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Some fixed point theorems for α_*-ψ-common rational type mappings on generalized metric spaces with application to fractional integral equations

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Abstract

Recently Hamed H Alsulami et al introduced the notion of (α-ψ)-rational type contractive mappings. They have been establish some fixed point theorems for the mappings in complete generalized metric spaces. In this paper, we introduce the notion of some fixed points theorems for α_*-ψ-common rational type mappings on generalized metric spaces with application to fractional integral equations and give a common fixed point result about fixed points of the set-valued mappings.

Keywords: Fixed points, α_*-common admissible, α_*-ψ-common rational type contractive, Partially ordered set, Generalized metric spaces, Weakly increasing, Fractional integral equations
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1. Introduction

You know, fixed point theory has many applications and was extended by several authors from different views (see for example [1]-[23]). Recently Hamed H Alsulami et al introduced the notion of $(\alpha - \psi)-$rational type contractive mappings ([2]). Denote with Ψ the family of upper semi-continuous, strictly increasing functions $\psi : [0, \infty) \to [0, \infty)$ such that $\{\psi^n(t)\}_{n\in N}$ converges to 0 as $n \to \infty$ and $\psi(t) < t$ for all $t > 0$ where ψ^n is the n-th iterate of ψ and $\psi \in \Psi$ ([2]). Let (X, d) be a generalized metric space, T a self-map on X, $\psi \in \Psi$ and $\alpha : X \times X \to [0, \infty)$ a function. Then

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T is called a (α-ψ)-rational type-I contraction mapping whenever $\alpha(x,y)d(Tx,Ty) \leq \psi(M_I(x,y))$ for all $x, y \in X$ where

$$M_I(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}, \frac{d(x,Tx)d(y,Ty)}{1+d(Tx,Ty)}\}, \qquad (1.1)$$

for all $x, y \in X$. Also, we say that T is α-admissible whenever $\alpha(x,y) \geq 1 \Rightarrow \alpha(Tx,Ty) \geq 1$. Also, we say that X has the property α-regular generalized metric space if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \geq 1$ and $x_n \to x$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all $k \geq 1$. Let (X,d) be a generalized complete metric space, T a α-admissible and (α-ψ)-rational type contractive mappings on X. Suppose that there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, $\alpha(x_0, T^2x_0) \geq 1$. If T is continuous or X has the property α-regular generalized metric space, then T has a fixed point (see [2]; Theorems (2.1) and (2.2)). The aim of this paper is to introduce the notion of some fixed points theorems for α_*-ψ-common rational type mappings on generalized metric spaces with application to fractional integral equations. Let 2^X denote the family of all nonempty subsets of X.

2. Preliminaries

In this section, we list some fundamental definitions that are useful tool in consequent analysis.

Definition 2.1. *[9] Let X be a nonempty set and $d : X \times X \to [0,\infty)$ satisfy the following conditions, for all $x, y \in X$ and all distinct $u, v \in X$ each of which is different from x and y:*
(GMS1) $d(x,y) = 0$ if and if $x = y$
(GMS2) $d(x,y) = d(y,x)$
(GMS3) $d(x,y) \leq d(x,u) + d(u,v) + d(v,y)$.
Then the map d is called a generalized metric and abbreviated as GM. Here, the pair (X,d) is called a generalized metric space and abbreviated as GMS.

In the abave definition, if d satisfies only $GMS1$ and $GMS2$, then it is called a semi-metric (see, e.g. [23]).

A sequence $\{x_n\}$ in a GMS (X,d) is GMS convergent to a limit x if and only if $d(x_n, x) \to 0$ as $n \to \infty$.

A sequence $\{x_n\}$ in a GMS (X,d) is GMS Cauchy if and if for every $\epsilon > 0$ there exists a positive integer $N(\epsilon)$ such that $d(x_n, x_m) < \epsilon$, for all $n > m > N(\epsilon)$.

A GMS (X,d) is called complete if every GMS Cauchy sequence in X is GMS convergent.

A mapping $T : (X,d) \to (X,d)$ is continuous if for any sequence $\{x_n\}$ in X such that $d(x_n, x) \to 0$ as $n \to \infty$, we have $d(Tx_n, Tx) \to 0$ as $n \to \infty$.

The following assumption was suggested by Wlilson ([23]) to replace the triangle inequality with the weakened condition.

(W) For each pair of (distinct) points u, v there is number $r_{u,v} > 0$ such that for every $z \in X$, $r_{u,v} < d(u,z) + d(z,v)$.

Proposition 2.2. *[19] Suppose that $\{x_n\}$ is a Cauchy sequence in a GMS (X,d) with $\lim_{n\to\infty} d(x_n, u) = 0$ where $u \in X$. Then $\lim_{n\to\infty} d(x_n, z) = d(u,z)$ for all $z \in X$. In particular, the sequence $\{x_n\}$ dose not converge to z if $z \neq u$.*

Definition 2.3. *Let F the family of functions $f : [0,\infty) \to \mathbb{R}$ satisfy:*
(i) $f(0) = 0$ and $f(t) > 0$ for all $t \in (0,+\infty)$;

(ii) f *is continuous;*
(iii) f *is nondecreasing on* $[0, +\infty)$*;*
(iv) $f(t_1 + t_2) \leq f(t_1) + f(t_2)$ *for all* $t_1, t_2 \in (0, +\infty)$*.*

Definition 2.4. *Let* (X, d) *be a* (GMS) *and* $T, S : X \to 2^X$ *with given multi-valued,* $\alpha : X \times X \to [0, +\infty)$*,*$\alpha_* : 2^X \times 2^X \to [0, +\infty)$*,* $\alpha_*(A, B) = \inf\{\alpha(a, b) : a \in A, b \in B\}$*,* $f \in F$*,* $\psi \in \Psi$*,* $D(s, Ts) = \inf\{d(s, z)/z \in Ts\}$*,* H *is the Hausdorff metric and let*

$$M_I(Ax, By) = \max\{d(x, y), D(x, Ax), D(y, By), \frac{D(x, Ax)D(y, By)}{1 + d(x, y)}, \frac{D(x, Ax)D(y, By)}{1 + H(Ax, By)}\}, \quad (2.1)$$

$$H(Ax, By) = \max\{\sup_{a \in Ax} D(a, By), \sup_{b \in By} D(Ax, b)\}.$$

One says that T, S *are* α_**-*ψ*-common rational type-I contractive set-valued mappings whenever*

$$\alpha_*(Ax, By) f(H(Ax, By)) \leq \psi(f(M_I(Ax, By))), \quad (2.2)$$

$A, B = T$ *or* S *for all* $x, y \in X$*.*

Definition 2.5. *Let* $T, S : X \to 2^X$ *and* $\alpha : X \times X \to [0, +\infty)$*. One says that* T, S *are an* α_**-common admissible if* $\alpha(x, y) \geq 1 \Rightarrow \alpha_*(Ax, By) \geq 1$*,* $A, B = T$ *or* S *for all* $x, y \in X$*.*

Definition 2.6. *A subset* $B \subseteq X$ *is said to be an approximation if for each given* $y \in X$*, there exists* $z \in B$ *such that* $D(B, y) = d(z, y)$*.*

