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Contractive gauge functions in strongly orthogonal metric spaces

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Abstract

Existence of fixed point in orthogonal metric spaces has been initiated recently by Eshaghi and et al. [On orthogonal sets and Banach fixed Point theorem, Fixed Point Theory, in press]. In this paper, we introduce the notion of the strongly orthogonal sets and prove a genuine generalization of Banach' fixed point theorem and Walter's theorem. Also, we give an example showing that our main theorem is a real generalization of these fixed point theorems.

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1. Introduction and preliminaries

Walter ([6]) generalized Banach's fixed point theorem with the following fact.

Theorem 1.1. (Walter [6]). Let (X, d) be complete metric space and suppose $T : X \to X$ has bounded orbits and satisfies the following condition.

For each $x \in X$, there exists $n(x) \in \mathbb{N}$ such that for all $n \geq n(x)$ and $y \in X$,

$$d(T^n(x), T^n(y)) \leq \phi(O(x, y))$$

where $O(x) = \{x, Tx, T^2x, \dots\}$ and $O(x, y) = O(x) \cup O(y)$. Then there exists a unique $z \in X$ such that

$$\lim_{n\to\infty} d(T^n(x), z) = 0$$

for all $x \in X$. Moreover, if T is continuous then T has a fixed point.

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In recent years, there has been recent interest in establishing fixed point theorems on ordered metric spaces with a contractivity condition which holds for all points that are related by partial ordering. In [4], Ran and Reurings established the fixed point theorem that extends the Banach contraction principle to the setting of ordered metric spaces. The existence of fixed point for contraction-type mappings on such spaces was considered by many authors (see [3, 5, 1, 7]).

Very recently, M. Eshaghi et al. [2] introduced the concept of the orthogonal sets as the following.

Definition 1.2. ([2]). Let $X \neq \emptyset$ and $\perp \subseteq X \times X$ be an binary relation. If $\perp$ satisfies the following condition

$$\exists x_0 : (\forall y, y \perp x_0) \ \ or \ \ (\forall y, x_0 \perp y),$$

then it is called an *orthogonal set* (briefly *O-set*). We denote this O-set by $(X, \perp)$.

They also proved a genuine generalization of Banach' fixed point theorem. They gave an example which say that the their main theorem is a real generalization of Banach's fixed point theorem.

The main result of [2] is the following theorem.

Theorem 1.3. (Eshaghi and et al. [4]) Let $(X, \perp, d)$ be an O-complete metric space (not necessarily complete metric space) and $0 < \lambda < 1$. Let $f : X \to X$ be $\perp$-continuous, $\perp$-contraction with Lipschitz constant λ and $\perp$-preserving. Then f has a unique fixed point $x^* \in X$. Also, f is a Picard operator, that is, $\lim f^n(x) = x^*$ for all $x \in X$.

In this paper, we introduce the concept the strongly orthogonal sets and analyze the existence of fixed points for generalized contractive operators in strongly metric spaces. Also, we prove our theorem is a substantial generalization of Walter's theorem, Theorem 1.3 of [2] and Banach contraction principle.

2. strongly orthogonal sets

We start our work with following definitions.

Definition 2.1. Let $(X, \perp)$ be O-set . A sequence $\{x_n\}_{n\in\mathbb{N}}$ is called strongly orthogonal sequence (briefly SO-sequence) if

$$(\forall n, k; x_n \perp x_{n+k}) \ \ or \ \ (\forall n, k; \forall x_{n+k} \perp x_n).$$

Definition 2.2. Let $(X, \perp, d)$ be an orthogonal metric space ($(X, \perp)$ is an O-set and (X, d) is a metric space) . Then $f : X \to X$ is strongly orthogonal continuous (SO-continuous) in $a \in X$ if for each SO-sequence $\{a_n\}_{n\in\mathbb{N}}$ in X if $a_n \to a$, then $f(a_n) \to f(a)$. Also f is SO-continuous on X if f is SO-continuous in each $a \in X$.

It is easy to see that every continuous mapping is SO-continuous . The following example shows that the converse is not true .

Example 2.3. Let $X = [0, 10)$, suppose $x \perp y$ if

$$x, y \in (\frac{3n+1}{3}, \frac{4n+1}{4}) \ for \ some \ n \in \mathbb{Z}, or$$

$$x = 0.$$

It is easy to see that $(X, \perp)$ is an O-set . Define $f : X \to X$ by $f(x) = [x]$. Then f is SO-continuous on X. Because if $\{x_k\}$ is an arbitrary SO-sequence in X such that $\{x_k\}$ converges to $x \in X$, then the following cases are hold :
case 1) $x_k = 0$ for all k . Then $x = 0$ and $f(x_k) = 0 \to 0 = f(x)$.
case 2) $x_{k_0} \neq 0$ for some k_0 . Then there exists $m \in \mathbb{Z}$ such that $x_k \in (m + \frac{1}{3}, m + \frac{2}{3})$ for all $k \geq k_0$. Thus $x \in [m + \frac{1}{3}, m + \frac{2}{3}]$ and $f(x_k) = m \to m = f(x)$.

