$

Let $S_1 : \mathbb{R}^2 \to \mathbb{R}^2$ respectively $S_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $S_1(\bar{x}) = (2x_1, -2x_2)$ and $S_2(\bar{x}) = (x_1, -x_2)$.

Let $\alpha_n = \frac{n}{2n+1}$, $r = \frac{1-4k^2}{||A||^2}$, $k = \frac{1}{2}$. Then, $\lambda_n = \frac{n+1}{10n+70}$. Hence, our Algorithm 3.1 becomes:
For $x_0, x_1 \in \mathbb{R}^2$

$$\begin{cases} u_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = u_n + \gamma J_1^{-1} A^* J_2 \left[\begin{pmatrix} \frac{1-\lambda_n}{1+\lambda_n} & 0 \\ 0 & 1 \end{pmatrix} - I \right] A u_n \quad n \geq 0, \\ x_{n+1} = \left(\frac{n}{2n+1}\right) y_n + \left(\frac{n+1}{2n+1}\right) \begin{pmatrix} (1+\lambda_n)(1-2\lambda_n) & \lambda_n(1-2\lambda_n) \\ \lambda_n(2\lambda_n - 1) & (1-\lambda_n)(1-2\lambda_n) \end{pmatrix} y_n, \quad n \geq 1. \end{cases} \tag{4.6}$$

Case I: $\bar{x}_0 = (0.1,\ 0.01)^T$, $\bar{x}_1 = (1,\ 2)^T$ and $\theta_n = \frac{n}{4n^5+1}$.
Case II: $\bar{x}_0 = (1,\ 2)^T$, $\bar{x}_1 = (0.1,\ 0.01)^T$ and $\theta_n = \frac{n}{2n^2+1}$.
Declaration
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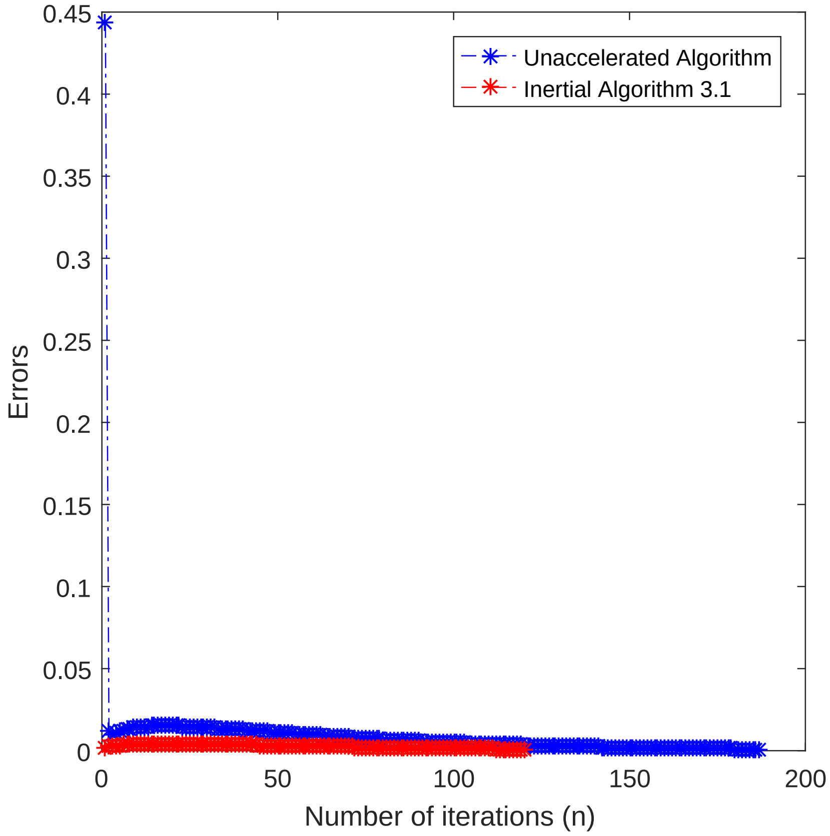

Figure 1: Errors vs number of iterations for **Case I**.