Definition 2.7. *A set-valued mapping* $T : X \longrightarrow 2^X$ *is said to have an approximate values in* X *if* Tx *is an approximation for each* $x \in X$*.*

Definition 2.8. *([22]) A set-valued operator* $T : X \to 2^X$ *is called order closed if for monotone sequences* $x_n \in X$ *and* $y_n \in Tx_n$*, with* $\lim_{n \to \infty} d(x_n, x) = 0$ *and* $\lim_{n \to \infty} d(y_n, y) = 0$*, implies* $y \in Tx$*.*

Definition 2.9. *Let* (X, d) *be a metric space. If* $T : X \to 2^X$ *is a set-valued mapping, then* $x \in X$ *is called fixed point for* T *if and only if* $x \in F(x)$*. The set* $Fix(T) := \{x \in X | x \in Tx\}$ *is called the fixed point set of* T*.*

Definition 2.10. *We say that* X *has the property* $\alpha-$*regular generalized metric space if, either*
(i) $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* n *and* $x_n \to x \in X$ *as* $n \to \infty$*, then there exists a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *such that* $\alpha(x_{n_k}, x) \geq 1$ *for all* k*. Or*
(ii) $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_{n+1}, x_n) \geq 1$ *for all* n *and* $x_n \to x \in X$ *as* $n \to \infty$*, then there exists a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *such that* $\alpha(x, x_{n_k}) \geq 1$ *for all* k*.*

Throughout this paper we always assume that all set-valued operators have approximate values. We have the following result. Finally, we should emphasize that throughout this paper we suppose that all set-valued mappings on a metric space (X, d) have closed values.

3. Main result

Now, we are ready to state and prove our main results. Fixed point theorems for order closed set-valued mappings.

Lemma 3.1. *Let (X,d) be a GMS and $T,S:X\to 2^X$ are α_*-ψ-common rational type-I contractive set-valued mappings. Then $Fix(T)=Fix(S)$.*

Proof . We first show that any fixed point of T is also a fixed point of S and conversely. Define $\alpha(x,y)=1$ for all $x,y\in X$. Since $Fix(T)\neq Fix(S)$, we may assume there exists $x^*\in X$ such that $x^*\in Fix(T)$, but $x^*\notin Fix(S)$. Since $D(x^*,Sx^*)>0$, we have

$$\begin{aligned} M_I(Tx^*,Sx^*) &= \max\{d(x^*,x^*),D(x^*,Tx^*),D(x^*,Sx^*) \\ &,\tfrac{D(x^*,Tx^*)D(x^*,Sx^*)}{1+d(x^*,x^*)},\tfrac{D(x^*,Tx^*)D(x^*,Sx^*)}{1+H(Tx^*,Sx^*)}\} \\ &= D(x^*,Sx^*) \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} f(D(x^*,Sx^*)) &\le f(H(Tx^*,Sx^*)) \\ &\le \alpha_*(Tx^*,Sx^*)f(H(Tx^*,Sx^*)) \\ &\le \psi(f(M_I(Tx^*,Sx^*))) \\ &\le \psi(f(D(x^*,Sx^*))) \\ &< f(D(x^*,Sx^*)). \end{aligned} \tag{3.2}$$

This contradiction establishes that $Fix(T)\subseteq Fix(S)$. A similar argument establishes the reverse containment, and therefore $Fix(T)=Fix(S)$. □

Theorem 3.2. *Let (X,d) be a complete GMS, $T,S:X\to 2^X$ be a α_*-ψ-common rational type-I contractive set-valued mappings and satisfies the following conditions:*
(i) T,S are α_*-common admissible;
(ii) there exists $x_0\in X$ such that

$$\alpha_*(\{x_0\},Tx_0)\ge 1,\alpha_*(\{x_0\},STx_0)\ge 1;$$

(iii) X has the property $\alpha-$regular generalized metric space
Then T,S have common fixed point $x^*\in X$. Further, for each $x_0\in X$, the iterated sequence $\{x_n\}$ with $x_{2n+1}\in Tx_{2n}$ and $x_{2n+2}\in Sx_{2n+1}$ converges to the common fixed point of T,S.

Proof . By lemma (3.1), we have $Fix(T)=Fix(S)$. Let $x_0\in X$ such that $\alpha_*(\{x_0\},Tx_0)\ge 1$ and $\alpha_*(\{x_0\},STx_0)\ge 1$. Define the sequence $\{x_n\}$ in X by $x_{2n+1}\in Tx_{2n}$ and $x_{2n+2}\in Sx_{2n+1}$ for all $n\in\mathbb{N}_0$. If $x_{n_0}=x_{n_0+1}$ for some $n_0>1$, then $x^*=x_{n_0}$ are a common fixed point for T,S. So, we can assume that $x_{2n}\notin Tx_{2n}$ and $x_{2n+1}\notin Sx_{2n+1}$ for all $n\in\mathbb{N}_0$. Since T,S are α_*-common admissible, we have

$$\alpha(x_0,x_1)\ge\alpha_*(\{x_0\},Tx_0)\ge 1\Rightarrow\alpha_*(Tx_0,Sx_1)\ge 1; \tag{3.3}$$

$$\alpha(x_1,x_2)\ge\alpha_*(Tx_0,Sx_1)\ge 1\Rightarrow\alpha_*(Sx_1,Tx_2)\ge 1. \tag{3.4}$$

Inductively, we have $\alpha(x_n,x_{n+1})\ge 1$ for all $n\in\mathbb{N}_0$. By similar arguments, since $\alpha_*(\{x_0\},STx_0)\ge 1$, we have

$$\alpha(x_0,x_2)\ge\alpha_*(\{x_0\},STx_0)\ge 1\Rightarrow\alpha_*(Tx_0,Tx_2)\ge 1; \tag{3.5}$$

$$\alpha(x_1,x_3)\ge\alpha_*(Tx_0,Tx_2)\ge 1\Rightarrow\alpha_*(Sx_1,Sx_3)\ge 1. \tag{3.6}$$

Inductively, we have $\alpha(x_n, x_{n+2}) \geq 1$ for all $n \in \mathbb{N}_0$. Without loss of generality, we may assume that $T, S : X \to 2^X$ be a α_*-ψ-common rational type-I contractive set-valued mappings. Consider equation (2.1), (2.2) with $x = x_{2n+1}$ and $y = x_{2n+2}$. Clearly, we have

$$\begin{aligned} f(d(x_{2n+1}, x_{2n+2})) &\leq f(H(Tx_{2n}, Sx_{2n+1})) \\ &\leq \alpha_*(Tx_{2n}, Sx_{2n+1}) f(H(Tx_{2n}, Sx_{2n+1})) \\ &\leq \psi(f(M_I(Tx_{2n}, Sx_{2n+1}))), \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} M_I(Tx_{2n}, Sx_{2n+1}) &= \max\{d(x_{2n}, x_{2n+1}), D(x_{2n}, Tx_{2n}), D(x_{2n+1}, Sx_{2n+1}), \\ &\quad \tfrac{D(x_{2n}, Tx_{2n}) D(x_{2n+1}, Sx_{2n+1})}{1 + d(x_{2n}, x_{2n+1})}, \tfrac{D(x_{2n}, Tx_{2n}) D(x_{2n+1}, Sx_{2n+1})}{1 + D(Tx_{2n}, Sx_{2n+1})}\} \\ &= \max\{d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \\ &\quad \tfrac{d(x_{2n}, x_{2n+1}) d(x_{2n+1}, x_{2n+2})}{1 + d(x_{2n}, x_{2n+1})}, \tfrac{d(x_{2n}, x_{2n+1}) d(x_{2n+1}, x_{2n+2})}{1 + d(x_{2n+1}, x_{2n+2})}\} \\ &= \max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\}, \end{aligned} \tag{3.8}$$

