



# Approximation of functional equations in intuitionistic fuzzy $C^*$ -algebras

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## Abstract

In this paper, we study a positive-additive functional equation in intuitionistic fuzzy  $C^*$ -algebras. Using fixed point methods, we approximate the positive-additive functional equation in intuitionistic fuzzy  $C^*$ -algebras.

*Keywords:* Functional equation, fixed point, generalized Hyers-Ulam stability, functional inequality, linear mapping, intuitionistic fuzzy  $C^*$ -algebra

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## 1. Introduction

The stability problem of functional equations was originated from a question of Ulam [1] concerning the stability of group homomorphisms. Hyers [2] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [3] for additive mappings and by Th.M. Rassias [4] for linear mappings by considering an unbounded Cauchy difference. The paper of Th.M. Rassias [4] has provided a lot of influence in the development of what we call *generalized Hyers-Ulam stability* or *Hyers-Ulam-Rassias stability* of functional equations. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of the Th.M. Rassias' approach. J.M. Rassias [6]–[8] followed the innovative approach of the Th.M. Rassias' theorem [4] in which he replaced the factor  $\|x\|^p + \|y\|^p$  by  $\|x\|^p \cdot \|y\|^q$  for  $p, q \in \mathbb{R}$  with  $p + q \neq 1$ . The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [9], [10], [11], [12], [13]).

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## 2. IFN Algebras

In this section by using the idea of IFMS introduced by Park [48] and Saadati-Park [50], we define a new notion of IFNS with the help of the notion of continuous  $t$ -representable [49].

**Lemma 2.1** ([51]). *Consider the set  $L^*$  and the order relation  $\leq_{L^*}$  defined by*

$$L^* = \{(x_1, x_2) | (x_1, x_2) \in [0, 1]^2, x_1 + x_2 \leq 1\},$$

$$(x_1, x_2) \leq_{L^*} (y_1, y_2) \iff x_1 \leq y_1, x_2 \geq y_2, \forall (x_1, x_2), (y_1, y_2) \in L^*.$$

*Then  $(L^*, \leq_{L^*})$  is a complete lattice.*

**Definition 2.2** ([51]). *An intuitionistic fuzzy set  $\mathcal{A}_{\zeta, \eta}$  in a universal set  $U$  is an object  $\mathcal{A}_{\zeta, \eta} = \{(\zeta(u), \eta(u)) | u \in U\}$ , where, for all  $u \in U$ ,  $\zeta_{\mathcal{A}}(u) \in [0, 1]$  and  $\eta_{\mathcal{A}}(u) \in [0, 1]$  are called the membership degree and the nonmembership degree, respectively, of  $u$  in  $\mathcal{A}_{\zeta, \eta}$  and, furthermore, they satisfy  $\zeta_{\mathcal{A}}(u) + \eta_{\mathcal{A}}(u) \leq 1$ .*

We denote its units by  $0_{L^*} = (0, 1)$  and  $1_{L^*} = (1, 0)$ . Classically, a triangular norm  $T = *$  on  $[0, 1]$  is defined as an increasing, commutative, associative mapping  $T : [0, 1]^2 \rightarrow [0, 1]$  satisfying  $T(1, x) = 1 * x = x$  for all  $x \in [0, 1]$ . A triangular conorm  $S = \diamond$  is defined as an increasing, commutative, associative mapping  $S : [0, 1]^2 \rightarrow [0, 1]$  satisfying  $S(0, x) = 0 \diamond x = x$  for all  $x \in [0, 1]$ .

Using the lattice  $(L^*, \leq_{L^*})$ , these definitions can be extended in a straightforward manner.

**Definition 2.3** ([51]). *A triangular norm ( $t$ -norm) on  $L^*$  is a mapping  $\mathcal{T} : (L^*)^2 \rightarrow L^*$  satisfying the following conditions:*

- (a)  $(\forall x \in L^*)(\mathcal{T}(x, 1_{L^*}) = x)$  (boundary condition);
- (b)  $(\forall (x, y) \in (L^*)^2)(\mathcal{T}(x, y) = \mathcal{T}(y, x))$  (commutativity);
- (c)  $(\forall (x, y, z) \in (L^*)^3)(\mathcal{T}(x, \mathcal{T}(y, z)) = \mathcal{T}(\mathcal{T}(x, y), z))$  (associativity);
- (d)  $(\forall (x, x', y, y') \in (L^*)^4)(x \leq_{L^*} x' \text{ and } y \leq_{L^*} y' \implies \mathcal{T}(x, y) \leq_{L^*} \mathcal{T}(x', y')$  (monotonicity).