Definition 2.4. Let $(X, \perp, d)$ be an orthogonal set with metric d . Then X is strongly orthogonal complete (briefly SO-complete) if every Cauchy SO-sequence is convergent .

It is easy to see that every complete metric space is SO-complete and the converse is not true. In the before example, X is SO-complete and it is not complete.

3. Fixed point theorems in strongly orthogonal sets

Let (X, d) be a metric space and $T : X \to X$ be a mapping . For $x, y \in X$ consider $O(x) := \{x, T(x), T^2(x), \ldots\}$ and $O(x, y) := O(x) \cup O(y)$. We say that $O(x, y)$ is an orbit in X and following we denote

$$D(x, y) := \sup\{d(T^k(x), T^l(y)); \quad k \leq l \ \ and \ \ k, l = 0, 1, 2, \ldots\}.$$

From now on , we suppose that ϕ is a contractive gauge function on $\mathbb{R}^+$. That is, $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, nondecreasing and satisfies $\phi(s) < s$ for $s > 0$.

Theorem 3.1. *Let $(X, \perp, d)$ be SO-complete and $T : X \to X$ be a self mapping with bounded orbits. Let T be orthogonal preserving, that is,*

$$x, y \in X, \qquad x \perp y \quad \Longrightarrow \quad T(x) \perp T(y)$$

and satisfies the following condition:
For each $x \in X$, there exists $n(x) \in \mathbb{N}$ such that for all $n \geq n(x)$ and $y \in X$ with $x \perp y$

$$d(T^n(x), T^n(y)) \leq \phi(D(x, y)) \tag{3.1}$$

Then there exists a unique $z \in X$ such that

$$\lim_{n \to \infty} d(T^n(x), z) = 0$$

for all $x \in X$. Moreover, if T is SO- continuous then T is a Picard operator.

Proof . For $x \in X$, we put $x^k := T^k(x)$, $k = 0, 1, 2, \ldots$. By definition of orthogonality

$$\exists x_0 \in X, \quad (\forall y \in X, x_0 \perp y) \ \ or(\ \forall y \in X, y \perp x_0).$$

We let $x_0 \perp y$ for all $y \in X$. Since T is orthogonality preserving, then $\{x_0^n\}$ is SO-sequence. Define sequences $\{k(i)\} \subseteq \mathbb{N}$ and $\{A_i\}_i \subseteq 2^X$ by

$$k(0) := 0 \quad , \qquad k(i + 1) := k(i) + n(x_0^{k(i)})$$

and

$$A_i := O(x_0^{k(i)}).$$

To complete the proof , we need the following two steps :

Step 1. $lim_{n\to\infty} diam(A_i) = 0$.

Proof. Since T is orthogonality preserving, we deduce that $x_0^{k(i)} \perp x_0^{k(i)+k}$ for $i, k = 0, 1, 2, \ldots$. Let $n \geq n(x_0^{k(i)})$. It follows from (3.1) that for $k = 0, 1, 2, \ldots$

$$\begin{aligned} d(x_0^{n+k(i)}, x_0^{n+k(i)+k}) &= d(T^n(x_0^{k(i)}), T^n(x_0^{k(i)+k})) \\ &\leq \phi(D(x_0^{k(i)}, x_0^{k(i)+k})) \\ &\leq \phi(diam(A_i)). \end{aligned}$$

This shows that $diam(A_{i+1}) \leq \phi(diam(A_i))$. Let $lim_{i\to\infty} diam(A_i) = a$. Then

$$\begin{aligned} a &= \lim_{i\to\infty} diam(A_{i+1}) \leq \lim_{i\to\infty} \phi(diam(A_i)) \\ &= \phi(\lim_{i\to\infty} diam(A_i)) = \phi(a). \end{aligned}$$

Since $\phi(t) < t$ if $t > 0$, it must be case that $a = 0$. This clearly implies that $\{x_0^i\}$ is a SO-Cauchy sequence. Since X is SO-complete, it follows that $\lim_{n\to\infty} x_0^n = z$ for some $z \in X$. Let $y \in X$ be arbitrary . We define sequences $\{m(i)\}_i \subseteq \mathbb{N}$ and $\{B_i\}_i \subseteq \mathbb{R}$ by

$$m(0) := 0 \quad , \quad m(i+1) := m(i) + \max\{n(x_0^{m(i)}), n(y^{m(i)})\}$$

and

$$B_i := D(x_0^{m(i)}, y^{m(i)}).$$

Step 2. $\lim_{i\to\infty} B_i = 0$.

