Int. J. Nonlinear Anal. Appl. 12 (2021) No. 1, 215-229
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.4762

Four Step Hybrid Block Method for the Direct Solution of Fourth Order Ordinary Differential Equations

Raft Abdelrahim*

Department of Mathematics, College of Art and Sciences-Tabarjal, Jouf University, Saudi Arabia

(Communicated by Madjid Eshaghi Gordji)

Abstract

This paper proposes a direct four-step implicit hybrid block method for directly solving general fourth-order initial value problems of ordinary differential equations. In deriving this method, the approximate solution in the form of power series is interpolated at four points, i.e $x_n,\ x_{n+1}\ x_{n+2}\ x_{n+3}$ while its forth derivative is collocated at all grid points, i.e $x_n,\ x_{n+\frac{1}{4}},\ x_{n+1},\ x_{n+2},\ x_{n+\frac{5}{2}},\ x_{n+3},\ x_{n+\frac{7}{2}}$ and x_{n+4} to produce the main continuous schemes. In order to verify the applicability of the new method, the properties of the new method such as local truncation error, zero stability, order and convergence are also established. The performance of the newly developed method is then compared with the existing methods in terms of error by solving the same test problems. The numerical results reveal that the proposed method produces better accuracy than several existing methods when solving the same initial value problems (IVPs) of second order ODEs.

Keywords: Hybrid block method, Fourth initial value problem, Collocation and Interpolation, Four step

2010 MSC: 65L06, 65L05

1. Introduction

This paper focuses on solving the following initial value problem (IVPs) of fourth order ordinary differential equations (ODEs)

$$y^{iv} = f(x, y, y^{'}, y^{''}, y^{'''}),\ x \in [a, b] \tag{1.1}$$

*Corresponding author

Email address: Rafatshaab@yahoo.com (Raft Abdelrahim)

Received: 7 October 2020 *Revised:* 28 December 2020

with initial conditions

$$y(a) = \delta_0,\, y^{'}(a) = \delta_1, y^{''}(a) = \delta_2, y^{'''}(a) = \delta_3$$

Numerous physical problems such elasticity, deformation of structures, or soil settlement can be modelled in the form of Equation (1). However, the solutions to these problems may not be obtained analytically. Therefore, it is essential to develop numerical methods to cater this issue (see Twizell[13] and Henrici[9]).

Although Equation (1) can be solved by reducing it into the equivalent first order system of ODEs, this approach will increase the number of equations and as a result, more work is required. Consequently, more computational time is needed. To avoid this drawback, scholars have opted to solve (1) using direct methods instead. Direct methods have also been proven to produce more accurate numerical results compared to reduction methods (see Abdelrahim and Omar[1],[2]).

One of the direct methods available is block method. Block method is capable of finding numerical solution at more than one point simultaneously. Nonetheless, block method has zero-stability barriers. To overcome this setback, hybrid block methods were introduced (see Adessanya [4], Yap [14], Awoyemi[5], Awoyemi and kayode [6]). In literature, four-step implicit method to solve IVPs of fourth order ODEs was proposed by [12]. The basic properties of the method was investigated and found to be zero stable, consistent and convergent. Subsequently, a four-step hybrid method with three hybrid points was developed by Yap and Ismail [15]. The numerical results revealed that the accuracy of this method was better than the previous non-hybrid methods. Not only this method has good properties of numerical method, the results obtained were also accurate. The method was tested on some numerical examples and the accuracy of the method can still be improved.

In this Four-step hybrid block model, new strategy of selecting the hybrid values within the grid points was considered. Consequently, this new approach has given rise to a better results when compared with the previous method developed by Abdelrahim and Omar[3]. One of the major advantages of this novel developed numerical method over current method is its ability to solving fourth order ordinary differential equations effectively. In order to further examine the capability of the new method, the same number of block used in the previous method was also considered. The new method gave better result at every point and this demonstrated the superiority of the method over existing methods in terms of accuracy.

2. Methodology

In this section, a four-step block method with a specific three off-step point $x_{n+\frac{1}{4}}$, $x_{n+\frac{5}{2}}$ and $x_{n+\frac{7}{2}}$, for solving (1.1) is developed.
Let the approximate solution of equation (1.1) be the polynomial of the form:

$$y(x) = \sum_{i=0}^{s+v-1} a_i \left(\frac{x - x_n}{h}\right)^i. \tag{2.1}$$

to be the approximate solution of (1.1) where $x \in [x_n, x_{n+1}]$ for $n = 0, 1, 2, ..., N-1$, v represents the number of collocation points, s denotes of the number of interpolation points which is equal to the order of differential equation and $h = x_n - x_{n-1}$ is constant step size of partition of the interval

$[a, b]$ which is given by $a = x_0 < x_1 < ... < x_{N-1} < x_N = b$.
The fourth derivative of equation (2.1) is

$$y^{iv}(x) = f(x, y, y^{'}, y^{''}, y^{'''}) = \sum_{i=4}^{s+v-1} \frac{i(i-1)(i-2)(i-3)}{h^4} a_i \left(\frac{x - x_n}{h}\right)^{i-4}. \tag{2.2}$$

Equation (2.1) is interpolated at x_{n+i}, $i = 0(1)3$ while Equation(2.2) is collocated at all points in the selected interval. The producing equations are reformed in matrix mode as below:

$$KX = L \tag{2.3}$$

where

$$K = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 & 32 & 64 & 128 & 256 & 512 & 1024 & 2048 \\ 1 & 3 & 9 & 27 & 81 & 243 & 729 & 2187 & 6561 & 19683 & 59049 & 177147 \\ 0 & 0 & 0 & 0 & \frac{24}{h^4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{24}{h^4} & \frac{30}{h^4} & \frac{45}{(2h^4)} & \frac{105}{(8h^4)} & \frac{105}{(16h^4)} & \frac{189}{(64h^4)} & \frac{315}{(256h^4)} & \frac{495}{(1024h^4)} \\ 0 & 0 & 0 & 0 & \frac{24}{h^4} & \frac{120}{h^4} & \frac{360}{h^4} & \frac{840}{h^4} & \frac{1680}{h^4} & \frac{3024}{h^4} & \frac{5040}{h^4} & \frac{7920}{h^4} \\ 0 & 0 & 0 & 0 & \frac{24}{h^4} & \frac{300}{h^4} & \frac{2250}{h^4} & \frac{13125}{h^4} & \frac{65625}{h^4} & \frac{590625}{(2h^4)} & \frac{4921875}{(4h^4)} & \frac{38671875}{(8h^4)} \\ 0 & 0 & 0 & 0 & \frac{24}{h^4} & \frac{240}{h^4} & \frac{1440}{h^4} & \frac{6720}{h^4} & \frac{26880}{h^4} & \frac{96768}{h^4} & \frac{322560}{h^4} & \frac{1013760}{h^4} \\ 0 & 0 & 0 & 0 & \frac{24}{h^4} & \frac{420}{h^4} & \frac{4410}{h^4} & \frac{36015}{h^4} & \frac{252105}{h^4} & \frac{3176523}{(2h^4)} & \frac{37059435}{(4h^4)} & \frac{407653785}{(8h^4)} \\ 0 & 0 & 0 & 0 & \frac{24}{h^4} & \frac{360}{h^4} & \frac{3240}{h^4} & \frac{22680}{h^4} & \frac{136080}{h^4} & \frac{734832}{h^4} & \frac{3674160}{h^4} & \frac{17321040}{h^4} \\ 0 & 0 & 0 & 0 & \frac{24}{h^4} & \frac{480}{h^4} & \frac{5760}{h^4} & \frac{53760}{h^4} & \frac{430080}{h^4} & \frac{3096576}{h^4} & \frac{20643840}{h^4} & \frac{129761280}{h^4} \end{pmatrix}$$

$$X = [a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}]^T$$

$$L = \left[y_n, y_{n+1}, y_{n+2}, y_{n+3}, f_n, f_{n+\frac{1}{4}}, f_{n+1}, f_{n+2}, f_{n+\frac{5}{2}}, f_{n+3}, f_{n+\frac{7}{2}}, f_{n+4}\right]^T$$

