Int. J. Nonlinear Anal. Appl. 12 (2021) No. 1, 27–36
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.4652

A numerical solution of variable order diffusion and wave equations

Nematollah Kadkhoda[a,*], Hossein Jafari[b,c], Roghayeh Moallem Ganji[b]

[a] *Department of Mathematics, Faculty of Basic Sciences, Bozorgmehr University of Qaenat, Qaenat, Iran.*
[b] *Department of Mathematics, University of Mazandaran, Babolsar, Iran.*
[c] *Department of Mathematical Sciences, University of South Africa, UNISA0003, South Africa.*

Abstract

In this work, we consider variable order diffusion and wave equations. The derivative is described in the Caputo sense of variable order. We use the Genocchi polynomials as basic functions and obtain operational matrices via these polynomials. These matrices and collocation method help us to convert variable order diffusion and wave equations to an algebraic system. Some examples are given to show the validity of the presented method.

Keywords: Variable order diffusion and wave equations, Genocchi polynomials, Operational matrix, Collocation method.
2010 MSC: 47A52; 35K05; 35R30; 65M70.

1. Introduction

Differential equations have been successfully applied in physics and engineering such as earthquake analysis, bio-chemical, electric circuits, controller design, signal processing, viscoelasticity and so on [1]. During last few decades, several approximation and numerical methods have been applied for solving class of differential equations, for example the homotopy analysis method [15], Variational iteration method [16], Bernstein polynomials [17, 26], Legendre polynomials [6, 9], Bernoulli polynomials [21], Chebyshev polynomials [5, 8, 20], Genocchi polynomials (GPs) [22, 23]. In recent years using of fractional partial differential equations (FPDEs) in mathematical models has become increasing popular. For solving these types of equations have been proposed different methods [3, 12, 13, 14].

In 1993, Samko and Ross [24] have introduced the variable order derivative operator just as a generalization of the fractional order derivative and studied some of its main properties. In this operator, the order of derivative is a function of independent variables such as time and space variables. Soon after, a variety of definitions has been offered for variable order derivative and integral

*Corresponding author: kadkhoda@buqaen.ac.ir

Received: July 2020 *Revised:* August 2020

operators such as Riemann-Liouville (RL)[10], Coimbra [25], Caputo [30], Caputo and Fabrizio [30] and Atangana–Baleanu–Caputo [30] derivatives. Therefore, many problems in various field can be modeling via systems of variable order ordinary/ partial/ integro-differential equations. Since derivative operator has a kernel of the variable order, it is not simply task to obtain the solution of such equations. Therefore, developing an effective numerical algorithms for solving such equations is importance. Some researchers have proposed several approximation and numerical methods for solving such equations, for example Li and Wu, Yang *et al.* used the reproducing kernel method for solving the variable order fractional functional boundary value problems [19, 29]. Heydari et al. applied Chebyshev wavelets for solving multi-term variable-order time fractional diffusion-wave equation [13]. Ganji and Jafari applied Jacobi polynomials to obtain solution the multi-variable orders differential equations [4]. Yu and Ertürk applied a finite difference method to variable order fractional integro-differential equations [28]. Ganji *et al.* applied the fifth-kind Chebyshev polynomials to obtain solution variable orders differential equations [8]. Hassani and Naraghirad solved variable-order time fractional Burgers equation via generalized polynomials [12]. Jiang and Guo applied the reproducing kernel method for solving two-dimensional variable-order anomalous sub-diffusion equation [18]. Jafari *et al.* used operational matrix (OM) based on Bernstein polynomials for solving variable order differential equations [17].

In this work, we study the following type of variable order diffusion and wave equations

$$\partial_t^{\kappa(x,t)} X(x,t) + \xi(x,t)\partial_x^{\upsilon(x,t)} X(x,t) = F\left(x,t,X(x,t)\right), \tag{1.1}$$

with the initial and boundary conditions

$$X(x,0) = f_0(x), \quad X(0,t) = f_1(t), \quad X(1,t) = f_2(t), \quad 0 < x,t < 1, \tag{1.2}$$

where $0 < \kappa(x,t) \le 1$, $1 < \upsilon(x,t) \le 2$. $\xi(x,t) \in L^2\left([0,1]\times[0,1]\right)$ is given known function. $X(x,t)$ is an unknown function to be determined. $\partial_t^{\kappa(x,t)} X(x,t)$ and $\partial_x^{\upsilon(x,t)} X(x,t)$ are the time-fractional and the space-fractional derivatives of variable order which are defined [7]

$$\partial_t^{\kappa(x,t)} X(x,t) = \frac{1}{\Gamma(1-\kappa(x,t))}\int_0^t (t-s)^{-\kappa(x,t)}\, \partial_s' X(x,s)\, ds,$$
$$\partial_x^{\upsilon(x,t)} X(x,t) = \frac{1}{\Gamma(2-\upsilon(x,t))}\int_0^x (x-s)^{1-\upsilon(x,t)}\, \partial_s^2 X(s,t)\, ds.$$