since

$$\tfrac{d(x_{2n}, x_{2n+1}) d(x_{2n+1}, x_{2n+2})}{1 + d(x_{2n}, x_{2n+1})} = \tfrac{d(x_{2n}, x_{2n+1})}{1 + d(x_{2n}, x_{2n+1})} \times d(x_{2n+1}, x_{2n+2}) \leq d(x_{2n+1}, x_{2n+2}) \tag{3.9}$$

and

$$\tfrac{d(x_{2n}, x_{2n+1}) d(x_{2n+1}, x_{2n+2})}{1 + d(x_{2n+1}, x_{2n+2})} = \tfrac{d(x_{2n+1}, x_{2n+2})}{1 + d(x_{2n+1}, x_{2n+2})} \times d(x_{2n}, x_{2n+1}) \leq d(x_{2n}, x_{2n+1}). \tag{3.10}$$

If

$$\max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\} = d(x_{2n+1}, x_{2n+2}). \tag{3.11}$$

So, in general,

$$f(d(x_{2n+1}, x_{2n+2})) \leq \psi(f(d(x_{2n+1}, x_{2n+2}))) < f(d(x_{2n+1}, x_{2n+2})), \tag{3.12}$$

which is contradiction since $d(x_{2n+1}, x_{2n+2}) > 0$. Thus

$$f(d(x_{2n+1}, x_{2n+2})) \leq \psi(f(d(x_{2n}, x_{2n+1}))). \tag{3.13}$$

Similarly,

$$f(d(x_{2n}, x_{2n+1})) \leq \psi(f(d(x_{2n-1}, x_{2n}))), \tag{3.14}$$

we have

$$f(d(x_{n+1}, x_{n+2})) \leq \psi(f(d(x_n, x_{n+1}))) \leq \ldots \leq \psi^n(f(d(x_0, x_1))), \tag{3.15}$$

for all $n \in \mathbb{N}$. From the property of ψ, we conclude that

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n), \tag{3.16}$$

for all $n \in \mathbb{N}$, it is clear that

$$\lim_{n \to \infty} d(x_{n+1}, x_{n+2}) = 0. \tag{3.17}$$

Consider equation (2.1), (2.2) with $x = x_{2n-1}$ and $y = x_{2n+1}$. Clearly, we have

$$\begin{aligned} f(d(x_{2n}, x_{2n+2})) &\leq f(H(Sx_{2n-1}, Sx_{2n+1})) \\ &\leq \alpha_*(Sx_{2n-1}, Sx_{2n+1}) f(H(Sx_{2n-1}, Sx_{2n+1})) \\ &\leq \psi(f(M_I(Sx_{2n-1}, Sx_{2n+1}))). \end{aligned} \tag{3.18}$$

where

$$\begin{aligned} M_I(Sx_{2n-1}, Sx_{2n+1}) &= \max\{d(x_{2n-1}, x_{2n+1}), D(x_{2n-1}, Sx_{2n-1}), D(x_{2n+1}, Sx_{2n+1}), \\ &\tfrac{D(x_{2n-1},Sx_{2n-1})D(x_{2n+1},Sx_{2n+1})}{1+d(x_{2n-1},x_{2n+1})}, \tfrac{D(x_{2n-1},Sx_{2n-1})D(x_{2n+1},Sx_{2n+1})}{1+H(Sx_{2n-1},Sx_{2n+1})}\} \\ &= \max\{d(x_{2n-1}, x_{2n+1}), d(x_{2n-1}, x_{2n}), d(x_{2n+1}, x_{2n+2}), \\ &\tfrac{d(x_{2n-1},x_{2n})d(x_{2n+1},x_{2n+2})}{1+d(x_{2n-1},x_{2n+1})}, \tfrac{d(x_{2n-1},x_{2n})d(x_{2n+1},x_{2n+2})}{1+d(x_{2n},x_{2n+2})}\}. \end{aligned} \tag{3.19}$$

From (3.16) we have

$$d(x_{2n+1}, x_{2n+2}) < d(x_{2n-1}, x_{2n}). \tag{3.20}$$

Define $a_{2n} = d(x_{2n}, x_{2n+2})$ and $b_{2n} = d(x_{2n}, x_{2n+1})$. Then

$$M_I(Sx_{2n-1}, Sx_{2n+1}) = \max\{a_{2n-1}, b_{2n-1}, \frac{b_{2n-1}b_{2n+1}}{1+a_{2n-1}}, \frac{b_{2n-1}b_{2n+1}}{1+a_{2n}}\}. \tag{3.21}$$

If $M_I(Sx_{2n-1}, Sx_{2n+1}) = b_{2n-1}$, or $\frac{b_{2n-1}b_{2n+1}}{1+a_{2n-1}}$ or $\frac{b_{2n-1}b_{2n+1}}{1+a_{2n}}$ then taking lim sup as $n \to \infty$ in (3.16) and using (3.17) and upper semi-continuity of ψ we get

$$\begin{aligned} 0 \le \limsup_{n\to\infty} a_{2n} &\le \limsup_{n\to\infty} \psi(M_I(Sx_{2n-1}, Sx_{2n+1})) \\ &= \psi(\limsup_{n\to\infty} M_I(Sx_{2n-1}, Sx_{2n+1})) \\ &= \psi(0) = 0 \end{aligned} \tag{3.22}$$

and hence,

$$\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} d(x_{2n}, x_{2n+2}) = 0.$$

If $M_I(Sx_{2n-1}, Sx_{2n+1}) = a_{2n-1}$, then (3.17) implies $a_{2n} \le \psi(a_{2n-1}) < a_{2n-1}$ and similarly $a_{2n+1} \le \psi(a_{2n}) < a_{2n}$. By induction, we get $a_n \le \psi(a_{n-1}) < a_{n-1}$, due to the property of ψ. In other words, the sequence a_n is positive monotone decreasing, and hence, it converges to some $t \ge 0$. Assume that $t > 0$. Now, by (3.17), we get

$$t = \limsup_{n\to\infty} a_n = \limsup_{n\to\infty} \psi(a_n) = \psi(\limsup_{n\to\infty} a_{n-1}) = \psi(t) < t. \tag{3.23}$$

which is a contradiction. Therefore,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} d(x_n, x_{n+2}) = 0. \tag{3.24}$$