The unknown values of $a_i's, i = 0(1)11$ can be obtained by using Gaussian elimination method and then substituted these values back into Equation (2.1) to produce a continuous implicit scheme with its derivatives of the form:

$$y^{(j)}(x) = \sum_{i=0}^{3} \alpha_i^{(j)}(x) y_{n+i} + \sum_{i=0}^{4} \beta_i^{(j)}(x) f_{n+i} + \sum_{s \in \frac{1}{4}, \frac{5}{2}, \frac{7}{2}} \beta_s^{(j)}(x) f_{n+s}, \; j = 0(1)3. \tag{2.4}$$

where the values of α_i, $\beta_i, i = 1(1)4$ and $\beta_s, s \in \frac{1}{4}, \frac{5}{2}\ \frac{7}{2}$, in equation (5), are

$$\alpha_0 = \frac{(x - x_n)^2}{h^2} - \frac{(x - x_n)^3}{(6h^3)} - \frac{11(x - x_n)}{(6h)} + 1$$

$$\alpha_1 = \frac{(x - x_n)^3}{(2h^3)} - \frac{5(x - x_n)^2}{(2h^2)} + \frac{3(x - x_n)}{h}$$

$$\alpha_2 = \frac{2(x - x_n)^2)}{h^2} - \frac{(x - x_n)^3}{(2h^3)} - \frac{3(x - x_n)}{(2h)}$$

$$\alpha_3 = \frac{(x - x_n)^3}{(6h^3)} - \frac{(x - x_n)^2}{(2h^2)} + \frac{(x - x_n)}{(3h)}$$

$$\beta_0 = \frac{(x-x_n)^4)}{24)} - \frac{(521h^2(x-x_n)^2)}{56448} - \frac{(2843(x-x_n)^5)}{(50400h)} + \frac{(473(x-x_n)^6)}{(12096h^2)}$$
$$-\frac{(5737(x-x_n)^7)}{(352800h^3)} + \frac{(239(x-x_n)^8)}{(56448h^4)} - \frac{(431(x-x_n)^9)}{(635040h^5)} + \frac{(13(x-x_n)^10)}{(211680h^6)}$$
$$-\frac{(x-x_n)^{11})}{(415800h^7)} - \frac{(37321h(x-x_n)^3)}{6350400} + \frac{(3761h^3(x-x_n))}{1108800}$$
$$\beta_1 = \frac{18067h^2(x-x_n)^2}{64800} - \frac{7(x-x_n)^5}{(270h)} + \frac{2423(x-x_n)^6}{(48600h^2)} - \frac{997(x-x_n)^7}{(32400h^3)}$$
$$+\frac{(899(x-x_n)^8)}{(90720h^4)} - \frac{(37(x-x_n)^9)}{(20412h^5)} + \frac{(61(x-x_n)^10)}{(340200h^6)} - \frac{(x-x_n)^{11})}{(133650h^7)}$$
$$-\frac{(128293h(x-x_n)^3)}{1020600} - \frac{(77107h^3(x-x_n))}{498960}$$
$$\beta_2 = \frac{(58363h^2(x-x_n)^2)}{423360} + \frac{((x-x_n)^5)}{(24h)} - \frac{(2633(x-x_n)^6)}{(30240h^2)} + \frac{(383(x-x_n)^7)}{(5880h^3)}$$
$$-\frac{(3439(x-x_n)^8)}{(141120h^4)} + \frac{(317(x-x_n)^9)}{(63504h^5)} - \frac{(19(x-x_n)^{10})}{(35280h^6)} + \frac{((x-x_n)^{11})}{(41580h^7)}$$
$$-\frac{(20039h(x-x_n)^3)}{635040} - \frac{(3923h^3(x-x_n))}{36960}$$
$$\beta_3 = \frac{(713h^2(x-x_n)^2)}{36960} + \frac{(7(x-x_n)^5)}{(198h)} - \frac{(901(x-x_n)^6)}{(11880h^2)} + \frac{(3341(x-x_n)^7)}{(55440h^3)}$$
$$-\frac{(2711(x-x_n)^8)}{(110880h^4)} + \frac{(34((x-x_n)^9)}{(6237h^5)} - \frac{(53(x-x_n)^{10})}{(83160h^6)} + \frac{((x-x_n)^{11})}{(32670h^7)}$$
$$+\frac{(17h(x-x_n)^3)}{3564)} - \frac{(2111h^3(x-x_n))}{87120}$$

$$\beta_4 = \frac{(689h^2(x-x_n)^2)}{1814400} + \frac{(7(x-x_n)^5)}{(4320h)} - \frac{(1369(x-x_n)^6)}{(388800h^2)} + \frac{(1307(x-x_n)^7)}{(453600h^3)}$$
$$-\frac{(443(x-x_n)^8)}{(362880h^4)} + \frac{(47(x-x_n)^9)}{(163296h^5)} - \frac{(7(x-x_n)^{10})}{(194400h^6)} + \frac{((x-x_n)^{11})}{(534600h^7)}$$
$$+\frac{(2921h(x-x_n)^3)}{8164800} - \frac{(1499h^3(x-x_n))}{1995840}$$
$$\beta_{\frac{1}{4}} = \frac{(3554816h^2(x-x_n)^2)}{42567525)} + \frac{(4096(x-x_n)^5)}{(57915h)} - \frac{(1190912(x-x_n)^6)}{(18243225h^2)}$$
$$+\frac{1290752(x-x_n)^7}{(42567525h^3)} - \frac{71168(x-x_n)^8}{(8513505h^4)} + \frac{21248(x-x_n)^9}{(15324309h^5)} - \frac{16384(x-x_n)^{10}}{(127702575h^6)}$$
$$+\frac{1024(x-x_n)^{11}}{(200675475h^7)} - \frac{33535232h(x-x_n)^3}{383107725} - \frac{(47104h^3(x-x_n))}{1911195}$$
$$\beta_{\frac{5}{2}} = \frac{(428(x-x_n)^6)}{(3645h^2)} - \frac{(112(x-x_n)^5)}{(2025h)} - \frac{(82h^2(x-x_n)^2)}{1701)} - \frac{(3874(x-x_n)^7)}{(42525h^3)}$$
$$+\frac{(304(x-x_n)^8)}{(8505h^4)} - \frac{(587(x-x_n)^9)}{(76545h^5)} + \frac{(22(x-x_n)^{10})}{(25515h^6)} - \frac{(8(x-x_n)^{11})}{(200475h^7)}$$
$$-\frac{(851h(x-x_n)^3)}{382725} + \frac{(3376h^3(x-x_n))}{66825}$$