This work is organized in five sections. Section (2) Genocchi polynomials with their properties are reviewed. We proposed a numerical scheme for solving variable order diffusion and wave equations in Section (3). Section (4) shows many numerical examples and Section (5) consists of a brief summary.

2. The Genocchi polynomials and their properties

In this section, we recall some basic properties of GPs and explain how to approximate a function in terms of these basis functions and obtain the OMs based on the GPs. Finally, we discusse convergence analysis using the GPs.

2.1. The GPs

The Genocchi numbers (g_i) and GPs $(G_i(t))$ are respectively given by

$$\frac{2x}{e^x+1} = \sum_{i=0}^{\infty} g_i \frac{x^i}{i!}, \quad |x| < \pi,$$
$$\frac{2xe^{xt}}{e^x+1} = \sum_{i=0}^{\infty} G_i(t) \frac{x^i}{i!}, \quad |x| < \pi.$$

The analytic form of the GPs of degree i is defined by

$$G_i(t) = \sum_{k=0}^{i} \binom{i}{k} g_{i-k} t^k.$$

It is easy to show the GPs satisfy the following properties

$$\begin{aligned}
&G_i(1) + G_i(0) = 0, && i > 1,\\
&\frac{dG_i(t)}{dt} = iG_{i-1}(t), && i \geq 1,\\
&\int_0^1 G_i(t)G_j(t)dt = \frac{2(-1)^i i!j!}{(i+j)!} g_{i+j}, && i, j \geq 1.
\end{aligned} \tag{2.1}$$

2.2. Function approximation

Any arbitrary function $X(x,t)$ in $L^2((0,1)\times(0,1))$ can be approximated in terms of the GPs as

$$X(x,t) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} c_{ij}\, G_i(x)\, G_j(t) = \varphi(x)^T\, C\, \varphi(t). \tag{2.2}$$

By taking only the first $n+1$ terms in (2.2), $X(x,t)$ can be approximated as

$$X(x,t) \simeq X_n(x,t) = \sum_{i=1}^{n+1}\sum_{j=1}^{n+1} c_{ij}\, G_i(x)\, G_j(t) = \varphi(x)^T\, C\, \varphi(t), \tag{2.3}$$

where

$$\begin{aligned}
\varphi(x) &= [G_1(x), G_2(x), \ldots, G_{n+1}(x)]^T,\\
\varphi(t) &= [G_1(t), G_2(t), \ldots, G_{n+1}(t)]^T,
\end{aligned}$$

and C can be computed by

$$C = Q^{-1}\, \langle \varphi(x),\, \langle X_n(x,t),\, \varphi(t)\rangle\rangle\, Q^{-1},$$

where $Q = [q_{ij}]$ is an $(n+1)\times(n+1)$ matrix whose elements are given by (2.1).

2.3. The OMs based on the GPs

In order to introduce the operational matrix, we first rewrite the basis vectors $\varphi(x)$ and $\varphi(t)$ in terms of the Taylor basis functions as

$$\begin{aligned}
\varphi(x) &= AT_n(x),\\
\varphi(t) &= AT_n(t),
\end{aligned} \tag{2.4}$$

where

$$\begin{aligned}
T_n(x) &= [1, x, \cdots, x^n]^T,\\
T_n(t) &= [1, t, \cdots, t^n]^T,
\end{aligned}$$

are the Taylor basis vectors and $A=[a_{i,j}],\ i,j=1,\cdots,n+1$, with

$$a_{i,j}=\begin{cases}\binom{i}{j-1}g_{i-j+1}, & i\geq j,\\ 0, & i<j.\end{cases}$$

1) By applying the first order derivative to both sides of (2.4), we get

$$\frac{d}{dt}\varphi(t)=D\varphi(t),$$

where D is the operational matrix of derivative based on the GPs given by

$$D=A\begin{bmatrix}0&0&0&\cdots&0&0\\1&0&0&\cdots&0&0\\0&2&0&\cdots&0&0\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\0&0&0&\cdots&M&0\end{bmatrix}A^{-1}.$$