**Definition 2.4** ([51]). *A continuous  $t$ -norm  $\mathcal{T}$  on  $L^*$  is said to be continuous  $t$ -representable if there exist a continuous  $t$ -norm  $*$  and a continuous  $t$ -conorm  $\diamond$  on  $[0, 1]$  such that, for all  $x = (x_1, x_2), y = (y_1, y_2) \in L^*$ ,*

$$\mathcal{T}(x, y) = (x_1 * y_1, x_2 \diamond y_2).$$

**Example 2.5.** *For all  $a = (a_1, a_2), b = (b_1, b_2) \in L^*$ , consider*

$$\mathcal{T}(a, b) = (a_1 b_1, \min\{a_2 + b_2, 1\})$$

$$\mathcal{M}(a, b) = (\min\{a_1, b_1\}, \max\{a_2, b_2\}).$$

*Then  $\mathcal{T}(a, b)$  and  $\mathcal{M}(a, b)$  are continuous  $t$ -representable.*

Now, we define a sequence  $\mathcal{T}^n$  recursively by  $\mathcal{T}^1 = \mathcal{T}$  and

$$\mathcal{T}^n(x^{(1)}, \dots, x^{(n+1)}) = \mathcal{T}(\mathcal{T}^{n-1}(x^{(1)}, \dots, x^{(n)}), x^{(n+1)})$$

for all  $n \geq 2$  and  $x^{(i)} \in L^*$ .

**Definition 2.6.** A negator on  $L^*$  is any decreasing mapping  $\mathcal{N} : L^* \rightarrow L^*$  satisfying  $\mathcal{N}(0_{L^*}) = 1_{L^*}$  and  $\mathcal{N}(1_{L^*}) = 0_{L^*}$ . If  $\mathcal{N}(\mathcal{N}(x)) = x$  for all  $x \in L^*$ , then  $\mathcal{N}$  is called an involutive negator. A negator on  $[0, 1]$  is a decreasing mapping  $N : [0, 1] \rightarrow [0, 1]$  satisfying  $N(0) = 1$  and  $N(1) = 0$ .  $N_s$  denotes the standard negator on  $([0, 1], \leq)$  defined by  $N_s = 1 - x$  for all  $x \in [0, 1]$ .

**Definition 2.7 ([51]).** The triple  $(X, \mathcal{P}, \mathcal{T})$  is said to be an IFNS if  $X$  is a vector space,  $\mathcal{T}$  is a continuous  $t$ -representable, and  $\mathcal{P}$  is a mapping  $X \times (0, \infty) \rightarrow L^*$ , satisfying the following conditions for all  $x, y \in X$  and  $t, s > 0$ :

- (i)  $\mathcal{P}(x, t) > 0_{L^*}$ ;
- (ii)  $\mathcal{P}(x, t) = 1_{L^*}$  if and only if  $x = 0$ ;
- (iii)  $\mathcal{P}(\alpha x, t) = \mathcal{P}(x, \frac{t}{|\alpha|})$  for all  $\alpha \neq 0$ ;
- (iv)  $\mathcal{P}(x + y, t + s) \geq_{L^*} \mathcal{T}(\mathcal{P}(x, t), \mathcal{P}(y, t))$ ;
- (v)  $\mathcal{P}(x, \cdot) : (0, \infty) \rightarrow L^*$  is continuous;
- (vi)  $\lim_{t \rightarrow \infty} \mathcal{P}(x, t) = 1_{L^*}$ .