$$\beta_{\frac{7}{2}} = \frac{(1556(x-x_n)^6)}{(61425h^2)} - \frac{(16(x-x_n)^5)}{(1365h)} - \frac{(1754h^2(x-x_n)^2)}{429975)} - \frac{(326(x-x_n)^7)}{(15925h^3)}$$
$$+\frac{(244(x-x_n)^8)}{(28665h^4)} - \frac{(101(x-x_n)^9)}{(51597h^5)} + \frac{(34(x-x_n)^{10})}{(143325h^6)} - \frac{(8(x-x_n)^{11})}{(675675h^7)}$$
$$-\frac{(2881h(x-x_n)^3)}{1289925} + \frac{(32h^3(x-x_n))}{5005}$$

For $j=0$, Equation (2.4) is evaluated at the non-interpolating point $x_{n+\frac{1}{4}},\ x_{n+\frac{5}{2}},\ x_{n+\frac{7}{2}},\ x_{n+4}$ while for $j=1(1)3$ Equation (2.4) are evaluated at all points in the selected interval to produce the discrete schemes with its derivatives. Discrete schemes and its derivatives are combined on a block of the form

$$AY_m = BR_1 + h^4[DR_2 + ER_3] \tag{2.5}$$

where,

$$A =$$

$$\begin{pmatrix}
1 & \frac{-77}{128} & 0 & \frac{33}{128} & 0 & \frac{-7}{128} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{5}{16} & 1 & \frac{-15}{16} & 0 & \frac{-5}{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-21}{16} & 0 & \frac{35}{16} & 1 & \frac{-35}{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -4 & 0 & 6 & 0 & -4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-3}{h} & 0 & \frac{3}{(2h)} & 0 & \frac{-1}{(3h)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-59}{(32h)} & 0 & \frac{19}{(32h)} & 0 & \frac{-11}{(96h)} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{(2h)} & 0 & \frac{-1}{h} & 0 & \frac{1}{(6h)} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{(8h)} & 0 & \frac{7}{(8h)} & 0 & \frac{-23}{(24h)} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{h} & 0 & \frac{-1}{(2h)} & 0 & \frac{-1}{(3h)} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-31}{(8h)} & 0 & \frac{47}{(8h)} & 0 & \frac{-71}{(24h)} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-3}{(2h)} & 0 & \frac{3}{h} & 0 & \frac{-11}{(6h)} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-7}{h} & 0 & \frac{19}{(2h)} & 0 & \frac{-13}{(3h)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{5}{h^2} & 0 & \frac{-4}{h^2} & 0 & \frac{1}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{17}{(4h^2)} & 0 & \frac{-13}{(4h^2)} & 0 & \frac{3}{(4h^2)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{2}{h^2} & 0 & \frac{-1}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-5}{(2h^2)} & 0 & \frac{7}{(2h^2)} & 0 & \frac{-3}{(2h^2)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-1}{h^2} & 0 & \frac{2}{h^2} & 0 & \frac{-1}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-11}{(2h^2)} & 0 & \frac{13}{(2h^2)} & 0 & \frac{-5}{(2h^2)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-4}{h^2} & 0 & \frac{5}{h^2} & 0 & \frac{-2}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-7}{h^2} & 0 & \frac{8}{h^2} & 0 & \frac{-3}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-3}{h^3} & 0 & \frac{3}{h^3} & 0 & \frac{-1}{h^3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-3}{h^3} & 0 & \frac{3}{h^3} & 0 & \frac{-1}{h^3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-3}{h^3} & 0 & \frac{3}{h^3} & 0 & \frac{-1}{h^3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-3}{h^3} & 0 & \frac{3}{h^3} & 0 & \frac{-1}{h^3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & \frac{-3}{h^3} & 0 & \frac{3}{h^3} & 0 & \frac{-1}{h^3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & \frac{-3}{h^3} & 0 & \frac{3}{h^3} & 0 & \frac{-1}{h^3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & \frac{-3}{h^3} & 0 & \frac{3}{h^3} & 0 & \frac{-1}{h^3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & \frac{-3}{h^3} & 0 & \frac{3}{h^3} & 0 & \frac{-1}{h^3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},$$

$$Y_m = \begin{pmatrix} y_{n+\frac{1}{4}} \\ y_{n+1} \\ y_{n+\frac{5}{2}} \\ y_{n+2} \\ y_{n+\frac{7}{2}} \\ y_{n+3} \\ y_{n+4} \end{pmatrix}, \; B = \begin{pmatrix} \frac{77}{128} & 0 & 0 & 0 \\ \frac{1}{16} & 0 & 0 & 0 \\ \frac{-5}{16} & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \frac{-11}{(6h)} & -1 & 0 & 0 \\ \frac{-131}{(96h)} & 0 & 0 & 0 \\ \frac{-1}{(3h)} & 0 & 0 & 0 \\ \frac{1}{(24h)} & 0 & 0 & 0 \\ \frac{1}{(6h)} & 0 & 0 & 0 \\ \frac{-23}{(24h)} & 0 & 0 & 0 \\ \frac{-1}{(3h)} & 0 & 0 & 0 \\ \frac{-11}{(6h)} & 0 & 0 & 0 \\ \frac{2}{h^2} & 0 & -1 & 0 \\ \frac{7}{(4h^2)} & 0 & 0 & 0 \\ \frac{1}{h^2} & 0 & 0 & 0 \\ \frac{-1}{(2h^2)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{-3}{(2h^2)} & 0 & 0 & 0 \\ \frac{-1}{h^2} & 0 & 0 & 0 \\ \frac{-2}{h^2} & 0 & 0 & 0 \\ \frac{-1}{h^3} & 0 & 0 & -1 \\ \frac{-1}{h^3} & 0 & 0 & 0 \\ \frac{-1}{h^3} & 0 & 0 & 0 \\ \frac{-1}{h^3} & 0 & 0 & 0 \\ \frac{-1}{h^3} & 0 & 0 & 0 \\ \frac{-1}{h^3} & 0 & 0 & 0 \\ \frac{-1}{h^3} & 0 & 0 & 0 \\ \frac{-1}{h^3} & 0 & 0 & 0 \end{pmatrix}, \; R_1 = \begin{pmatrix} y_n \\ y_n' \\ y_n'' \\ y_n''' \end{pmatrix}, \; D = \begin{pmatrix} \frac{781079h^4}{2642411520} \\ \frac{-18929h^4}{36126720} \\ \frac{403h^4}{147456} \\ \frac{311h^4}{35280} \\ \frac{3761h^3}{1108800} \\ \frac{-75578915h^3}{122079412224} \\ \frac{4981h^3}{4656960} \\ \frac{-1112357h^3}{2980454400} \\ \frac{-30953h^3}{23284800} \\ \frac{719827h^3}{85155840} \\ \frac{3167h^3}{1108800} \\ \frac{75997h^3}{4656960} \\ \frac{-521h^2}{28224} \\ \frac{-23985149h^2}{2477260800} \\ \frac{527h^2}{302400} \\ \frac{113671h^2}{27095040} \\ \frac{-23h^2}{43200} \\ \frac{1809751h^2}{135475200} \\ \frac{6239h^2}{705600} \\ \frac{39701h^2}{2116800} \\ \frac{-37321h}{1058400} \\ \frac{559192093h}{8670412800} \\ \frac{-20971h}{1058400} \\ \frac{294853h}{33868800} \\ \frac{10919h}{1058400} \\ \frac{272053h}{33868800} \\ \frac{10469h}{1058400} \\ \frac{15479h}{1058400} \end{pmatrix},$$