Also, for the i-th order derivative of the vector $\varphi(t)$, we have

$$\frac{d^i\varphi(t)}{dt^i}=D^i\,\varphi(t),\quad i\geq 2.$$

Similarly, we can write

$$\frac{d^i\varphi(x)}{dx^i}=D^i\varphi(x),\quad i\geq 2.$$

2) By applying the derivative operator of variable order, $p-1\leq\zeta(x,t)\leq p$, to both sides of (2.4) yields

$$\partial_t^{\zeta(x,t)}\varphi(t)=\partial_t^{\zeta(x,t)}\left(AT_n(t)\right)=A\,\partial_t^{\zeta(x,t)}T_n(t)=A\,\partial_t^{\zeta(x,t)}[1\ t\ \cdots\ t^{p-1}\ t^p\ \ldots\ t^n]^T. \tag{2.5}$$

The Caputo derivative of variable order of the function t^i is given by

$$\partial_t^{\zeta(x,t)}t^i=\begin{cases}\frac{\Gamma(i+1)}{\Gamma(i-\zeta(x,t)+1)}t^{i-\zeta(x,t)}, & i\in N\ and\ i\geq\lceil\zeta(x,t)\rceil\ or\ i\notin N\ and\ i>\lfloor\zeta(x,t)\rfloor,\\ 0, & i\in N\cup\{0\}\ and\ i<\lceil\zeta(x,t)\rceil.\end{cases} \tag{2.6}$$

By utilizing (2.6) in (2.5), we obtain

$$\partial_t^{\zeta(x,t)}\varphi(t)=A[0\ 0\ \cdots\ 0\ \frac{\Gamma(p+1)}{\Gamma(p+1-\zeta(x,t))}t^{p-\zeta(x,t)}\ \cdots\ \frac{\Gamma(n+1)}{\Gamma(n+1-\zeta(x,t))}t^{n-\zeta(x,t)}]^T=A\Psi_tT_n(t),$$

where

$$\Psi_t=[\rho_t^{i,j}],\quad i,j=0,1,\cdots,n,$$

with

$$\rho_t^{i,j}=\begin{cases}\frac{\Gamma(i+1)}{\Gamma(i+1-\zeta(x,t))}t^{-\zeta(x,t)}, & i=j\ \&\ i\geq p,\\ 0, & otherwise.\end{cases}$$

From (2.4), we have

$$\partial_t^{\zeta(x,t)}\varphi(t)=A\Psi_tA^{-1}\varphi(t)=\Omega_t\varphi(t),$$

where $\Omega_t=A\Psi_tA^{-1}$ is called the OM of the Caputo derivative of variable order based on the GPs. Similarly, we can write

$$\partial_x^{\zeta(x,t)}\varphi(x)=A\Psi_xA^{-1}\varphi(x)=\Omega_x\varphi(x).$$

2.4. Convergence analysis

Here our aim is to obtain an analytic expression for the error norm of the best approximation of a smooth function $X(x,t) \in I = [0,1] \times [0,1]$ by its expansion in terms of double GPs. Consider

$$\Pi_n = span\{G_i(x)\ G_j(t),\ i,j = 1,2,\dots,n+1\}.$$

Let that P_n is the interpolating polynomials to X at points (x_i, t_j), where $x_i, i = 1,2,\cdots,n+1$ and $t_j, j = 1,2,\cdots,n+1$ are the roots of $(n+1)$-degree shifted Chebysheve polynomials on $[0,1]$. Then [2, 11, 13, 27]

$$\begin{aligned} X(x,t) - P_n(x,t) =& \frac{1}{(n+1)!}\frac{\partial^{n+1}X(\xi,t)}{\partial x^{n+1}}\prod_{i=1}^{n+1}(x-x_i) + \frac{1}{(n+1)!}\frac{\partial^{n+1}X(x,\eta)}{\partial t^{n+1}}\prod_{j=1}^{n+1}(t-t_j) \\ &- \frac{1}{((n+1)!)^2}\frac{\partial^{2n+2}X(\xi',\eta')}{\partial x^{n+1}\partial t^{n+1}}\prod_{i=1}^{n+1}(x-x_i)\prod_{j=1}^{n+1}(t-t_j), \end{aligned}$$