In this case,  $\mathcal{P}$  is called an IFN on  $X$ . Given  $\mu$  and  $\nu$ , membership and nonmembership degrees of an IF set from  $X \times (0, \infty)$  to  $[0, 1]$ , such that

$$\mu(x, t) + \nu(x, t) \leq 1$$

for all  $x \in X$  and  $t > 0$ , we write

$$\mathcal{P}_{\mu, \nu}(x, t) = (\mu(x, t), \nu(x, t)).$$

**Example 2.8 ([52]).** Let  $(X, \|\cdot\|)$  be a normed space,

$$\mathcal{T}(a, b) = (a_1 b_1, \min\{a_2 + b_2, 1\})$$

for all  $a = (a_1, a_2), b = (b_1, b_2) \in L^*$ , and  $\mu, \nu$  be membership and nonmembership degree of an IF set defined by

$$\mathcal{P}_{\mu, \nu}(x, t) = (\mu(x, t), \nu(x, t)) = \left( \frac{t}{t + \|x\|}, \frac{\|x\|}{t + \|x\|} \right) \quad \forall t \in \mathbb{R}^+.$$

Then  $(X, \mathcal{P}_{\mu, \nu}, \mathcal{T})$  is an IFNS.

In Example 2.8,  $\mu(x, t) + \nu(x, t) = 1$  for all  $x \in X$ . We present an example in which  $\mu(x, t) + \nu(x, t) < 1$  for  $x \neq 0$ . This example is a modification of the example of Saadati and Park [50].

**Example 2.9 ([52]).** Let  $(X, \|\cdot\|)$  be a normed space,

$$\mathcal{T}(a, b) = (a_1 b_1, \min\{a_2 + b_2, 1\})$$

for all  $a = (a_1, a_2), b = (b_1, b_2) \in L^*$ , and  $\mu, \nu$  be membership and non-membership degree of an IF set defined by

$$\mathcal{P}_{\mu, \nu}(x, t) = (\mu(x, t), \nu(x, t)) = \left( \frac{t}{t + m\|x\|}, \frac{\|x\|}{t + \|x\|} \right)$$

for all  $t \in \mathbb{R}^+$  in which  $m > 1$ . Then  $(X, \mathcal{P}_{\mu, \nu}, \mathcal{T})$  is an IFNS. Here,

$$\begin{aligned} \mu(x, t) + \nu(x, t) &= 1, \text{ for } x = 0; \\ \mu(x, t) + \nu(x, t) &< 1, \text{ for } x \neq 0. \end{aligned}$$

**Lemma 2.10 ([51]).** *Let  $\mathcal{P}_{\mu,\nu}$  be an intuitionistic fuzzy norm on  $X$ . Then  $\mathcal{P}_{\mu,\nu}(x, t)$  is nondecreasing with respect to  $t$  for all  $x \in X$ .*

The concepts of convergence and Cauchy sequences in an IFNS are studied in [50].

Let  $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$  be an IFNS. Then, a sequence  $\{x_n\}$  is said to be IF convergent to a point  $x \in X$  (denoted by  $x_n \rightarrow x$ ) if  $\mathcal{P}_{\mu,\nu}(x_n - x, t) \rightarrow 1_{L^*}$  as  $n \rightarrow \infty$  for every  $t > 0$ . The sequence  $\{x_n\}$  is said to be IF Cauchy sequence if for every  $\varepsilon > 0$  and  $t > 0$ , there exists  $n_0 \in \mathbb{N}$  such that  $\mathcal{P}_{\mu,\nu}(x_n - x_m, t) >_{L^*}(N_s(\varepsilon), \varepsilon)$  for all  $n, m \geq n_0$ , where  $N_s$  is the standard negator.  $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$  is said to be complete if every IF Cauchy sequence in  $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$  is IF convergent in  $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ . A complete IFNS is called an IF Banach space.

**Definition 2.11.** *An IFN algebra  $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T}, \mathcal{T}')$  is an IFNS  $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$  with algebraic structure such that*

(vii)  $\mathcal{P}_{\mu,\nu}(xy, ts) \geq_L \mathcal{T}'(\mathcal{P}_{\mu,\nu}(x, t), \mathcal{P}_{\mu,\nu}(y, s))$  for all  $x, y \in X$  and all  $t, s > 0$ . in which  $\mathcal{T}'$  is a continuous  $t$ -norm.