Now, we shall prove that $x_n \neq x_m$ for all $n \neq m$. Assume on the contrary that $x_n = x_m$ for some $m, n \in \mathbb{N}$ with $n \neq m$. Since $d(x_p, x_{p+1}) > 0$ for each $p \in \mathbb{N}$, without loss of generality, we may assume that $m > n+1, m = 2k$ and $n = 2l$ for $k, l \in \mathbb{N}$. Substitute again $x = x_{2l} = x_{2k}$ and $y = x_{2l+1} = x_{2k+1}$ in (2.1), (2.2) which yields

$$\begin{aligned} f(d(x_{2l}, x_{2l+1})) &= f(d(x_{2k}, x_{2k+1})) \le f(H(Sx_{2k-1}, Tx_{2k})) \\ &\le \alpha_*(Sx_{2k-1}, Tx_{2k}) f(H(Sx_{2k-1}, Tx_{2k})) \\ &\le \psi(f(M_I(Sx_{2k-1}, Tx_{2k}))). \end{aligned} \tag{3.25}$$

where

$$\begin{aligned} M_I(Sx_{2k-1}, Tx_{2k}) &= \max\{d(x_{2k-1}, x_{2k}), D(x_{2k-1}, Sx_{2k-1}), D(x_{2k}, Tx_{2k}), \\ &\tfrac{D(x_{2k-1},Sx_{2k-1})D(x_{2k},Tx_{2k})}{1+d(x_{2k-1},x_{2k})}, \tfrac{D(x_{2k-1},Sx_{2k-1})D(x_{2k},Tx_{2k})}{1+H(Sx_{2k-1},Tx_{2k})}\} \\ &= \max\{d(x_{2k-1}, x_{2k}), d(x_{2k-1}, x_{2k}), d(x_{2k}, x_{2k+1}), \\ &\tfrac{d(x_{2k-1},x_{2k})d(x_{2k},x_{2k+1})}{1+d(x_{2k-1},x_{2k})}, \tfrac{d(x_{2k-1},x_{2k})d(x_{2k},x_{2k+1})}{1+d(x_{2k},x_{2k+1})}\} \\ &= \max\{d(x_{2k-1}, x_{2k}), d(x_{2k}, x_{2k+1})\}. \end{aligned} \tag{3.26}$$

If $M_I(Sx_{2k-1}, Tx_{2k}) = d(x_{2k-1}, x_{2k})$, then from (3.26), implies

$$f(d(x_{2l}, x_{2l+1})) \leq \psi(f(d(x_{2k-1}, x_{2k}))) \leq \psi^{2k-2l}(f(d(x_{2l}, x_{2l+1}))). \tag{3.27}$$

If on the other hand $M_I(Sx_{2k-1}, Tx_{2k}) = d(x_{2k}, x_{2k+1})$, then from (18) we have

$$f(d(x_{2l}, x_{2l+1})) \leq \psi(f(d(x_{2k}, x_{2k+1}))) \leq \psi^{2k-2l+1}(f(d(x_{2l}, x_{2l+1}))). \tag{3.28}$$

Using the property of ψ, the two inequalities (3.27) and (3.28) imply

$$d(x_{2l}, x_{2l+1}) < d(x_{2l}, x_{2l+1}),$$

which is impossible. Now, we shall prove that $\{x_n\}$ is a Cauchy sequence, that is,

$$\lim_{n\to\infty} d(x_n, x_{n+k}) = 0,$$

for all $k \in \mathbb{N}$. We have already proved the cases for $k = 1$ and $k = 2$ in (3.16) and (3.19), respectively. Take arbitrary $k \geq 3$. We discuss two cases.

Case 1. Suppose that $k = 2m + 1$, where $m \geq 1$. Using the quadrilateral inequality $(GMS3)$, we have

$$f(d(x_n, x_{n+1})) \leq \psi(f(d(x_{n-1}, x_n))) \leq ... \leq \psi^n(f(d(x_0, x_1))), \text{ for all } n \in \mathbb{N}_0. \tag{3.29}$$

And

$$\begin{aligned} f(d(x_n, x_{n+2m+1})) &\leq f(d(x_n, x_{n+1})) + f(d(x_n, x_{n+2})) + ...\\ &+f(d(x_{n+2m}, x_{n+2m+1}))\\ &\leq \textstyle\sum_{p=n}^{n+2m} \psi^p(f(d(x_0, x_1)))\\ &\leq \textstyle\sum_{p=n}^{+\infty} \psi^p(f(d(x_0, x_1))) \to 0 \end{aligned} \tag{3.30}$$

as $n \to \infty$.

Case 2. Suppose that $k = 2m$, where $m \geq 2$. Using the quadrilateral inequality $(GMS3)$, we have

$$\begin{aligned} f(d(x_n, x_{n+2m})) &\leq f(d(x_n, x_{n+2})) + f(d(x_{n+2}, x_{n+3})) + ...\\ &+f(d(x_{n+2m-1}, x_{n+2m}))\\ &\leq d(x_n, x_{n+2}) + \textstyle\sum_{p=n+2}^{n+2m-1} \psi^p(f(d(x_0, x_1)))\\ &\leq d(x_n, x_{n+2}) + \textstyle\sum_{p=n}^{+\infty} \psi^p(f(d(x_0, x_1))) \to 0 \end{aligned} \tag{3.31}$$

as $n \to \infty$. In both of the abave cases, we have

$$\lim_{n\to\infty} d(x_n, x_{n+k}) = 0,$$

for all $k \geq 3$. Fix $\epsilon > 0$ and let $n(\epsilon) \in \mathbb{N}_0$ such that

$$\sum_{n=n(\epsilon)}^{\infty} \psi^n(f(d(x_0, x_1))) < \epsilon. \tag{3.32}$$

Let $n, m \in \mathbb{N}_0$ with $m > n > n(\epsilon)$. Using the quadrilateral inequality $(GMS3)$, we obtain

$$\begin{aligned} f(d(x_n, x_m)) &\leq f(d(x_n, x_{n+1})) + f(d(x_{n+1}, x_{n+2})) + f(d(x_{n+2}, x_m))\\ &\leq f(d(x_n, x_{n+1})) + f(d(x_{n+1}, x_{n+2})) + f(d(x_{n+2}, x_{n+3}))\\ &+f(d(x_{n+3}, x_{n+4})) + f(d(x_{n+4}, x_m))\\ &\leq f(d(x_n, x_{n+1})) + ... + f(d(x_{m-1}, x_m))\\ &= \textstyle\sum_{k=n}^{m-1} f(d(x_k, x_{k+1}))\\ &\leq \textstyle\sum_{k=n}^{m-1} \psi^k(f(d(x_0, x_1)))\\ &\leq \textstyle\sum_{n=n(\epsilon)}^{\infty} \psi^n(f(d(x_0, x_1))) < \epsilon. \end{aligned} \tag{3.33}$$

Thus we proved that $\{x_n\}$ is a Cauchy sequence in the metric space (X,d). Since (X,d) is complete metric space, there exists $x^* \in X$ such that

$$\lim_{n\to\infty} d(x_n, x^*) = 0$$

and condition (iii), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\alpha_*(\{x_{2n_k+1}\}, \{x^*\}) \geq \alpha_*(Tx_{2n_k}, Sx^*) \geq 1 \text{ for all } k. \tag{3.34}$$