$$R_2 = \left[\; f_n \;\right], \; R_3 = \begin{pmatrix} f_{n+\frac{1}{4}} \\ f_{n+1} \\ f_{n+\frac{5}{2}} \\ f_{n+2} \\ f_{n+\frac{7}{2}} \\ f_{n+3} \\ f_{n+4} \end{pmatrix} \; \textit{and}$$

$$E = \begin{pmatrix}
\frac{-11233967h^4}{4981616640} & \frac{-19693903h^4}{849346560} & \frac{152117h^4}{15925248} & \frac{-24296183h^4}{1321205760} & \frac{17473h^4}{13418496} & \frac{-4939417h^4}{1038090240} & \frac{-107059h^4}{679477248} \\
\frac{1931h^4}{1702701} & \frac{-3371h^4}{290304} & \frac{13043h^4}{1088640} & \frac{-123749h^4}{3612672} & \frac{3301h^4}{1467648} & \frac{-11003h^4}{1419264} & \frac{-377h^4}{1327104} \\
\frac{-23h^4}{3861} & \frac{16339h^4}{276480} & \frac{-133h^4}{6912} & \frac{14447h^4}{73728} & \frac{-889h^4}{74880} & \frac{2597h^4}{50688} & \frac{661h^4}{442368} \\
\frac{-32768h^4}{1702701} & \frac{2153h^4}{11340} & \frac{-32h^4}{1701} & \frac{11413h^4}{17640} & \frac{-160h^4}{5733} & \frac{2987h^4}{13860} & \frac{31h^4}{6480} \\
\frac{-47104h^3}{1911195} & \frac{-77107h^3}{498960} & \frac{3376h^3}{66825} & \frac{-3923h^3}{36960} & \frac{32h^3}{5005} & \frac{-2111h^3}{87120} & \frac{-1499h^3}{1995840} \\
\frac{2506718009h^3}{1438441804800} & \frac{-7644152563h^3}{196199055360} & \frac{26768713h^3}{1051066368} & \frac{-2608006057h^3}{61039706112} & \frac{474073927h^3}{123986903040} & \frac{-91487603h^3}{6851395584} & \frac{-1878528901h^3}{3923981107200} \\
\frac{-464384h^3}{280945665} & \frac{334973h^3}{7484400} & \frac{-298h^3}{8019} & \frac{143483h^3}{2328480} & \frac{-26762h^3}{4729725} & \frac{35947h^3}{1829520} & \frac{4259h^3}{5987520} \\
\frac{46040h^3}{56189133} & \frac{-153667h^3}{19160064} & \frac{420401h^3}{359251200} & \frac{-9381137h^3}{298045440} & \frac{362279h^3}{242161920} & \frac{-74533h^3}{14636160} & \frac{-20909h^3}{109486080} \\
\frac{1334272h^3}{468242775} & \frac{-75577h^3}{2494800} & \frac{2452h^3}{66825} & \frac{-177167h^3}{2328480} & \frac{27644h^3}{4729725} & \frac{-5299h^3}{261360} & \frac{-2447h^3}{3326400} \\
\frac{-246832h^3}{13378365} & \frac{3630043h^3}{19958400} & \frac{-90403h^3}{3421440} & \frac{26201207h^3}{42577920} & \frac{-6119551h^3}{172972800} & \frac{3121739h^3}{16727040} & \frac{130157h^3}{28385280} \\
\frac{-415232h^3}{66891825} & \frac{156061h^3}{2494800} & \frac{-19466h^3}{467775} & \frac{22133h^3}{110880} & \frac{-2762h^3}{225225} & \frac{26539h^3}{609840} & \frac{2201h^3}{1425600} \\
\frac{-50120704h^3}{1404728325} & \frac{523147h^3}{1496880} & \frac{11152h^3}{280665} & \frac{567547h^3}{465696} & \frac{-9088h^3}{945945} & \frac{180941h^3}{365904} & \frac{295327h^3}{29937600} \\
\frac{7109632h^2}{42567525} & \frac{18067h^2}{32400} & \frac{-164h^2}{1701} & \frac{58363h^2}{211680} & \frac{-3508h^2}{429975} & \frac{713h^2}{18480} & \frac{689h^2}{907200} \\
\frac{39934439h^2}{778377600} & \frac{679403297h^2}{1857945600} & \frac{-331423h^2}{3110400} & \frac{57858169h^2}{247726080} & \frac{-3254351h^2}{251596800} & \frac{68203649h^2}{1362493440} & \frac{11092369h^2}{7431782400} \\
\frac{-15872h^2}{2027025} & \frac{-1817h^2}{25200} & \frac{46h^2}{14175} & \frac{-47h^2}{6048} & \frac{-2h^2}{61425} & \frac{-101h^2}{166320} & \frac{h^2}{43200} \\
\frac{-386588h^2}{42567525} & \frac{1348801h^2}{14515200} & \frac{-62593h^2}{544320} & \frac{3700877h^2}{13547520} & \frac{-493307h^2}{27518400} & \frac{650413h^2}{10644480} & \frac{131711h^2}{58060800} \\
\frac{8192h^2}{6081075} & \frac{-43h^2}{9072} & \frac{8h^2}{6075} & \frac{-17h^2}{224} & \frac{8h^2}{4095} & \frac{-1087h^2}{166320} & \frac{-227h^2}{907200} \\
\frac{-414844h^2}{14189175} & \frac{66011h^2}{230400} & \frac{69989h^2}{907200} & \frac{922247h^2}{903168} & \frac{-37533h^2}{1019200} & \frac{4821521h^2}{10644480} & \frac{128353h^2}{19353600} \\
\frac{-54784h^2}{2837835} & \frac{4787h^2}{25200} & \frac{-422h^2}{14175} & \frac{136903h^2}{211680} & \frac{-16906h^2}{429975} & \frac{8573h^2}{55440} & \frac{59h^2}{12096} \\
\frac{-1752064h^2}{42567525} & \frac{87701h^2}{226800} & \frac{9332h^2}{42525} & \frac{291779h^2}{211680} & \frac{81988h^2}{429975} & \frac{122737h^2}{166320} & \frac{23497h^2}{907200} \\
\frac{-67070464h}{127702575} & \frac{-128293h}{170100} & \frac{-1702h}{127575} & \frac{-20039h}{105840} & \frac{-5762h}{429975} & \frac{17h}{594} & \frac{2921h}{1360800} \\
\frac{-1471524053h}{4086482400} & \frac{-2174268307h}{2786918400} & \frac{-33700027h}{522547200} & \frac{-130327357h}{867041280} & \frac{-42426827h}{1761177600} & \frac{41588339h}{681246720} & \frac{40330147h}{11147673600} \\
\frac{7041536h}{127702575} & \frac{-85511h}{340200} & \frac{45158h}{127575} & \frac{-52427h}{105840} & \frac{24418h}{429975} & \frac{-15977h}{83160} & \frac{-9889h}{1360800} \\
\frac{-2435464h}{127702575} & \frac{2081123h}{10886400} & \frac{-53857h}{2041200} & \frac{2608127h}{3386880} & \frac{-225317h}{6879600} & \frac{55357h}{532224} & \frac{185347h}{43545600} \\
\frac{-2927104h}{127702575} & \frac{34067h}{170100} & \frac{-44902h}{127575} & \frac{11429h}{21168} & \frac{-20162h}{429975} & \frac{3439h}{20790} & \frac{7961h}{1360800} \\
\frac{-2243464h}{127702575} & \frac{2053523h}{10886400} & \frac{296903h}{2041200} & \frac{2621711h}{3386880} & \frac{1034563h}{6879600} & \frac{285623h}{380160} & \frac{-14573h}{43545600} \\
\frac{-214528h}{9823275} & \frac{67009h}{340200} & \frac{35558h}{127575} & \frac{76837h}{105840} & \frac{-1814h}{33075} & \frac{29839h}{83160} & \frac{8591h}{1360800} \\
\frac{-4155904h}{127702575} & \frac{5261h}{24300} & \frac{8534h}{18225} & \frac{67129h}{105840} & \frac{301438h}{429975} & \frac{7123h}{20790} & \frac{211241h}{1360800}
\end{pmatrix}$$