Every normed algebra  $(X, \|\cdot\|)$  defines an IFN algebra  $(X, \mathcal{P}_{\mu,\nu}, \mathcal{M}, \mathcal{M})$ , where

$$\mathcal{P}_{\mu,\nu}(x, t) = \left( \frac{t}{t + \|x\|}, \frac{\|x\|}{t + \|x\|} \right)$$

for all  $t > 0$  if and only if

$$\|xy\| \leq \|x\|\|y\| + s\|y\| + t\|x\| \quad (x, y \in X; \quad t, s > 0).$$

This space is called the induced IFN algebra.

**Definition 2.12.** *Let  $(\mathcal{U}, \mathcal{P}_{\mu,\nu}, \mathcal{T}, \mathcal{T}')$  be a IF Banach algebra, then an involution on  $\mathcal{U}$  is a mapping  $u \rightarrow u^*$  from  $\mathcal{U}$  into  $\mathcal{U}$  which satisfies*

- (i)  $u^{**} = u$  for  $u \in \mathcal{U}$ ;
- (ii)  $(\alpha u + \beta v)^* = \bar{\alpha}u^* + \bar{\beta}v^*$ ;
- (iii)  $(uv)^* = v^*u^*$  for  $u, v \in \mathcal{U}$ .

If, in addition  $\mathcal{P}_{\mu,\nu}(u^*u, ts) = \mathcal{T}'(\mathcal{P}_{\mu,\nu}(u, t), \mathcal{P}_{\mu,\nu}(u, s))$  for  $u \in \mathcal{U}$  and  $t, s > 0$ , then  $\mathcal{U}$  is an IF  $C^*$ -algebra.

We recall a fundamental result in fixed point theory. Let  $\Omega$  be a set. A function  $d : \Omega \times \Omega \rightarrow [0, \infty]$  is called a *generalized metric* on  $\Omega$  if  $d$  satisfies

- (1)  $d(x, y) = 0$  if and only if  $x = y$ ;
- (2)  $d(x, y) = d(y, x)$  for all  $x, y \in \Omega$ ;
- (3)  $d(x, z) \leq d(x, y) + d(y, z)$  for all  $x, y, z \in \Omega$ .

**Theorem 2.13.** [15] *Let  $(\Omega, d)$  be a complete generalized metric space and let  $J : \Omega \rightarrow \Omega$  be a contractive mapping with Lipschitz constant  $K < 1$ . Then for each given element  $x \in \Omega$ , either  $d(J^n x, J^{n+1} x) = \infty$  for all nonnegative integers  $n$  or there exists a positive integer  $n_0$  such that*

- (1)  $d(J^n x, J^{n+1} x) < \infty, \quad \forall n \geq n_0$ ;
- (2) the sequence  $\{J^n x\}$  converges to a fixed point  $y^*$  of  $J$ ;
- (3)  $y^*$  is the unique fixed point of  $J$  in the set  $\Gamma = \{y \in \Omega \mid d(J^{n_0} x, y) < \infty\}$ ;
- (4)  $d(y, y^*) \leq \frac{1}{1-K}d(y, Jy)$  for all  $y \in \Gamma$ .

**Definition 2.14.** [36] Let  $(A, \mathcal{P}_{\mu, \nu}, \mathcal{T}, \mathcal{T}')$  be an intuitionistic fuzzy Banach algebra  $C^*$ -algebra and  $x \in A$  a self-adjoint element, i.e.,  $x^* = x$ . Then  $x$  is said to be positive if it is of the form  $yy^*$  for some  $y \in A$ .

The set of positive elements of  $A$  is denoted by  $A^+$ .

Note that  $A^+$  is a closed convex cone (see [36]).

It is well-known that for a positive element  $x$  and a positive integer  $n$  there exists a unique positive element  $y \in A^+$  such that  $x = y^n$ . We denote  $y$  by  $x^{\frac{1}{n}}$  (see [37]).

Kenary [38] introduced the following functional equation

$$f\left((\sqrt{x} + \sqrt{y})^2\right) = \left(\sqrt{f(x)} + \sqrt{f(y)}\right)^2$$

in the set of non-negative real numbers.

In this paper, we introduce the following functional equation

$$T\left(\left(x^{\frac{1}{m}} + y^{\frac{1}{m}}\right)^m\right) = \left(T(x)^{\frac{1}{m}} + T(y)^{\frac{1}{m}}\right)^m \tag{2.1}$$

for all  $x, y \in A^+$  and a fixed integer  $m$  greater than 1, which is called a *positive-additive functional equation*. Each solution of the positive-additive functional equation is called a *positive-additive mapping*.