Proof. Observe $x_0^{m(i)} \perp y^{m(i)+k}$ for $i, k = 0, 1, 2 \ldots$. Suppose

$$n \geq \max\{n(x_0^{m(i)}), n(y^{m(i)})\}$$

then for $k \in \mathbb{N}$,

$$\begin{aligned} d(x_0^{n+m(i)}, y^{n+m(i)+k}) &= d(T^n(x_0^{m(i)}), T^n(y^{m(i)+k})) \\ &\leq \phi(D(x_0^{m(i)}, y^{m(i)+k})) \\ &\leq \phi(D(x_0^{m(i)}, y^{m(i)})). \end{aligned}$$

Hence $B_{i+1} \leq \phi(B_i)$. Let $\lim_{i\to\infty} B_i = b \geq 0$. If $b > 0$, then

$$b = \lim_{i\to\infty} B_{i+1} \leq \phi(\lim_{i\to\infty} B_i) = \phi(b) < b.$$

Thus $b = 0$. This implies that $\lim_{i\to\infty} D(x_0^i, y^i) = 0$. Applying step 1, we have $\lim_{i\to\infty} y^i = z$. Let T be SO- continuous. Since $\{x_0^n\}$ is SO-sequence then by step(2) we have $T(z) = z$. □

Remark 3.2. Theorem 3.1 is a generalization Theorem 1.1 of Walter. To see this, suppose

$$x \perp y \iff \exists n \in \mathbb{N}; \;\; d(T^n(x), T^n(y)) \leq \phi(O(x, y)).$$

It is easy to see that for every x and y in X, $x \perp y$. Thus, $(X, d, \perp)$ is SO-complete metric space and T is $\perp$-preserving and SO-continuous mapping that satisfies condition (3.1). By applying Theorem 1.3, T is a Picard operator.

Remark 3.3. Theorem 3.1 is a generalization of Theorem 1.3 of Eshaghi and et al. In fact, by putting $\phi(t) = \lambda t$, and since every SO-sequence is O-sequence,we can see the result.

Remark 3.4. Theorem 3.1 is twofold generalization of Banach contraction principle. Note that in [2], it is proved that Theorem 1.3 is a real generalization of Banach contraction principle.

Now, we shall show that there is an example which shows that Theorem 3.1 is a genuine generalization of Theorem 1.3.

Example 3.5. Let $X = (0, \infty)$ and d be a usual metric. Suppose $x \perp y$ if $xy = x$. It is easy to see that $(X, \perp)$ is an O-set. Let $T : X \to X$ defined by

$$T(x) = \begin{cases} \frac{x+1}{2} & x \leq 1 \\ \frac{1}{2} & x > 1, \end{cases}$$

Now, we shall show that T satisfies all assumptions of our Theorem 3.1. We have the following items:

1. X is SO-complete (not complete). In fact, if $\{x_n\}$ is an arbitrary Cauchy SO-sequence in X, then there exists $n_0 \in \mathbb{N}$ such that $x_n = 1$ for all $n \geq n_0$. It follows that $\{x_n\}$ is the constant sequence 1 and hence x_n converges to $x = 1$.
2. T is SO-continuous (not continuous). In fact, let $\{x_n\}$ be an SO-sequence converging to a point $x \in X$. By using the step 1, there exists $n_0 \in \mathbb{N}$ such that $x_n = 1$ for all $n \geq n_0$ and $x = 1$. This implies that $T(x_n) \to 1 = T(x)$.
3. T is orthogonal preserving. In fact, if $x \perp y$, then $y = 1$. By definition of T, we see that $T(y) = 1$ and $T(x)\ T(y) = T(x)$, this implies that $T(x) \perp T(y)$.
4. T satisfies in condition (3.1). To see this, let x be an arbitrary element in X. If $x > 1$, we can choose integer number n_1 such that $\frac{1}{n_1} \leq \ln(x)$. For every $n \geq n_1$ and y with $x \perp y$, we have

$$d(T^n(x), T^n(y)) = \frac{1}{n} \leq \frac{1}{n_1} \leq \ln(x) \leq \phi(D(x, y)).$$

If $x < 1$, then there exists iteger n_2 such that $\frac{1-x}{n_2} \leq \ln(2 - x)$. For every $n \geq n_2$ and y with $x \perp y$, we have

$$d(T^n(x), T^n(y)) = \frac{|1 - x|}{n} \leq \frac{1 - x}{n_2} \leq \ln(2 - x) \leq \phi(D(x, y)).$$

If $x = 1$, then for all positive integer n and y with $y \perp x$, $d(T^n(x), T^n(y)) = 0 \leq \phi(x, y)$. Putting $n(x) = \max\{n_1, n_2\}$ we conclude condition (3.1).

Observe that all assumptions of Theorem 3.1 are satisfied. Thus T has a unique fixed point $x = 1$. We can also see that the mapping T does not satisfy assumptions of Theorem 1.3. Because, if $x = \frac{3}{2}$ and $y = 1$, then we see that $x \perp y$ and $d(T(x), T(y)) = \frac{1}{2} = d(x, y)$.

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