where ξ, ξ', η, $\eta' \in [0,1]$. Then we obtain

$$\begin{aligned} |X(x,t) - P_n(x,t)| \leq \quad & \max_{(x,t)\in I}\left|\frac{\partial^{n+1}X(\xi,t)}{\partial x^{n+1}}\right| \frac{\prod_{i=1}^{n+1}|x-x_i|}{(n+1)!} \\ & + \max_{(x,t)\in I}\left|\frac{\partial^{n+1}X(x,\eta)}{\partial t^{n+1}}\right| \frac{\prod_{j=1}^{n+1}|t-t_j|}{(n+1)!} \\ & + \max_{(x,t)\in I}\left|\frac{\partial^{2n+2}X(\xi',\eta')}{\partial x^{n+1}\ \partial t^{n+1}}\right| \frac{\prod_{i=1}^{n+1}|x-x_i|\prod_{j=1}^{n+1}|t-t_j|}{((n+1)!)^2}. \end{aligned} \tag{2.7}$$

Since $X(x,t)$ is a smooth function on I, then there exist constants σ_1, σ_2 and σ_3, such that

$$\max_{(x,t)\in I}\left|\frac{\partial^{n+1}X(x,t)}{\partial x^{n+1}}\right| \leq \sigma_1, \quad \max_{(x,t)\in I}\left|\frac{\partial^{n+1}X(x,t)}{\partial t^{n+1}}\right| \leq \sigma_2, \quad \max_{(x,t)\in I}\left|\frac{\partial^{2n+2}X(x,t)}{\partial x^{n+1}\ \partial t^{n+1}}\right| \leq \sigma_3. \tag{2.8}$$

By substituting (2.8) into (2.7) and employing the estimates for Chebysheve interpolation nodes, we have

$$|X(x,t) - P_n(x,t)| \leq \frac{1}{(n+1)!2^{2n+1}}\left(\sigma_1 + \sigma_2 + \frac{\sigma_3}{(n+1)!2^{2n+1}}\right). \tag{2.9}$$

Since X_n is the best approximation of X in Π_n, that is

$$\|X(x,t) - X_n(x,t)\|_2 \leq \|X(x,t) - X^*(x,t)\|_2\,,$$

where X^* is any arbitrary polynomial in Π_n. Then, using (2.9) we obtain

$$\begin{aligned} \|X(x,t) - X_n(x,t)\|_2^2 &= \int_0^1\int_0^1 |X(x,t) - X_n(x,t)|^2\,dxdt \\ &\leq \int_0^1\int_0^1 |X(x,t) - P_n(x,t)|^2\,dxdt \\ &= \int_0^1\int_0^1 \left(\tfrac{1}{(n+1)!2^{2n+1}}\left(\sigma_1 + \sigma_2 + \tfrac{\sigma_3}{(n+1)!2^{2n+1}}\right)\right)^2 dxdt \\ &= \left(\tfrac{1}{(n+1)!2^{2n+1}}\left(\sigma_1 + \sigma_2 + \tfrac{\sigma_3}{(n+1)!2^{2n+1}}\right)\right)^2. \end{aligned} \tag{2.10}$$

By taking the square root of both sides of (2.10) yields

$$\|X(x,t) - X_n(x,t)\|_2 \leq \frac{1}{(n+1)!2^{2n+1}}\left(\sigma_1 + \sigma_2 + \frac{\sigma_3}{(n+1)!2^{2n+1}}\right).$$

3. Proposed method for solving the equation (1.1)

. To find solution $X(x,t)$, do the following steps
Step 1. Consider the equation (1.1) under the initial and boundary conditions (1.2).
Step 2. Approximate the unknown function ($X(x,t)$) as (2.3) and substitute in the equations (1.1) and (1.2).
Step 3. Calculate the OMs and substitute in the equation (1.1).

• The outline of steps 2 and 3 are as

$$\varphi(x)^T\ C\ \Delta_t\ \varphi(t) + \xi(x,t)\ \left(\Pi_x\ \varphi(x)\right)^T\ C\ \varphi(t) = F\left(x,t,\varphi(x)^T\ C\ \varphi(t)\right),$$
$$\varphi(x)^T\ C\ \varphi(0) = f_0(x),$$
$$\varphi(0)^T\ C\ \varphi(t) = f_1(t), \quad \varphi(1)^T\ C\ \varphi(t) = f_2(t),$$

where

$$\Delta_t = \begin{cases} D, & \kappa(x,t) = 1, \\ \Omega_t, & 0 < \kappa(x,t) < 1, \end{cases} \qquad \Pi_x = \begin{cases} D^2, & \upsilon(x,t) = 2, \\ \Omega_x, & 1 < \kappa(x,t) < 2. \end{cases}$$

Step 4. Calculate the residual function.