Multiplying Equation (2.5) by inverse of A to have a hybrid block method of the form

$$A^{(0)}Y_m = A^{-1}BR_1 + h^4[A^{-1}DR_2 + A^{-1}ER_3] \tag{2.6}$$

Equation (2.6) can be clearly written as following:

$$\begin{aligned}
y_{n+\frac{1}{4}} = y_n + \frac{hy_n'}{4} + \frac{h^2y_n''}{32} + \frac{h^3y_n'''}{384} + h^4[\frac{1064747599f_n}{9155955916800} - \frac{43791943f_{n+1}}{2942985830400} \\
+ \frac{4221817f_{n+2}}{183119118336} + \frac{13908947f_{n+3}}{719396536320} + \frac{10371721f_{n+4}}{11771943321600} \\
+ \frac{946914649f_{n+\frac{1}{4}}}{17261301657600} - \frac{8384303f_{n+\frac{5}{2}}}{275904921600} - \frac{1485331f_{n+\frac{7}{2}}}{232475443200}],
\end{aligned}$$

$$y_{n+1} = y_n + hy_n^{'} + \frac{h^2y_n^{''}}{2} + \frac{h^3y_n^{'''}}{6} + h^4[\frac{1636651f_n}{139708800} + \frac{64091f_{n+1}}{44906400} - \frac{2227f_{n+2}}{13970880}$$
$$+\frac{373f_{n+3}}{2195424} + \frac{2437f_{n+4}}{179625600} + \frac{120825088f_{n+\frac{1}{4}}}{4214184975} - \frac{377f_{n+\frac{5}{2}}}{4209975} - \frac{1147f_{n+\frac{7}{2}}}{14189175}],$$
$$y_{n+2} = y_n + 2hy_n^{'} + +2h^2y_n^{''} + \frac{4h^3y_n^{'''}}{3}\frac{84208f_n}{1091475} + \frac{279896f_{n+1}}{1403325} - \frac{18926f_{n+2}}{218295}$$
$$h^4[\frac{22936f_{n+3}}{-343035} - \frac{808f_{n+4}}{280665} + \frac{350224384f_{n+\frac{1}{4}}}{842836995} + \frac{461312f_{n+\frac{5}{2}}}{4209975} + \frac{303616f_{n+\frac{7}{2}}}{14189175}],$$
$$y_{n+\frac{5}{2}} = y_n + \frac{5hy_n^{'}}{2} + \frac{25h^2y_n^{''}}{8} + \frac{125h^3y_n^{'''}}{48} + \frac{100507625f_n}{715309056} + \frac{2448125f_{n+1}}{4105728}$$
$$-\frac{49159375f_{n+2}}{357654528}\frac{-19990625h^4f_{n+3}}{140507136} - \frac{836875h^4f_{n+4}}{131383296} + \frac{153150625h^4f_{n+\frac{1}{4}}}{168567399}$$
$$+\frac{4779125f_{n+\frac{5}{2}}}{21555072} + \frac{6779375h^4f_{n+\frac{7}{2}}}{145297152},$$

$$y_{n+3} = y_n + 3hy_n^{'} + \frac{9h^2y_n^{''}}{2} + \frac{9h^3y_n^{'''}}{2} + \frac{399411f_n}{1724800} + \frac{2373f_{n+1}}{1760} - \frac{12123f_{n+2}}{172480}$$
$$h^4[-\frac{31131f_{n+3}}{135520} - \frac{381f_{n+4}}{35200} + \frac{3248384f_{n+\frac{1}{4}}}{1926925} + \frac{659f_{n+\frac{5}{2}}}{1925} + \frac{2727f_{n+\frac{7}{2}}}{35035}],$$
$$y_{n+\frac{7}{2}} = y_n + \frac{7hy_n^{'}}{2} + \frac{49h^2y_n^{''}}{8} + \frac{343h^3y_n^{'''}}{48} + \frac{129886897f_n}{364953600} + \frac{1056840967f_{n+1}}{410572800}$$
$$+\frac{8453921f_{n+2}}{36495360} - \frac{1529437h^4f_{n+3}}{5018112} - \frac{52118507h^4f_{n+4}}{3284582400}$$
$$+\frac{241701467h^4f_{n+\frac{1}{4}}}{86003775} + \frac{75413009h^4f_{n+\frac{5}{2}}}{153964800} + \frac{4132121h^4f_{n+\frac{7}{2}}}{37065600},$$

$$y_{n+4} = y_n + 4hy_n^{'} + 8h^2y_n^{''} + \frac{32h^3y_n^{'''}}{3} + \frac{566528f_n}{1091475} + \frac{6163456f_{n+1}}{1403325} + \frac{38656f_{n+2}}{43659}$$
$$\frac{-103424h^4f_{n+3}}{343035} - \frac{29728h^4f_{n+4}}{1403325} + \frac{18312331264h^4f_{n+\frac{1}{4}}}{4214184975} + \frac{2916352h^4f_{n+\frac{5}{2}}}{4209975}$$
$$+\frac{2195456h^4f_{n+\frac{7}{2}}}{14189175},$$
$$y_{n+\frac{1}{4}}^{'} = y_n^{'} + \frac{hy_n^{''}}{4} + \frac{h^2y_n^{'''}}{32} + h^3[\frac{94652819f_n}{55490641920} - \frac{7795423f_{n+1}}{29727129600} + \frac{11201017f_{n+2}}{27745320960}$$
$$+\frac{669631f_{n+3}}{1981808640} + \frac{1829147f_{n+4}}{118908518400} + \frac{460883f_{n+\frac{1}{4}}}{440294400} - \frac{39499f_{n+\frac{5}{2}}}{74317824} - \frac{6289399f_{n+\frac{7}{2}}}{56357683200}],$$