Thus

$$\begin{aligned} f(D(x^*, Sx^*)) &\leq f(d(x^*, x_{2n_k+1})) + f(D(x_{2n_k+1}, Sx^*)) \\ &\leq f(d(x^*, x_{2n_k+1})) + \alpha_*(Tx_{2n_k}, Sx^*) f(H(Tx_{2n_k}, Sx^*)) \\ &\leq f(d(x^*, x_{2n_k+1})) + \psi(f(\max\{d(x_{2n_k}, x^*), D(x_{2n_k}, Tx_{2n_k}), \\ &\quad D(x^*, Sx^*), \tfrac{D(x_{2n_k}, Tx_{2n_k}) D(x^*, Sx^*)}{1+d(x_{2n_k}, x^*)}, \tfrac{D(x_{2n_k}, Tx_{2n_k}) D(x^*, Sx^*)}{1+H(Tx_{2n_k}, Sx^*)}\} \\ &\leq f(d(x^*, x_{2n_k+1})) + \psi(f(\max\{d(x_{2n_k}, x^*), d(x_{2n_k}, x_{2n_k+1}), \\ &\quad d(x^*, Sx^*), \tfrac{d(x_{2n_k}, x_{2n_k+1}) d(x^*, Sx^*)}{1+d(x_{2n_k}, x^*)}, \tfrac{d(x_{2n_k}, x_{2n_k+1}) D(x^*, Sx^*)}{1+D(x_{2n_k+1}, Sx^*)}\} = 0 \end{aligned} \tag{3.35}$$

for all k. Hence, $D(x^*, Sx^*) = 0$ and so $x^* \in Sx^*$. By Lemma (3.1) we have x^* common fixed point of T, S. □

Corollary 3.3. *Let (X,d) be a complete GMS, $T : X \to 2^X$ be a α_*-ψ-rational type$-I$ contractive set-valued mappings and satisfies the following conditions:*
(i) T are α_*-admissible;
(ii) there exists $x_0 \in X$ such that

$$\alpha_*(\{x_0\}, Tx_0) \geq 1, \alpha_*(\{x_0\}, T^2x_0) \geq 1;$$

(iii) if X has the property α-regular generalized metric space.
Then T has fixed point $x^* \in X$. Further, for each $x_0 \in X$, the iterated sequence $\{x_n\}$ with $x_{n+1} \in Tx_n$ converges to the fixed point of T.

Example 3.4. *Let X be a finite set defined as $X = \{1,2,3,4\}$. Define $d : X \times X \to [0,\infty)$ as:*
$d(1,1) = d(2,2) = d(3,3) = d(4,4) = 0,$
$d(1,2) = d(2,1) = 3,$
$d(2,3) = d(3,2) = d(1,3) = d(3,1) = 1$ *and*
$d(1,4) = d(4,1) = d(2,4) = d(4,2) = d(3,4) = d(4,3) = \frac{1}{2}.$
The function d is not a metric on X. Indeed, note

$$3 = d(1,2) \leq d(1,3) + d(3,2) = 1 + 1 = 2,$$

that is, the triangle inequality is not satisfied. However, d is a generalized metric on X and moreover (X,d) is a complete generalized metric space. Define $T, S : X \to 2^X$ as: $T1 = T2 = T3 = \{2,4\}, T4 = \{1,3\}$ and $S1 = S2 = S4 = \{2,3\}, S3 = \{1,2\}$, $\alpha : X \times X \to [0,+\infty), \alpha_* = \inf \alpha$ as $\alpha(x,y) = 1$, $\psi(t) = \frac{2}{3}t$ and $f(t) = \sqrt{t}$. Clearly, T, S satisfies the conditions of Theorem (3.2) and has a common fixed point $x = 2$.

4. Fixed point theorems for weakly increasing set-valued mappings without order closed

In the following we provide set-valued versions of the preceding theorem. The results are related to those in ([22]). Let X be a topological space and $\preceq$ be a partial order on X.

Definition 4.1. *([5]). Let A, B be two nonempty subsets of X, the relations between A and B are definers follows:*
(r_1) If for every $a \in A$, there exists $b \in B$ such that $a \preceq b$, then $A \prec_1 B$.
(r_2) If for every $b \in B$ there exists $a \in A$, such that $a \preceq b$, then $A \prec_2 B$.
(r_3) If $A \prec_1 B$ and $A \prec_2 B$, then $A \prec B$.

Definition 4.2. *([20], [10]). Let $(X, \preceq)$ be a partially ordered set. Two mappings $f, g : X \to X$ are said to be weakly increasing if $fx \preceq gfx$ and $gx \preceq fgx$ hold for all $x \in X$.*

Definition 4.3. *([3]) Let $(X, \preceq)$ be a partially ordered set. Two mapping $F, G : X \to 2^X$ are said to be weakly increasing with respect to $\prec_1$ if for any $x \in X$ we have $Fx \prec_1 Gy$ for all $y \in Fx$ and $Gx \prec_1 Fy$ for all $y \in Gx$. Similarly two maps $F, G : X \to 2^X$ are said to be weakly increasing with respect to $\prec_2$ if for any $x \in X$ we have $Gy \prec_2 Fx$ for all $y \in Fx$ and $Fy \prec_2 Gx$ for all $y \in Gx$.*

Now we give some examples.

Example 4.4. *([3]) Let $X = [1, \infty)$ and $\leq$ be usual order on X. Consider two mappings $F, G : X \to 2^X$ defined by $Fx = [1, x^2]$ and $Gx = [1, 2x]$ for all $x \in X$. Then the pair of mappings F and G are weakly increasing with respect to $\prec_2$ but not $\prec_1$. Indeed, since*

$$Gy = [1, 2y] \prec_2 [1, x^2] = Fx \text{ for all } y \in Fx$$

and

$$Fy = [1, y^2] \prec_2 [1, 2x] = Gx \text{ for all } y \in Gx$$

so F and G are weakly increasing with respect to $\prec_2$ but $F2 = [1, 4] \not\prec_1 [1, 2] = G1$ for $1 \in F2$, so F and G are not weakly increasing with respect to $\prec_1$.

Example 4.5. *([3]) Let $X = [1, \infty)$ and $\leq$ be usual order on X. Consider two mappings $F, G : X \to 2^X$ defined by $Fx = [0, 1]$ and $Gx = [x, 1]$ for all $x \in X$. Then the pair of mappings F and G are weakly increasing with respect to $\prec_1$ but not $\prec_2$. Indeed, since*

$$Fx = [0, 1] \prec_1 [y, 1] = Gy \text{ for all } y \in Fx$$

and

$$Gx = [x, 1] \prec_1 [0, 1] = Fy \text{ for all } y \in Gx$$

so F and G are weakly increasing with respect to $\prec_1$ but $G1 = 1 \not\prec_2 0, 1 = F1$ for $1 \in F1$, so F and G are not weakly increasing with respect to $\prec_2$.

Theorem 4.6. *Let $(X, \preceq, d)$ be a partially ordered complete GMS. Suppose that $T, S : X \to 2^X$ are set-valued mappings and satisfies the following conditions:*
(i) $f(H(Ax, By)) \leq \psi(f(M_I(Ax, By)))$ for all $A, B = T$ or S;
(ii) T and S be a weakly increasing pair on X w.r.t $\prec_1$;
(iii) there exists $x_0 \in X$ such that $\{x_0\} \prec_1 Tx_0$ and $\{x_0\} \prec_1 STx_0$;
(iv) X has the property $\alpha-$regular generalized metric space
Then T, S have common fixed point $x^* \in X$. Further, for each $x_0 \in X$, the iterated sequence $\{x_n\}$ with $x_{2n+1} \in Tx_{2n}$ and $x_{2n+2} \in Sx_{2n+1}$ converges to the common fixed point of T, S.