• The residual function can be calculated as

$$R(x,t) = \varphi(x)^T\ C\ \Delta_t\ \varphi(t) + \xi(x,t)\ \left(\Pi_x\ \varphi(x)\right)^T\ C\ \varphi(t) - F\left(x,t,\varphi(x)^T\ C\ \varphi(t)\right).$$

Step 5. Let $x_i = \dfrac{i}{n}$ and $t_j = \dfrac{j}{n}$ for $i,j = 0,1,\ldots,n$ be the collocation points, then solve obtained system and obtain the uniqu coefficients. Finally, substitute the results into step 2.

• By solving below system, coefficients c_{ij} can be calculated.

$$R(x_i,t_j) = 0, \qquad i = 1,2,\ldots,n-1, \quad j = 1,2,\ldots,n,$$

$$\begin{array}{lr} \varphi(x_i)^T\ C\ \varphi(0) = f_0(x_i), & i = 0,1,\ldots,n, \\ \varphi(0)^T\ C\ \varphi(t_j) = f_1(t_j), \quad \varphi(1)^T\ C\ \varphi(t_j) = f_2(t_j), & j = 1,2,\ldots,n. \end{array}$$

4. Test Examples

We present three examples to compare the approximate solution with exact solution. The absolute errors are defined as

$$Error = \left|X(x,t) - \varphi(x)^T\ C\ \varphi(t)\right|, \quad (x,t) \in [0,1] \times [0,1].$$

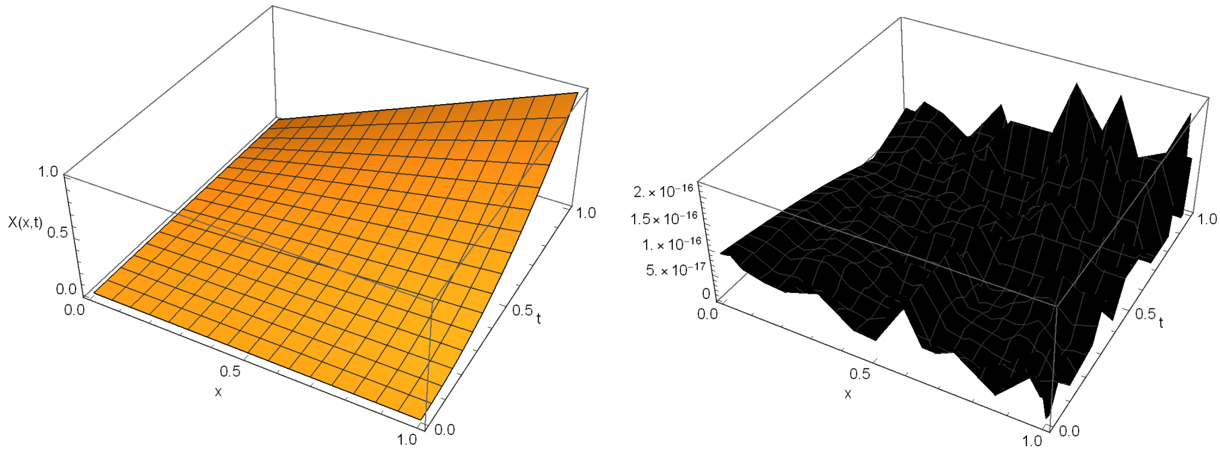
Example 4.1. *Consider the following problem*

$$\frac{\partial^{\kappa(x,t)} X(x,t)}{\partial t^{\kappa(x,t)}} - \frac{\partial^2 X(x,t)}{\partial x^2} = \frac{2\ x\ t^{2-\kappa(x,t)}}{\Gamma(3-\kappa(x,t))},$$

where $\kappa(x,t) = \sin(xt)$ *and with the exact solution* $X(x,t) = xt^2$ *and*

$$X(x,0) = 0, \qquad X(0,t) = 0, \quad X(1,t) = t^2, \quad 0 < x,t < 1.$$

By using the presented method, we obtained the approximate solution. The numerical results are shown in Figure 1 and Table 1.



(a) (b)

Figure 1: (a) The exact solution (b) The absolute error ($n = 3$).

Table 1: Comparison absolute error for various $\kappa(x,t)$ ($n = 3$).