$$y_{n+1}^{'} = y_n^{'} + hy_n^{''} + \frac{h^2y_n^{'''}}{2} + h^3[\frac{1787f_n}{52920} + \frac{4259f_{n+1}}{226800} - \frac{703f_{n+2}}{52920} - \frac{1499f_{n+3}}{166320}$$
$$-\frac{f_{n+4}}{2700} + \frac{5047552f_{n+\frac{1}{4}}}{42567525} + \frac{131f_{n+\frac{5}{2}}}{8505} + \frac{1207f_{n+\frac{7}{2}}}{429975}],$$
$$y_{n+\frac{5}{2}}^{'} = y_n^{'} + \frac{5hy_n^{''}}{2} + \frac{25h^2y_n^{'''}}{8} + h^3[\frac{1653625f_n}{10838016} + \frac{644125f_{n+1}}{580608} - \frac{124375f_{n+2}}{5419008}$$
$$-\frac{44375f_{n+3}}{266112} - \frac{37375f_{n+4}}{4644864} + \frac{709000f_{n+\frac{1}{4}}}{567567} + \frac{33875f_{n+\frac{5}{2}}}{145152} + \frac{252625f_{n+\frac{7}{2}}}{4402944}],$$
$$y_{n+2}^{'} = y_n^{'} + 2hy_n^{''} + 2h^2y_n^{'''} + h^3[\frac{151f_n}{1470} + \frac{22004f_{n+1}}{42525} - \frac{944f_{n+2}}{6615} - \frac{452f_{n+3}}{3465}$$

$$-\frac{493f_{n+4}}{85050}+\frac{7307264f_{n+\frac{1}{4}}}{9823275}+\frac{5248f_{n+\frac{5}{2}}}{25515}+\frac{1408f_{n+\frac{7}{2}}}{33075}],$$

$$y'_{n+\frac{7}{2}}=y'_n+\frac{7hy''_n}{2}+\frac{49h^2y'''_n}{8}+h^3[\frac{105301f_n}{368640}+\frac{2336173f_{n+1}}{777600}+\frac{506611f_{n+2}}{552960}$$
$$-\frac{16807f_{n+3}}{168960}-\frac{521017f_{n+4}}{49766400}+\frac{6876464f_{n+\frac{1}{4}}}{2606175}+\frac{319333f_{n+\frac{5}{2}}}{933120}+\frac{154693f_{n+\frac{7}{2}}}{2246400}],$$

$$y'_{n+3}=y'_n+3hy''_n+\frac{9h^2y'''_n}{2}+h^3[\frac{837f_n}{3920}+\frac{5427f_{n+1}}{2800}+\frac{81f_{n+2}}{245}-\frac{99f_{n+3}}{560}$$
$$-\frac{27f_{n+4}}{2800}+\frac{2304f_{n+\frac{1}{4}}}{1225}+\frac{9f_{n+\frac{5}{2}}}{35}+\frac{81f_{n+\frac{7}{2}}}{1225}],$$

$$y'_{n+4}=y'_n+4hy''_n+8h^2y'''_n+h^3[\frac{488f_n}{1323}+\frac{6784f_{n+1}}{1575}+\frac{11488f_{n+2}}{6615}+\frac{128f_{n+3}}{945}$$
$$-\frac{136f_{n+4}}{14175}+\frac{1048576f_{n+\frac{1}{4}}}{297675}+\frac{4096f_{n+\frac{5}{2}}}{8505}+\frac{4096f_{n+\frac{7}{2}}}{33075}],$$

$$y''_{n+\frac{1}{4}}=y''_n+\frac{hy'''_n}{4}+\frac{305073173f_n}{17340825600}-\frac{18915997f_{n+1}}{5573836800}+\frac{8977231f_{n+2}}{1734082560}+\frac{168203f_{n+3}}{38928384}$$
$$+\frac{4379827f_{n+4}}{22295347200}+\frac{254779379f_{n+\frac{1}{4}}}{16345929600}-\frac{444427f_{n+\frac{5}{2}}}{65318400}-\frac{2511401f_{n+\frac{7}{2}}}{1761177600},$$

$$y''_{n+1}=y''_n+hy'''_n+\frac{58703f_n}{1058400}+\frac{42353f_{n+1}}{340200}-\frac{1993f_{n+2}}{21168}-\frac{5639f_{n+3}}{83160}-\frac{3923f_{n+4}}{1360800}$$
$$+\frac{3441664f_{n+\frac{1}{4}}}{9823275}+\frac{14416f_{n+\frac{5}{2}}}{127575}+\frac{712f_{n+\frac{7}{2}}}{33075},$$

$$y''_{n+\frac{5}{2}}=y''_n+\frac{5hy'''_n}{2}+\frac{600475f_n}{5419008}+\frac{2474875f_{n+1}}{1741824}+\frac{1275625f_{n+2}}{2709504}-\frac{104375f_{n+3}}{2128896}$$
$$-\frac{26875f_{n+4}}{6967296}+\frac{5807500f_{n+\frac{1}{4}}}{5108103}+\frac{4825f_{n+\frac{5}{2}}}{326592}+\frac{26125f_{n+\frac{7}{2}}}{1100736},$$

$$y''_{n+2}=y''_n+2hy'''_n+\frac{5851f_n}{66150}+\frac{40232f_{n+1}}{42525}+\frac{179f_{n+2}}{6615}-\frac{152f_{n+3}}{1485}-\frac{451f_{n+4}}{85050}$$
$$+\frac{112984064f_{n+\frac{1}{4}}}{127702575}+\frac{15872f_{n+\frac{5}{2}}}{127575}+\frac{15872f_{n+\frac{7}{2}}}{429975},$$

$$y''_{n+\frac{7}{2}}=y''_n+\frac{7hy'''_n}{2}+\frac{429191f_n}{2764800}+\frac{14734937f_{n+1}}{6220800}+\frac{77861f_{n+2}}{55296}+\frac{477799f_{n+3}}{1520640}$$
$$-\frac{40817f_{n+4}}{24883200}+\frac{4279268f_{n+\frac{1}{4}}}{2606175}+\frac{256907f_{n+\frac{5}{2}}}{1166400}+\frac{10241f_{n+\frac{7}{2}}}{561600},$$

$$y''_{n+3}=y''_n+3hy'''_n+\frac{5217f_n}{39200}+\frac{379f_{n+1}}{200}+\frac{3681f_{n+2}}{3920}+\frac{93f_{n+3}}{3080}-\frac{13f_{n+4}}{5600}$$
$$+\frac{730112f_{n+\frac{1}{4}}}{525525}+\frac{8f_{n+\frac{5}{2}}}{75}+\frac{144f_{n+\frac{7}{2}}}{15925},$$