Table 1: Showing numerical results for **Case I**.

No. of iterations	Accelerated Algorithm 3.1	Unaccelerated Algorithm
1		
2	0.0021	0.4435
3	0.0025	0.0120
4	0.0029	0.0100
5	0.0032	0.0110
6	0.0034	0.0119
7	0.0036	0.0125
8	0.0038	0.0131
9	0.0039	0.0136
10	0.0041	0.0140
11	0.0041	0.0143
12	0.0042	0.0145
13	0.0043	0.0147
14	0.0043	0.0149
15	0.0044	0.0150
16	0.0044	0.0151
17	0.0044	0.0151
18	0.0044	0.0152
19	0.0044	0.0152
20	0.0044	0.0152

Table 2: Showing numerical results for **Case II**.

No. of iterations	Accelerated Algorithm 3.1	Unaccelerated Algorithm
1		
2	0.0236	0.7576
3	0.0283	0.1216
4	0.0323	0.0515
5	0.0356	0.0516
6	0.0383	0.0548
7	0.0405	0.0577
8	0.0423	0.0601
9	0.0438	0.0621
10	0.0451	0.0637
11	0.0461	0.0651
12	0.0469	0.0662
13	0.0476	0.0671
14	0.0481	0.0677
15	0.0485	0.0682
16	0.0487	0.0686
17	0.0489	0.0688
18	0.0490	0.0689
19	0.0490	0.0690
20	0.0490	0.0689
21	0.0489	0.0687
22	0.0487	0.0685
23	0.0485	0.0682
24	0.0483	0.0679
25	0.0480	0.0675
26	0.0477	0.0670
27	0.0473	0.0665
28	0.0470	0.0660
29	0.0466	0.0655
30	0.0462	0.0649
31	0.0458	0.0643

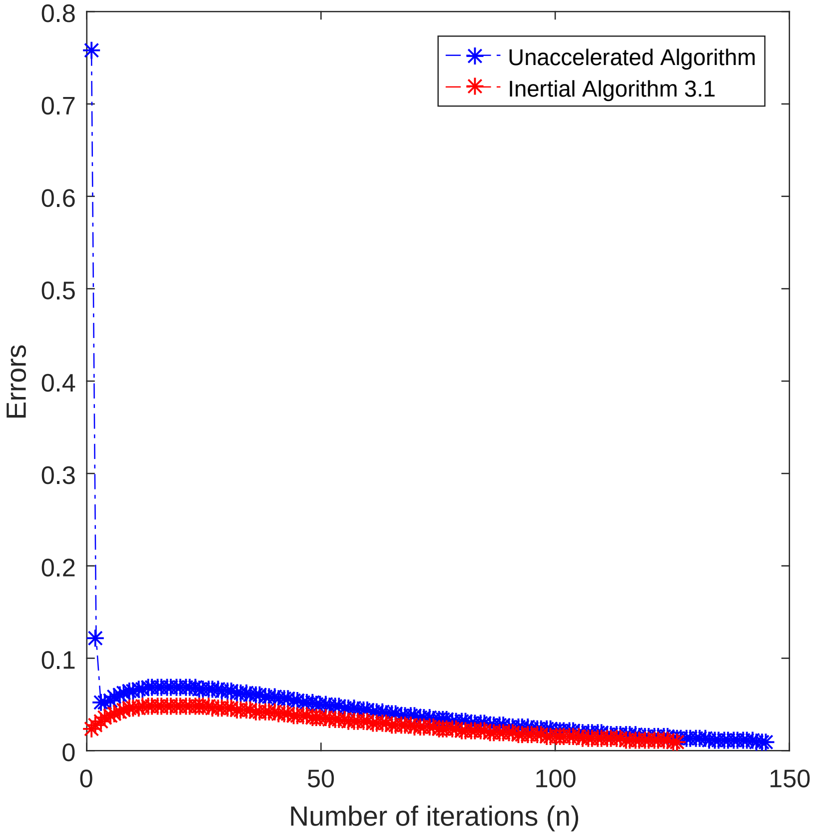

Figure 2: Errors vs number of iterations for **Case 2**.