Proof . Define the sequence x_n in X by $x_{2n+1} \in Tx_{2n}$ and $x_{2n+2} \in Sx_{2n+1}$ for all $n \in \mathbb{N}_0$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}_0$, then $x^* = x_n$ is a common fixed point for T, S. Using that the pair of set-valued mappings T and S is weakly increasing and by define $\alpha : X \times X \to [0, +\infty)$

$$\alpha(x,y) = \begin{cases} 1 & ifx \preceq y \\ 0 & ifx \succ y. \end{cases}$$

It can be easily shown that the sequence x_n is nondecreasing w.r.t, $\preceq$ i.e; and

$$\alpha_*(\{x_0\}, Tx_0) \geq 1 \Rightarrow \exists x_1 \in Tx_0, \text{ such that } \alpha(x_0, x_1) \geq 1 \Rightarrow x_0 \preceq x_1.$$

Now since T and S are weakly increasing with respect to $\prec_1$, we have $x_1 \in Tx_0 \prec_1 Sx_1$. Thus there exist some $x_2 \in Sx_1$ such that $x_1 \preceq x_2$. Again since T and S are weakly increasing with respect to $\prec_1$, we have $x_2 \in Sx_1 \prec_1 Tx_2$. Thus there exist some $x_3 \in Tx_2$ such that $x_2 \preceq x_3$. Continue this process, we will get a nondecreasing sequence $\{x_n\}_{n=1}^{\infty}$ which satisfies $x_{2n+1} \in Tx_{2n}$ and $x_{2n+2} \in Sx_{2n=1}$, $n = 0, 1, 2, 3, \cdots$ We get

$$x_0 \preceq x_1 \preceq x_2 \preceq \cdots \preceq x_{2n} \preceq x_{2n+1} \preceq x_{2n+2} \preceq \cdots$$

In particular x_n, x_{n+k} are comparable for all $k \in \mathbb{N}$. $\alpha(x_n, x_{n+k}) \geq 1$ for all $n \in \mathbb{N}_0$ and by equation (2.1) and (2.2) we have $\lim_{n\to\infty} d(x_n, x_{n+k}) = 0$. Following the proof of Theorem (3.2), we know that $\{x_n\}$ is a Cauchy sequence in the partially ordered complete GMS $(X, \preceq, d)$. There exists $x^* \in X$ such that $\lim_{n\to+\infty} d(x_n, x^*) = 0$. and condition (iv), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{2n_k+1}, x^*) \geq \alpha_*(Tx_{2n_k}, Sx^*) \geq 1$ for all k. Thus,

$$\begin{aligned} f(D(x^*, Sx^*)) &\leq f(d(x^*, x_{2n_k+1})) + f(D(x_{2n_k+1}, Sx^*)) \\ &\leq f(d(x^*, x_{2n_k+1})) + \alpha_*(Tx_{2n_k}, Sx^*) f(H(Tx_{2n_k}, Sx^*)) \\ &\leq f(d(x^*, x_{2n_k+1})) + \psi(f(\max\{d(x_{2n_k}, x^*), D(x_{2n_k}, Tx_{2n_k}), \\ &\quad D(x^*, Sx^*), \tfrac{D(x_{2n_k}, Tx_{2n_k}) D(x^*, Sx^*)}{1+d(x_{2n_k}, x^*)}, \tfrac{D(x_{2n_k}, Tx_{2n_k}) D(x^*, Sx^*)}{1+H(Tx_{2n_k}, Sx^*)}\})) \\ &\leq f(d(x^*, x_{2n_k+1})) + \psi(f(\max\{d(x_{2n_k}, x^*), d(x_{2n_k}, x_{2n_k+1}, \\ &\quad D(x^*, Sx^*), \tfrac{d(x_{n_{2k}}, x_{2n_k+1}) D(x^*, Sx^*)}{1+d(x_{n_{2k}}, x^*)}, \tfrac{d(x_{2n_k}, x_{2n_k}) d(x^*, Sx^*)}{1+D(x_{2n_k+1}, Sx^*)}\})) \\ &< f(D(x^*, Sx^*)) \end{aligned} \tag{4.1}$$

for all k. Hence, $D(x^*, Sx^*) = 0$ and so $x^* \in Sx^*$. □

Theorem 4.7. *Let $(X, \preceq, d)$ be a partially ordered complete GMS. Suppose that $T, S : X \to 2^X$ are set-valued mappings and satisfies the following conditions:*
(i) $f(H(Ax, By)) \leq \psi(f(M_I(Ax, By)))$ for all $A, B = T$ or S;
$(ii)F$ and G be a weakly increasing pair on X w.r.t $\prec_2$;
(iii) there exists $x_0 \in X$ such that $Tx_0 \prec_2 \{x_0\}$ and $STx_0 \prec_2 \{x_0\}$;
(iv)if X has the property $\alpha-$regular generalized metric space
Then T, S have common fixed point $x^* \in X$. Further, for each $x_0 \in X$, the iterated sequence $\{x_n\}$ with $x_{2n+1} \in Tx_{2n}$ and $x_{2n+2} \in Sx_{2n+1}$ converges to the common fixed point of T, S.

Proof . Define the sequence x_n in X by $x_{2n+1} \in Tx_{2n}$ and $x_{2n+2} \in Sx_{2n+1}$ for all $n \in N_0$. If $x_n = x_{n+1}$ for some $n \in N_0$, then $x^* = x_n$ is a common fixed point for T, S. Using that the pair of multi-valued mappings T and S is weakly increasing and by define

$$\alpha(x,y) = \begin{cases} 1 & ifx \preceq y \\ 0 & ifx \succ y \end{cases}$$

It can be easily shown that the sequence x_n is non-increasing w.r.t, $\preceq$ i.e; and

$$\alpha_*(Tx_0, \{x_0\}) \geq 1 \Rightarrow \exists x_1 \in Tx_0, \text{ such that } \alpha(x_1, x_0) \geq 1 \Rightarrow x_1 \preceq x_0;$$

Now since T and S are weakly increasing with respect to $\prec_2$, we have $Sx_1 \prec_2 Tx_0$. Thus there exist some $x_2 \in Sx_1$ such that $x_2 \preceq x_1$. Again since T and S are weakly increasing with respect to $\prec_2$, we have $Tx_2 \preceq_2 Sx_1$. Thus there exist some $x_3 \in Tx_2$ such that $x_3 \preceq x_2$. Continue this process, we will get a non-increasing sequence $\{x_n\}_{n=1}^{\infty}$ which satisfies $x_{2n+1} \in Tx_{2n}$ and $x_{2n+2} \in Sx_{2n+1}$, $n = 0, 1, 2, 3, \cdots$ We get

$$x_0 \succeq x_1 \succeq x_2 \succeq \cdots \succeq x_{2n} \succeq x_{2n+1} \succeq x_{2n+2} \succeq \cdots$$