$$y''_{n+4}=y''_n+4hy'''_n+\frac{5896f_n}{33075}+\frac{121024f_{n+1}}{42525}+\frac{12304f_{n+2}}{6615}+\frac{1216f_{n+3}}{2079}+\frac{704f_{n+4}}{42525}$$
$$+\frac{241696768f_{n+\frac{1}{4}}}{127702575}+\frac{47104f_{n+\frac{5}{2}}}{127575}+\frac{108544f_{n+\frac{7}{2}}}{429975},$$

$$y'''_{n+\frac{1}{4}}=y'''_n+h[\frac{11532343f_n}{115605504}-\frac{4821053f_{n+1}}{185794560}+\frac{11277377f_{n+2}}{289013760}+\frac{7363793f_{n+3}}{227082240}$$
$$+\frac{156203f_{n+4}}{106168320}+\frac{44982053f_{n+\frac{1}{4}}}{272432160}-\frac{1781909f_{n+\frac{5}{2}}}{34836480}-\frac{251009f_{n+\frac{7}{2}}}{23482368}],$$

$$\begin{aligned}
y'''_{n+1} &= y'''_n + h[\frac{109f_n}{7056} + \frac{2281f_{n+1}}{4536} - \frac{2699f_{n+2}}{8820} - \frac{6119f_{n+3}}{27720} - \frac{61f_{n+4}}{6480} \\
&\quad + \frac{988160f_{n+\frac{1}{4}}}{1702701} + \frac{3124f_{n+\frac{5}{2}}}{8505} + \frac{2012f_{n+\frac{7}{2}}}{28665}], \\
y'''_{n+\frac{5}{2}} &= y'''_n + h[\frac{19855f_n}{451584} + \frac{137225f_{n+1}}{145152} + \frac{216625f_{n+2}}{225792} + \frac{13375f_{n+3}}{177408} + \frac{175f_{n+4}}{82944} \\
&\quad + \frac{861800f_{n+\frac{1}{4}}}{1702701} - \frac{355f_{n+\frac{5}{2}}}{27216} - \frac{1775f_{n+\frac{7}{2}}}{91728}], \\
y'''_{n+2} &= y'''_n + h[\frac{67f_n}{1470} + \frac{902f_{n+1}}{945} + \frac{536f_{n+2}}{735} + \frac{158f_{n+3}}{1155} + \frac{f_{n+4}}{270} + \frac{475136f_{n+\frac{1}{4}}}{945945} \\
&\quad - \frac{64f_{n+\frac{5}{2}}}{189} - \frac{64f_{n+\frac{7}{2}}}{1911}], \\
y'''_{n+\frac{7}{2}} &= y'''_n + h[\frac{133f_n}{3072} + \frac{6517f_{n+1}}{6912} + \frac{7399f_{n+2}}{7680} + \frac{30527f_{n+3}}{42240} - \frac{343f_{n+4}}{138240} \\
&\quad + \frac{1960f_{n+\frac{1}{4}}}{3861} + \frac{343f_{n+\frac{5}{2}}}{2160} + \frac{511f_{n+3}}{3120}], \\
y'''_{n+3} &= y'''_n + h[\frac{177f_n}{3920} + \frac{799f_{n+1}}{840} + \frac{897f_{n+2}}{980} + \frac{1017f_{n+3}}{3080} + \frac{31744f_{n+\frac{1}{4}}}{63063} \\
&\quad + \frac{f_{n+4}}{240} + \frac{92f_{n+\frac{5}{2}}}{315} - \frac{132f_{n+\frac{7}{2}}}{3185}] \text{ and} \\
y'''_{n+4} &= y'''_n + h[\frac{22f_n}{441} + \frac{2752f_{n+1}}{2835} + \frac{1816f_{n+2}}{2205} + \frac{1088f_{n+3}}{3465} + \frac{62f_{n+4}}{405} \\
&\quad + \frac{4194304f_{n+\frac{1}{4}}}{8513505} + \frac{4096f_{n+\frac{5}{2}}}{8505} + \frac{4096f_{n+\frac{7}{2}}}{5733}]
\end{aligned}$$

3. Properties of the Method

3.1. Order of the Method

The linear difference operator L associated with equation (2.6) is defined as

$$L[y(x);h] = Y_m - A^{-1}BR_1 - h^4\left[A^{-1}DR_2 + A^{-1}ER_3\right] \quad (3.1)$$

Expanding the terms Y_m and R_3 using Taylor series about x_n respectively and then collecting their like elements to the powers of h gives

$$L[y(x),h] = \bar{C}_0 y(x) + \bar{C}_1 h y'(x) + \bar{C}_2 h^2 y''(x) + \cdots \quad (3.2)$$

Definition 3.1. *The hybrid block method (2.6) and its linear operator (3.1) are said to have order p if, in(9),* $\bar{C}_0 = \bar{C}_1 = \bar{C}_2 = \cdots = \bar{C}_{p+3} = 0$ *and* $\bar{C}_{p+4} \neq 0$ *with error constants vector* $\bar{C}_{p+4}$.

Expanding equation(2.6) about $x = x_n$ using Taylor series and then comparing the coefficients of h lead to $\bar{C}_0 = \bar{C}_1 = \bar{C}_2 = \cdots = \bar{C}_{11} = 0$ and $\bar{C}_{12} \neq 0$. Hence the order of the new method is $[8, 8, 8, 8, 8, 8, 8]^T$. This implies that, the new block method is consistent since its order is greater than one.

3.2. Zero stability

In order to find the zero stability of the method, we have extended the definition in Fatunla [8]. That is, the hybrid block method (7) is said to be zero stable if the first characteristic polynomial $\Pi(t)$ having roots such that $t_r \leq 1$; and if $t_r = 1$; then the multiplicity of t_r must not greater than 4. Applying this in our main block method $\left[y_{n+1}, y_{n+\frac{1}{4}}, y_{n+1}, y_{n+2}, y_{n+\frac{5}{2}}, y_{n+3}, y_{n+\frac{7}{2}}, y_{n+4}\right]^T$ where I is 7×7 identity matrix and $\bar{B}$ is coefficients matrix of y_n,

$$\begin{aligned}
\Pi(t) &= |t\, I - \bar{B}| \\
&= \left| t \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right| \\
&= t^6(t-1)
\end{aligned}$$

which gives $t = 0, 0, 0, 0, 0, 0, 1$. Hence our method is zero stable.

3.3. Convergence

According to Henrici [9], zero stability and consistency are sufficient conditions of the numerical method to be convergent. Hence, the new method (2.6) is convergent.

4. Numerical Results

4.1. Numerical Examples

In order to show the accuracy of the new method, three fourth order IVPs problems are solved and compared with existing methods as shown in **Table 1-3** where
h: Step size.
Step: total number of steps taken to obtain solution.
E(x): magnitude of the maximum error of the computed solution.
Error: absolute error.