(x,t)	$\kappa(x,t) = \frac{1-(xt)^4}{5}$	$\kappa(x,t) = x^2t^2$	$\kappa(x,t) = \cos(xt)$
$(0.1, 0.1)$	$2.146720e-17$	$4.098284e-17$	$7.567731e-17$
$(0.3, 0.3)$	$1.040834e-18$	$1.387779e-17$	$3.469447e-18$
$(0.5, 0.5)$	$5.551115e-17$	$8.326673e-17$	0
$(0.7, 0.7)$	$5.551115e-17$	$1.110223e-16$	$5.551115e-17$
$(0.9, 0.9)$	$1.110223e-16$	0	$1.110223e-16$

Example 4.2. *Consider the following problem*

$$\frac{\partial^{\kappa(x,t)}X(x,t)}{\partial t^{\kappa(x,t)}} + \frac{\partial^{\upsilon(x,t)}X(x,t)}{\partial x^{\upsilon(x,t)}} = \frac{x^2\ t^{1-\kappa(x,t)}}{\Gamma(2-\kappa(x,t))} + \frac{2\ t\ x^{2-\upsilon(x,t)}}{\Gamma(3-\upsilon(x,t))},$$

and

$$X(x,0) = 0, \quad X(0,t) = 0, \quad X(1,t) = t, \quad 0 < x,t < 1,$$

where $\kappa(x,t)$ *and* $\upsilon(x,t)$ *are* $\sin(xt)$ *and* $1+\cos(xt)$ *respectively. We applied the presented method for solving this example. The exact solution* ($X(x,t) = x^2t$) *and the absolute error are shown in Figure 2.*

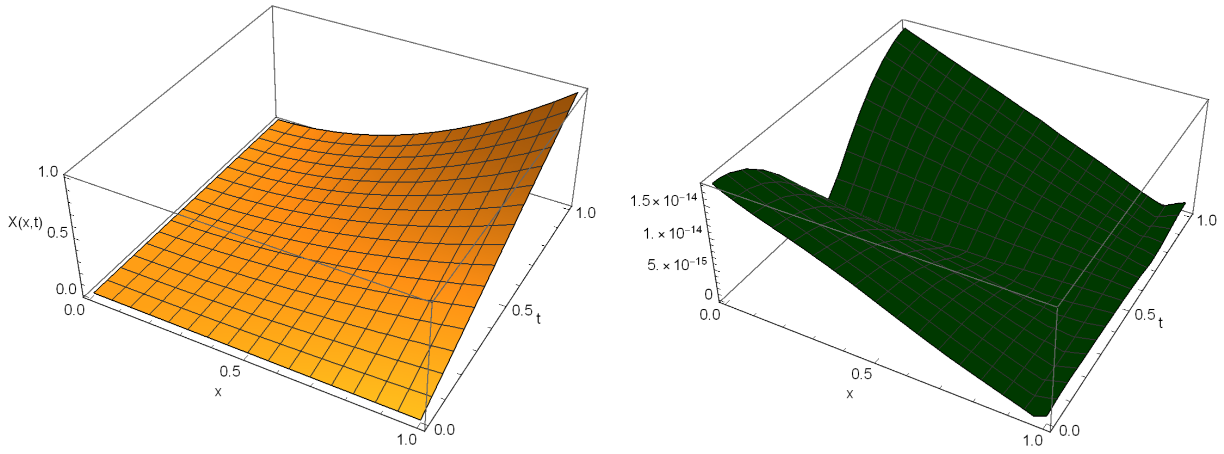
Example 4.3. *Consider the following problem*

$$\frac{\partial^{\kappa(x,t)}X(x,t)}{\partial t^{\kappa(x,t)}} - \frac{1}{2}\ x^2\frac{\partial^2 X(x,t)}{\partial x^2} = x^2\ e^t(\frac{\Gamma(\sin(xt)) - \Gamma(\sin(xt),t)}{\Gamma(\sin(xt))} - 1),$$