Note that the function  $f(x) = cx$ ,  $c \geq 0$ , in the set of non-negative real numbers is a solution of the functional equation (1.1).

In 1996, G. Isac and Th.M. Rassias [39] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [40], [41], [42]–[45]).

Throughout this paper, let  $A^+$  and  $B^+$  be the sets of positive elements in intuitionistic fuzzy  $C^*$ -algebras  $(A, N)$  and  $(B, N)$ , respectively. Assume that  $m$  is a fixed integer greater than 1.

### 3. Stability of the positive-additive functional equation (1.1): fixed point approach

In this section, we investigate the positive-additive functional equation (1.1) in intuitionistic fuzzy  $C^*$ -algebras.

**Lemma 3.1.** [46] Let  $T : A^+ \rightarrow B^+$  be a positive-additive mapping satisfying (1.1). Then  $T$  satisfies

$$T(2^{mn}x) = 2^{mn}T(x)$$

for all  $x \in A^+$  and all  $n \in \mathbb{Z}$ .

Using the fixed point method, we prove the Hyers-Ulam stability of the positive-additive functional equation (1.1) in intuitionistic fuzzy  $C^*$ -algebras.

Note that the fundamental ideas in the proofs of the main results in this section are contained in [14, 40, 41].

**Theorem 3.2.** Let  $\varphi : A^+ \times A^+ \times (0, \infty) \rightarrow L^*$  be a function such that there exists an  $K < 1$  with

$$\varphi(x, y, t) \geq_L \varphi\left(2^m x, 2^m y, \frac{2^m t}{K}\right) \tag{3.1}$$

for all  $x, y \in A^+$  and  $t > 0$ . Let  $f : A^+ \rightarrow B^+$  be a mapping satisfying

$$\mathcal{P}_{\mu,\nu} \left( f \left( \left( x^{\frac{1}{m}} + y^{\frac{1}{m}} \right)^m \right) - \left( f(x)^{\frac{1}{m}} + f(y)^{\frac{1}{m}} \right)^m, t \right) \geq_L \varphi(x, y, t) \tag{3.2}$$

for all  $x, y \in A^+$  and  $t > 0$ . Then there exists a unique positive-additive mapping  $T : A^+ \rightarrow A^+$  satisfying (1.1) and

$$\mathcal{P}_{\mu,\nu}(f(x) - T(x), t) \geq_L \varphi \left( x, x, \frac{(2^m - 2^m L)t}{L} \right) \tag{3.3}$$

for all  $x \in A^+$  and  $t > 0$ .

**Proof .** Letting  $y = x$  in (2.2), we get

$$\mathcal{P}_{\mu,\nu}(f(2^m x) - 2^m f(x), t) \geq_L \varphi(x, x, t) \tag{3.4}$$

for all  $x \in A^+$  and  $t > 0$ .

Consider the set

$$X := \{g : A^+ \rightarrow B^+\}$$

and introduce the generalized metric on  $X$ :

$$d(g, h) = \inf \{ \mu \in \mathbb{R}_+ : \mathcal{P}_{\mu,\nu}(g(x) - h(x), t) \geq_L \varphi \left( x, x, \frac{t}{\mu} \right), \forall x \in A^+, t > 0 \},$$

where, as usual,  $\inf \emptyset = +\infty$ .

It is easy to show that  $(X, d)$  is complete (see [47]).

Now, we consider the linear mapping  $J : X \rightarrow X$  such that

$$Jg(x) := 2^m g \left( \frac{x}{2^m} \right)$$

for all  $x \in A^+$ .

Let  $g, h \in X$  be given such that  $d(g, h) = \varepsilon$ . Then

$$\mathcal{P}_{\mu,\nu}(g(x) - h(x), t) \geq \varphi(x, x, t)$$

for all  $x \in A^+$  and  $t > 0$ . Hence

$$\mathcal{P}_{\mu,\nu}(Jg(x) - Jh(x), t) = \mathcal{P}_{\mu,\nu} \left( 2^m g \left( \frac{x}{2^m} \right) - 2^m h \left( \frac{x}{2^m} \right), t \right) \geq_L \varphi \left( x, x, \frac{t}{L} \right)$$

for all  $x \in A^+$  and  $t > 0$ . So  $d(g, h) = \varepsilon$  implies that  $d(Jg, Jh) \leq L\varepsilon$ . This means that

$$d(Jg, Jh) \leq Ld(g, h)$$

for all  $g, h \in X$ .