Problem 1: $y^{iv} = -y^{''}, \quad y(0) = 0, \quad y^{'}(0) = \dfrac{-1.1}{72 - 50\pi},\ y^{''}(0) = \dfrac{1}{144 - 100\pi},$
$y^{'''}(0) = \dfrac{1.2}{144 - 100\pi} \qquad x \in [0, 101325]$
Exact solution : $\quad y(x) = 1 - x - \cos x - \dfrac{1.2 \sin x}{144 - 100\pi}$

Problem 2: $y^{iv} = y^{'''} + y^{''} + y^{'} + 2y = 0, \quad y(0) = 1,$
$y^{'}(0) = 0, \quad y^{''}(0) = 0, \quad y^{'''}(0) = 30, \qquad x \in [0, 2].$
Exact solution : $\quad y(x) = 2e^{2x} - 5e^{-x} + 3\cos x - 9 \sin x$

Problem 3: $y^{iv} = 4y^{''}, \quad y(0) = 1, \quad y^{'}(0) = 3,\ y^{''}(0) = 0,\ y^{'''}(0) = 16$
Exact solution : $\quad y(x) = 1 - x + e^{2x} - e^{-2x}, \qquad x \in [0, 1]$

In general, the obtained numerical results show the efficiency of the developed method in term of accuracy. Its performance is clear better than several existing method through Table 1, Table 2 and Table 3.

Table 1: Comparison of the new method with some existing methods for solving problem 1

h	Method	Error at x=1.01325	Steps.
	New method	5.28×10^{-18}	3
0.103125	Method in [15]	2.95×10^{-17}	3
	Method in[6]	5.37×10^{-8}	2
h		Error at x=1	
	New method	1.45×10^{-18}	25
0.01	Method in[4]	8.04×10^{-16}	20

Table 2: Comparison of the new method with some existing methods for solving problem 2

h	Method	error at x=2	Step
	New method	1.70×10^{-10}	5
	Method in [15]	8.07×10^{-10}	5
	Method in [15]	1.74×10^{-8}	5
0.1	Adams Method	2.11×10^{-3}	
	Method in [10]	1.26×10^{-4}	20
	New method	4.15×10^{-12}	10
	Method in [15]	8.45×10^{-11}	10
0.05	Adams Method	5.37×10^{-4}	
	Method in[10]	1.91×10^{-6}	40
	New method	2.13×10^{-13}	20
	Method in [15]	3.69×10^{-13}	20
0.025	Adams Method	5.09×10^{-5}	
	Method in[10]	2.96×10^{-8}	80

Table 3: Comparison of the new method with some existing methods for solving problem 3

h	Method	Error at x=0.031250.	Step
	New method	9.17×10^{-17}	3
0.003125	Method in [15]	3.29×10^{-15}	3
	Method in[6]	3.60×10^{-13}	2

4.2. Application

The new method is applied for solving physical status appear in ship dynamics. In particular, Twizell [13] and Cortell[7] have presented numerical solution of this problem which is fourth order IVPs describes how the sinusoidal wave of frequency Ω passes along a ship or offshore structure to relate fluids action with time t as below.

$$y'''' = -3y'' - y(2 + \epsilon\cos(\Omega t)) \tag{4.1}$$

which is imposed to the following conditions: $y(0) = 1,\ y'(0) = 0, y''(0) = 0,\ y'''(0) = 0$. where $\epsilon = 0$ for the existence of the theoretical solution, $y(t) = 2\cos t - \cos(t\sqrt{2})$. The theoretical solution is undefined when $\epsilon \neq 0$.

In Table 4, the accuracy of the new method is compared with [13] and [7] for solving previous application at the end point $t = 15$, where $h = 0.25$ and $h = 0.1$.

Table 4: Comparison of the new method with [13] and [7] for solving Equation (4.1)

h	Method	Error at t=15.
	New method	8.15×10^{-8}
0.25	Method in [15]	5.2×10^{-7}
	Method in [13]	1.9×10^{-4}
	New method	3.6×10^{-11}
0.1	Method in [15]	2.8×10^{-10}
	Method in [7]	3.7×10^{-5}

5. Conclusion

Four steps block method with three hybrid points for solving fourth order initial value problems has been developed. The implementation of the method was then made through direct self-starting block method. Since this method is capable of solving forth order IVPs directly, it is more robust and flexible. Besides having good numerical properties the method is also claim superior to the existing method in terms of error.

Conflicts of Interest: The author declare no conflict of interest.

References

[1] R. Abdelrahim and Z. Omar, *Direct solution of second-order ordinary differential equation using a single-step hybrid block method of order five*, Math. Comput. Appl. 21 (2016) 1–7.

[2] R. Abdelrahim and Z. Omar, *Solving third order ordinary differential equations using hybrid block method of order five*, Int. J. Appl. Engin. Res. 10 (2015) 44307–44310.

[3] R. Abdelrahim and Z. Omar, *A four-step implicit block method with three generalized off-step points for solving fourth order initial value problems directly*, J. King Saud Univ.- Sci. 4 (2017) 401–412.

[4] O. Adesanya, M. Momoh, Adamu and A.O. Tahir, *Five steps block method for the solution of fourth order ordinary differential equations*, Int. J. Engin. Res. Appl. 2 (2012) 991–998 .

[5] D.O. Awoyemi, *Algorithmic collocation approach for direct solution of fourth-order initial-value problems of ordinary differential equations*, Int. J. Comput. Math. 47 (2005) 321–329.

[6] D.O. Awoyemi, S.J. Kayode and O.A Adoghe, *Six-Step Continuous Multistep Method For The Solution Of General Fourth Order Initial Value Problems Of Ordinary Differential Equations.* Journal of Natural Sciences Research. 5(2005) 131-138.

[7] R. Cortell, *Application of the fourth-order Runge-Kutta method for the solution of high-order general initial value problems*, Comput. Struct. 49 (1993) 897–900.

[8] S.O. Fatunla, *Block method for second order ODES*, Int. J. Comput. Math. 41 (1991) 55–63.

[9] P. Henrici, *Discrete Variable Methods in Ordinary Differential Equations*, John Wiley & Sons, New York, NY, USA, 1962.

[10] S.N. Jator, *Numerical integrators for fourth order initial and boundary value problems*, Int. J. Pure Appl. Math. 47 (2008) 563–576.

[11] J.D. Lambert, *Numerical Methods for Ordinary Differential Systems*, Wiley, 1991.

[12] Z. Omar and R. Abdelrahim, *Application of single step with three generalized hybrid points block method for solving third order ordinary differential equations*, Int. J. Appl. Engin. Res. 9 (2016) 2705–2717.

[13] E.H. Twizell, *A family of numerical methods for the solution of high-order general initial value problems*, Comput. Meth. Appl. Mech. Engin. 67 (1988) 15–25.

[14] L.K. Yap, F. Ismail and N. Senu, *An accurate block hybrid collocation method for third order ordinary differential equations*, J. Appl. Math. 2014 (2014) 9.

[15] L.K. Yap and F. Ismail, *Block Hybrid Collocation Method with Application to Fourth Order Differential Equations*, Math. Probl. Engin. 2015 (2015) 1–6.