where $\kappa(t) = 1 - \sin(xt)$ *and*

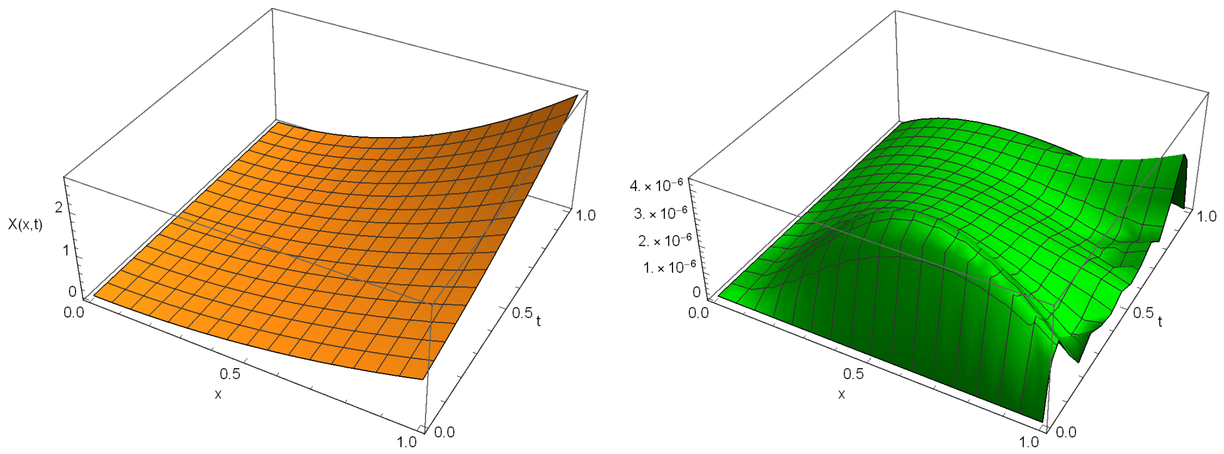
$$X(x,0) = x^2, \qquad X(0,t) = 0, \quad X(1,t) = e^t, \quad 0 < x,t < 1.$$

By applying the presented method, we obtained the approximate solution. The exact solution ($X(x,t) = x^2e^t$) *and the absolute error are shown in Figure 3 and Table 2.*



(a) (b)

Figure 2: (a) The exact solution (b) The absolute error ($n = 5$).



(a) (b)

Figure 3: (a) The exact solution (b) The absolute error ($n = 5$).

5. Conclusion

The purpose of the current paper was to numerical solutions of variable order diffusion and wave equations by using operational matrix based on Genocchi polynomials. First, we approximated the unknown functions and its derivatives in terms of the Genocchi basis. Then, we substituted these approximations in variable order diffusion and wave equations, and got an algebraic system. By applying collocation method, we obtained the approximate solution.

Acknowledgment

The authors would like to acknowledge the financial support of Bozorgmehr University of Qaenat for this research under contract number 39157.

References

[1] T. M. Atanackovic, S. Pilipovic, B. Stankovic, and D. Zorica, *Fractional Calculus with Applications in Mechanics*, Wiley, London, 2014.

Table 2: Comparison absolute error for various n ($\kappa(x,t) = 1 - \sin(xt)$).

(x,t)	$n=2$	$n=3$	$n=4$	$n=5$
$(0.1, 0.1)$	$8.803923e-4$	$1.049905e-4$	$5.514766e-6$	$2.517008e-7$
$(0.3, 0.3)$	$5.303140e-3$	$4.959848e-4$	$3.289236e-5$	$2.045076e-6$
$(0.5, 0.5)$	$6.105720e-3$	$3.335951e-4$	$2.313688e-5$	$1.728718e-6$
$(0.7, 0.7)$	$1.699835e-3$	$1.584021e-4$	$1.811637e-5$	$6.000226e-7$
$(0.9, 0.9)$	$8.182640e-3$	$7.047627e-4$	$3.447973e-5$	$1.846946e-6$

[2] A. H. Bhrawya, and M. A. Zakyc, *A method based on the Jacobi tau approximation for solving multi-term time-space fractional partial differential equations*, J. Comput. Phys. 281 (2015), 876–895.

[3] M. A. Firoozjaee, and S. A. Yousefi, *A numerical approach for fractional partial differential equations by using Ritz approximation*, Appl. Math. Comput. 338 (2018), 711–721.

[4] R. M. Ganji, and H. Jafari, *A numerical approach for multi-variable orders differential equations using Jacobi polynomials*, Int. J. Appl. Comput. Math. 5 (2019).

[5] R. M. Ganji, and H. Jafari, *Numerical solution of variable order integro-differential equations*, Adv. Math. Mod. Appl. *4(1)* (2019), 64–69.

[6] R. M. Ganji, and H. Jafari, *A new approach for solving nonlinear Volterra integro-differential equations with Mittag–Leffler kernel*, Proce. Inst. Math. Mech. 46(1) (2020), 144–158.

[7] R. M. Ganji, H. Jafari, and A. R. Adem, *A numerical scheme to solve variable order diffusion-wave equations*, Thermal Sci. 23 (2019), 371–371.

[8] R. M. Ganji, H. Jafari and D. Baleanu, *A new approach for solving multi variable orders differential equations with Mittag-Leffler kernel*, Chaos Solitons Fractals 130 (2020), 109405.