In particular x_{n+k}, x_n are comparable for all $k \in \mathbb{N}$, $\alpha(x_{n+k}, x_n) \geq 1$ for all $n \in \mathbb{N}_0$ and by equation (2.1) and (2.2) we have $\lim_{n\to\infty} d(x_{n+k}, x_n) = 0$. Following the proof of Theorem (3.2), we know that $\{x_n\}$ is a Cauchy sequence in the partially ordered complete (GMS) $(X, \preceq, d)$. There exists $x^* \in X$ such that $\lim_{n\to+\infty} d(x_n, x^*) = 0$. In the case, suppose that, for example, T is a order closed multi-valued mappings then we have that $\lim_{n\to+\infty} d(Tx_n, Tx^*) = 0$, which (taking n even) implies that $x^* \in Tx^*$. The proof is similar when S is a order closed multi-valued mappings. Then x^* is a common fixed point of T, S and condition (iv), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x^*, \{x_{2n_k+1}\}) \geq \alpha_*(Sx^*, Tx_{2n_k}) \geq 1$ for all k. Thus,

$$\begin{aligned} f(D(Sx^*, x^*)) &\leq f(D(Sx^*, x_{2n_k+1})) + f(d(x_{2n_k+1}, x^*)) \\ &\leq \alpha_*(Sx^*, Tx_{2n_k}) f(H(Sx^*, Tx_{2n_k})) + f(d(x_{2n_k+1}, x^*)) \\ &\leq \psi(f(\max\{d(x^*, x_{2n_k}), D(x^*, Sx^*), D(x_{2n_k}, Tx_{2n_k}), \\ &\tfrac{D(x^*,Sx^*)D(x_{2n_k},Tx_{2n_k})}{1+d(x^*,x_{2n_k})}, \tfrac{D(x^*,Sx^*)D(x_{2n_k},Tx_{2n_k})}{1+H(Sx^*,Tx_{2n_k})}\})) + f(d(x_{2n_k+1}, x^*)) \\ &\tfrac{D(x^*,Sx^*)d(x_{2n_k},x_{2n_k+1})}{1+d(x^*,x_{2n_k})}, \tfrac{D(x^*,Sx^*)d(x_{2n_k},x_{2n_k+1})}{1+D(Sx^*,x_{2n_k+1})}\})) + f(d(x_{2n_k+1}, x^*)) \\ &< f(D(Sx^*, x^*)) \end{aligned} \tag{4.2}$$

for all k. Hence, $D(Sx^*, x^*) = 0$ and so $x^* \in Sx^*$. □

Corollary 4.8. *Let $(X, \preceq, d)$ be a partially ordered complete (GMS). Suppose that $T : X \to 2^X$ is set-valued mapping and satisfies the following conditions:*
(i) $f(H(Tx, Ty)) \leq \psi(f(M_I(Tx, Ty)))$;
(ii) T and i_x be a weakly increasing pair on X w.r.t $\prec_1$;
(iii) there exists $x_0 \in X$ such that $\{x_0\} \prec_1 Tx_0$ and $\{x_0\} \prec_1 T^2x_0$;
(iv)X has the property $\alpha-$regular generalized metric space
Then T has fixed point $x^* \in X$. Further, for each $x_0 \in X$, the iterated sequence $\{x_n\}$ with $x_{n+1} \in Tx_n$ converges to the fixed point of T.

Corollary 4.9. *Let $(X, \preceq, d)$ be a partially ordered complete (GMS). Suppose that $T : X \to 2^X$ is set-valued mappings and satisfies the following conditions:*
(i) $f(H(Tx, TBy)) \leq \psi(f(M_I(TAx, TBy)))$;
(ii)T and i_x be a weakly increasing pair on X w.r.t $\prec_2$;
(iii) there exists $x_0 \in X$ such that $Tx_0 \prec_2 \{x_0\}$ and $STx_0 \prec_2 \{x_0\}$;
(iv)if X has the property $\alpha-$regular generalized metric space
Then T has fixed point $x^* \in X$. Further, for each $x_0 \in X$, the iterated sequence $\{x_n\}$ with $x_{n+1} \in Tx_n$ converges to fixed point of T.

5. Coupled fixed point theorem

Recall that a function $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be super-additive if $\eta(s) + \eta(t) \leq \eta(s+t)$ for all $s, t \in \mathbb{R}_+$.
It is well-known that every nondecreasing, convex function $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ with $\eta(0) = 0$ is super-additive; cf.Theorem in [4].

Definition 5.1. *[21] Let $F : X \times X \to X$ be a mapping, where (X, d) is a metric space. We say that $(x, y) \in X \times X$ is a coupled fixed point of F if*

$$x = F(x, y),\ y = F(y, x).$$

Note that if (x, y) is a coupled fixed point of F then (y, x) are coupled fixed points of F too. Our results are based on the following simple lemma.

Lemma 5.2. *[18] Let $F : X \times X \to X$ be a given mapping. Define the mapping $T_F : X \times X \to X \times X$ by $T_F(x, y) = (F(x, y), F(y, x))$ for all $(x, y) \in X \times X$. Then, (x, y) is a coupled fixed point of F if and only if (x, y) is a fixed point of T_F.*

Theorem 5.3. *Let (X, d) be a complete metric space and $F : X \times X \to X$ be a given mapping. Assume there are exist nondecreasing functions $\psi_i : [0, +\infty) \to [0, +\infty)$, $i = 1, 2$, such that $\psi = \psi_1 + \psi_2$ is convex, $\psi(0) = 0$, $\lim_{n\to+\infty} \psi^n(t) = 0$ for all $t > 0$, a function $\alpha : X^2 \times X^2 \to [0, +\infty)$ and satisfies the following conditions:*
(i) for all (x, y), $(u, v) \in X \times X$,

$$\alpha((x, y), (u, v))d(F(x, y), F(u, v)) \leq \psi_1(d(x, u)) + \psi_2(d(y, v));$$

(ii) if for all (x, y), $(u, v) \in X \times X$,

$$\alpha((x, y), (u, v)) \geq 1 \Rightarrow \alpha(T_F(x, y), T_F(u, v)) \geq 1;$$

(iii) there exists $(x_0, y_0) \in X \times X$ such that

$$\alpha((x_0, y_0), T_F(x_0, y_0)) \geq 1 \quad \text{and} \quad \alpha(T_F(y_0, x_0), (y_0, x_0) \geq 1; \quad \text{or}$$

$(iii)^*$ there exists $(x_0, y_0) \in X \times X$ such that

$$\alpha(T_F(x_0, y_0), (x_0, y_0)) \geq 1 \quad \text{and} \quad \alpha((y_0, x_0), T_F(y_0, x_0)) \geq 1;$$

(iv) if $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\alpha(x_n, x_{n+1}) \geq 1$, $\alpha(y_n, y_{n+1}) \geq 1$, for all n, $x_n \to x \in X$, $y_n \to y \in X$ as $n \to \infty$, then there are exist subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $\{y_{n_k}\}$ of $\{y_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ and $\alpha(y_{n_k}, y) \geq 1$ for all k; or
$(iv)^*$ if $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\alpha(x_{n+1}, x_n) \geq 1$, $\alpha(y_{n+1}, y_n) \geq 1$, for all n, $x_n \to x \in X$, $y_n \to y \in X$ as $n \to \infty$, then there are exist subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $\{y_{n_k}\}$ of $\{y_n\}$ such that $\alpha(x, x_{n_k}) \geq 1$ and $\alpha(y, y_{n_k}) \geq 1$ for all k.
Then, F has a coupled fixed point, that is, there exists $(x^*, y^*) \in X \times X$ such that $x^* = F(x^*, y^*)$ and $y^* = F(y^*, x^*)$.