It follows from (2.4) that

$$\mathcal{P}_{\mu,\nu} \left( f(x) - 2^m f \left( \frac{x}{2^m} \right), t \right) \geq_L \varphi \left( x, x, \frac{2^m t}{L} \right)$$

for all  $x \in A^+$  and  $t > 0$ . So  $d(f, Jf) \leq \frac{L}{2^m}$ .

By Theorem 1.1, there exists a mapping  $T : A^+ \rightarrow B^+$  satisfying the following:

(1)  $T$  is a fixed point of  $J$ , i.e.,

$$T\left(\frac{x}{2^m}\right) = \frac{1}{2^m}T(x) \tag{3.5}$$

for all  $x \in A^+$ . The mapping  $T$  is a unique fixed point of  $J$  in the set

$$M = \{g \in X : d(f, g) < \infty\}.$$

This implies that  $T$  is a unique mapping satisfying (2.5) such that there exists a  $\beta \in (0, \infty)$  satisfying

$$\mathcal{P}_{\mu,\nu}(f(x) - T(x), t) \geq_L \varphi\left(x, x, \frac{t}{\beta}\right)$$

for all  $x \in A^+$  and  $t > 0$ ;

(2)  $d(J^n f, T) \rightarrow 0$  as  $n \rightarrow \infty$ . This implies the equality

$$\lim_{n \rightarrow \infty} 2^{mn} f\left(\frac{x}{2^{mn}}\right) = T(x)$$

for all  $x \in A^+$ ;

(3)  $d(f, T) \leq \frac{1}{1-K}d(f, Jf)$ , which implies the inequality

$$d(f, T) \leq \frac{K}{2^m - 2^m K}.$$

This implies that the inequality (2.3) holds.

By (2.1) and (2.2),

$$\begin{aligned} & \mathcal{P}_{\mu,\nu}\left(f\left(\frac{\left(x^{\frac{1}{m}} + y^{\frac{1}{m}}\right)^m}{2^{mn}}\right) - \left(\left(2^{mn} f\left(\frac{x}{2^{mn}}\right)\right)^{\frac{1}{m}} + \left(2^{mn} f\left(\frac{y}{2^{mn}}\right)\right)^{\frac{1}{m}}\right)^m, \frac{t}{2^{mn}}\right) \\ & \geq_L \varphi\left(\frac{x}{2^{mn}}, \frac{y}{2^{mn}}, \frac{t}{2^{mn}}\right) \\ & \geq_L \varphi\left(x, y, \frac{t}{K^{mn}}\right) \end{aligned}$$

for all  $x, y \in A^+$ , all  $n \in \mathbb{N}$  and  $t > 0$ . So

$$\mathcal{P}_{\mu,\nu}\left(T\left(\left(x^{\frac{1}{m}} + y^{\frac{1}{m}}\right)^m\right) - \left(T(x)^{\frac{1}{m}} + T(y)^{\frac{1}{m}}\right)^m, t\right) = 1_{L^*}$$

for all  $x, y \in A^+$  and  $t > 0$ . Thus the mapping  $T : A^+ \rightarrow B^+$  is positive-additive, as desired.  $\square$

**Corollary 3.3.** *Let  $p > 1$  and  $\theta_1, \theta_2$  be non-negative real numbers, and let  $f : A^+ \rightarrow B^+$  be a mapping such that*

$$\begin{aligned} & \mathcal{P}_{\mu,\nu}\left(f\left(\left(x^{\frac{1}{m}} + y^{\frac{1}{m}}\right)^m\right) - \left(f(x)^{\frac{1}{m}} + f(y)^{\frac{1}{m}}\right)^m, t\right) \\ & \geq_L \left(\frac{t}{t + \theta_1(\|x\|^p + \|y\|^p) + \theta_2 \cdot \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}}}, \frac{\theta_1(\|x\|^p + \|y\|^p) + \theta_2 \cdot \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}}}{t + \theta_1(\|x\|^p + \|y\|^p) + \theta_2 \cdot \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}}}\right) \end{aligned} \tag{3.6}$$