[9] R. M. Ganji, H. Jafari, and S. Nemati, *A new approach for solving integro-differential equations of variable order*, J. Comput. Appl. Math. 379 (2020), 112946.

[10] E. Doha, M. Abdelkawy, A. Z. M. Amin and D. Baleanu, *Spectral technique for solving variable-order fractional Volterra integro-differential equations*, Numer. Meth. Partial Differ. Equ., 34(5) (2018), 1659–1677.

[11] M. Gasca, and T. Sauer, *On the history of multivariate polynomial interpolation*, J. Comput. Appl. Math. 122(1-2) (2000), 23–35.

[12] H. Hassani, and E. Naraghirad, *A new computational method based on optimization scheme for solving variable-order time fractional Burgers equation*, Math. Comput. Simul. 162 (2019), 1–17.

[13] M. H. Heydari, Z. Avazzadeh, and M. F. Haromi, *A wavelet approach for solving multi-term variable-order time fractional diffusion-wave equation*, Appl. Math. Comput. 341 (2019), 215–228.

[14] V. R. Hosseini, E. Shivanian, and W. Chen, *Local integration of 2-D fractional telegraph equation via local radial point interpolant approximation*, Eur. Phys. J. Plus 130(33) (2015).

[15] H. Jafari, M. Saeidy, and J. Vahidi, *The homotopy analysis method for solving fuzzy system of linear equation*, Int. J. Fuzzy Syst. 11(4) (2009), 308–313.

[16] H. Jafari, H. Tajadodi, and D. Baleanu, *A modified variational Iteration method for solving fractional Riccati differential equation by Adomian polynomials*, Fractional Calculus Appl. Anal. 16 (2013), 109–122.

[17] H. Jafari, H. Tajadodi, and R. M. Ganji, *A numerical approach for solving variable order differential equations based on Bernstein polynomials*, Comput. Math. Meth. 1(5) (2019).

[18] W. Jiang, and B. Guo, *A new numerical method for solving two-dimensional variable-order anomalous sub-diffusion equation*, Thermal Sci. 20(3) (2016), 701–710.

[19] X. Li, and B. Wu, *A numerical technique for variable fractional functional boundary value problems*, Appl. Math. Lett. 43 (2014), 108–113.

[20] K. Maleknejad, K. Nouri, and L. Torkzadeh, *Operational matrix of fractional integration based on the shifted second kind chebyshev polynomials for solving fractional differential equations*, Mediter. J. Math. 13 (2016), 1377–1390.

[21] P. Rahimkhani, Y. Ordokhani, and E. Babolian, *Fractional-order Bernoulli functions and their applications in solving fractional Fredholem-Volterra integro-differential equations*, Appl. Numer. Math. 122 (2017), 66–81.

[22] S. Sadeghi Roshan, H. Jafari, and D. Baleanu, *Solving FDEs with Caputo-Fabrizio derivative by operational matrix based on Genocchi polynomials*, Math. Meth. Appl. Sci. 41(18) (2018), 9134–9141.

[23] S. Sadeghi Roshan, H. Jafari, and S. Nematia, *Operational matrix for Atangana–Baleanu derivative based on Genocchi polynomials for solving FDEs*, Chaos Solitons Fractals 135 (2020), 109736.

[24] S.G. Samko, and B. Ross, *Integration and differentiation to a variable fractional order*, Integral Transf. Spec. Funct. 1 (1993), 277–300.

[25] C.M. Soon, C.F. M. Coimbra, and M. H. Kobayashi, *The variable viscoelasticity oscillator*, Ann. Phys. 14(6) (2005), 378–389.

[26] N.H. Tuan, S. Nemati, R.M. Ganji, and H. Jafari, *Numerical solution of multi-variable order fractional integro-differential equations using the Bernstein polynomials*, Engin. with Comput. 38 (2020), 139–147.

[27] J. de. Villiers, *Mathematics of Approximation*, Atlantis Press, 2012.

[28] Y. Xu, and V. Suat Ertürk, *A Finite Difference Technique For Solving Variable-Order Fractional Integro-Differential Equations*, Bull. Iran. Math. Soc. 40(3) (2014), 699–712.

[29] J. Yanga, *An efficient numerical method for variable order fractional functional differential equation*, Appl. Math. Let. 76 (2018), 221–226.

[30] X. J. Yang, *Fractional derivatives of constant and variable orders applied to anomalous relaxation models in heat-transfer problems*, Thermal Sci. 21(3) (2017), 1161–1171.