for all  $x, y \in A^+$  and  $t > 0$ . Then there exists a unique positive-additive mapping  $T : A^+ \rightarrow B^+$  satisfying (1.1) and

$$\mathcal{P}_{\mu,\nu}(f(x) - T(x), t) \geq_L \left( \frac{t}{t + \frac{2\theta_1 + \theta_2}{2^{mp} - 2^m}} \|x\|^p, \frac{\frac{2\theta_1 + \theta_2}{2^{mp} - 2^m}}{t + \frac{2\theta_1 + \theta_2}{2^{mp} - 2^m}} \|x\|^p \right)$$

for all  $x \in A^+$  and  $t > 0$ .

**Proof .** The proof follows from Theorem 3.2 by taking

$$\varphi(x, y, t) = \frac{t}{t + \theta_1(\|x\|^p + \|y\|^p) + \theta_2 \cdot \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}}}$$

for all  $x, y \in A^+$  and  $t > 0$ . Then we can choose  $K = 2^{m-mp}$  and we get the desired result.  $\square$

**Theorem 3.4.** Let  $\varphi : A^+ \times A^+ \times (0, \infty) \rightarrow L^*$  be a function such that there exists an  $K < 1$  with

$$\varphi(x, y, t) \geq_L \varphi\left(\frac{x}{2^m}, \frac{y}{2^m}, \frac{t}{2^m K}\right)$$

for all  $x, y \in A^+$  and  $t > 0$ . Let  $f : A^+ \rightarrow B^+$  be a mapping satisfying (2.2). Then there exists a unique positive-additive mapping  $T : A^+ \rightarrow A^+$  satisfying (1.1) and

$$\mathcal{P}_{\mu,\nu}(f(x) - T(x), t) \geq_L \varphi(x, x, (2^m - 2^m K)t)$$

for all  $x \in A^+$  and  $t > 0$ .

**Proof .** Let  $(X, d)$  be the generalized metric space defined in the proof of Theorem 3.2.

Consider the linear mapping  $J : X \rightarrow X$  such that

$$Jg(x) := \frac{1}{2^m} g(2^m x)$$

for all  $x \in A^+$ .

It follows from (2.4) that

$$\mathcal{P}_{\mu,\nu}\left(f(x) - \frac{1}{2^m} f(2^m x), t\right) \geq_L \varphi(x, x, 2^m t)$$

for all  $x \in A^+$  and  $t > 0$ . So  $d(f, Jf) \leq \frac{1}{2^m}$ .

The rest of the proof is similar to the proof of Theorem 3.2.  $\square$

**Corollary 3.5.** Let  $0 < p < 1$  and  $\theta_1, \theta_2$  be non-negative real numbers, and let  $f : A^+ \rightarrow B^+$  be a mapping satisfying (2.6). Then there exists a unique positive-additive mapping  $T : A^+ \rightarrow B^+$  satisfying (1.1) and

$$\mathcal{P}_{\mu,\nu}(f(x) - T(x), t) \geq_L \left( \frac{t}{t + \frac{2\theta_1 + \theta_2}{2^m - 2^{mp}}} \|x\|^p, \frac{\frac{2\theta_1 + \theta_2}{2^m - 2^{mp}}}{t + \frac{2\theta_1 + \theta_2}{2^m - 2^{mp}}} \|x\|^p \right)$$

for all  $x \in A^+$  and  $t > 0$ .

**Proof .** The proof follows from Theorem 3.4 by taking

$$\varphi(x, y, t) = \left( \frac{t}{t + \theta_1(\|x\|^p + \|y\|^p) + \theta_2 \cdot \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}}}, \frac{\theta_1(\|x\|^p + \|y\|^p) + \theta_2 \cdot \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}}}{t + \theta_1(\|x\|^p + \|y\|^p) + \theta_2 \cdot \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}}} \right)$$

for all  $x, y \in A^+$  and  $t > 0$ . Then we can choose  $K = 2^{mp-m}$  and we get the desired result.  $\square$



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