Proof . The idea consists in transporting the problem to the complete metric space (Y, δ), where $Y = X \times X$ and $\delta((x, y),(u, v)) = d(x, u) + d(y, v)$, for all (x, y), $(u, v) \in X \times X$. From condition (i), we have

$$\alpha((x, y),(u, v))d(F(x, y), F(u, v)) \leq \psi_1(d(x, u)) + \psi_2(d(y, v)) \tag{5.1}$$

and

$$\alpha((v, u),(y, x))d(F(v, u), F(y, x)) \leq \psi_1(d(v, y)) + \psi_2(d(u, x)) \tag{5.2}$$

for all $x, y, u, v \in X$. Adding (5.1) to (5.2), we get (note that ψ is super-additive)

$$\begin{aligned} \beta(\xi, \eta)\delta(T_F\xi, T_F\eta) &\leq \psi_1(d(\xi_1, \eta_1)) + \psi_2(d(\xi_2, \eta_2)) + \psi_1(d(\eta_2, \xi_2)) + \psi_2(d(\eta_1, \xi_1)) \\ &\leq \psi_1(d(\xi_1, \eta_1) + d(\eta_2, \xi_2)) + \psi_2(d(\xi_2, \eta_2) + d(\eta_1, \xi_1)) \\ &= \psi(d(\xi_1, \eta_1) + d(\eta_2, \xi_2)) \\ &= \psi(\delta(\xi, \eta)) \end{aligned} \tag{5.3}$$

for all $\xi = (\xi_1, \xi_2)$, $\eta = (\eta_1, \eta_2) \in Y$, where $\beta : Y \times Y \to [0, +\infty)$ is the function defined by

$$\beta((\xi_1, \xi_2),(\eta_1, \eta_2)) \quad = \min\{\alpha((\xi_1, \xi_2),(\eta_1, \eta_2)), \alpha((\eta_2, \eta_1),(\xi_2, \xi_1))\} \tag{5.4}$$

and $T_F : Y \to Y$ is given by lemma (5.2). Let $\{(x_n, y_n)\}$ be a sequence in $Y = X \times X$ such that

$$\beta((x_n, y_n),(x_{n+1}, y_{n+1})) \geq 1 \text{ and } (x_n, y_n) \to (x, y)$$

as $n \to +\infty$. Using the condition (iv), we obtain easily there exists a subsequence $\{(x_{n_k}, y_{n_k})\}$ of $\{(x_n, y_n)\}$ such that $\beta((x_{n_k}, y_{n_k}),(x, y)) \geq 1$ for all k. Then all the hypotheses of Theorem (3.2) are satisfied. We deduce the existence of a fixed point of T_F that gives us from Lemma (5.2) the existence of a coupled fixed point of F. $\square$

6. Application

In this section, an existence result for a fractional integral equation

$$y(t) = \frac{f(t, x(t), y(t))}{\Gamma(\alpha)} \int_0^t \frac{h'(s)g(s, x(s), y(s))}{(h(t) - h(s))^{1-\alpha}} ds, \quad t \in [0, T], \tag{6.1}$$

where $T > 0$, $\alpha \in (0, 1)$ and $h : [0, T] \to \mathbb{R}$. We suppose that the following conditions are satisfied.

(i) The function $f : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous.

(ii)There exists an upper semi-continuous function $\psi_i : [0, +\infty) \to [0, +\infty)$, $i = 1, 2$, are nondecreasing functions such that $\psi = \psi_1 + \psi_2$ is convex, $\psi(0) = 0$, and $\lim_{n\to\infty} \psi^n(t) = 0$ for each $t > 0$,

$$|f(t, x(t), y(t)) - f(t, u(t), v(t))| \leq \psi_1(x - u) + \psi_2(y - v), \tag{6.2}$$

for all $(t, x(t), y(t))$ and $(t, u(t), v(t)) \in [0, T] \times \mathbb{R} \times \mathbb{R}$.

(iii)The function $h : [0, T] \to \mathbb{R}$ is C^1 and nondecreasing.

(iv)The function $g : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a nondecreasing function $\omega : [0, \infty) \to [0, \infty)$ such that

$$|g(t, x(t), y(t))| \leq \omega(|(x(t), y(t))|) \quad (t, x(t), y(t)) \in [0, T] \times \mathbb{R} \times \mathbb{R}.$$

(v)There exists $r_0 > 0$ such that

$$(\psi(r_0) + F_0)\omega(r_0)(g(T) - g(0)))^\alpha \leq r_0\Gamma(\alpha + 1) \text{ and } \frac{\omega(r_0)}{\Gamma(\alpha + 1)} \times (g(T) - g(0))^\alpha \leq 1 \tag{6.3}$$

where $F_0 = \max\{|f(t, 0, 0)| : t \in [0, T]\}$.

Theorem 6.1. *Consider fractional integral equation (6.1) with $g \in C([0,T] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ is C^1 and nondecreasing in the third variables. Suppose that for $x \geq u$ and $y \geq v$, we have*

$$0 \leq g(t,x,y) - g(t,u,v) \leq \frac{\Gamma(\alpha+1)}{F_0(h(t)-h(s))^\alpha}(\psi_1(x-u)+\psi_2(y-v)), \tag{6.4}$$

Then the fractional integral equation (6.1) with the assumptions $(i-v)$ has at least one solution $y^* \in C([0,T], \mathbb{R})$.

Proof . Let $X = C([0,T], \mathbb{R})$ is partially ordered if we define the following order relation in X:

$$x, y \in X, \quad x \leq y \Leftrightarrow x(t) \leq y(t), \text{ for all } t \in [0,T].$$

It is well-known that (X,d) is a complete metric space with the metric

$$d(x,y) = \sup_{t\in[0,T]} |x(t)-y(t)|, \quad x,y \in C([0,T],\mathbb{R}).$$

Suppose $\{x_n\}$ is a nondecreasing sequence in X that converges to $x \in X$. Then for every $t \in [0,T]$, the sequence of the real numbers

$$x_1(t) \leq x_2(t) \leq \cdots \leq x_n(t) \leq \cdots,$$

converges to $x(t)$. Therefore, for all $t \in I$ and $n \in \mathbb{N}$, we have $x_n(t) \leq x(t)$. Hence $x_n \leq x$, for all $n \in \mathbb{N}$. Also, $X \times X$ is a partially ordered set if we define the following order relation in $X \times X$:

$$(x,y) \leq_r (u,v) \Leftrightarrow x(t) \leq u(t) \text{ and } y(t) \leq v(t), \text{ for all } t \in [0,T],$$

for all (x,y), $\quad (u,v) \in X \times X$. For any x, $y \in X$, $\max\{x(t),u(t)\}$ for all $t \in [0,T]$ is in X and is the upper bound of x, u. Therefore, for every (x,y) and $(u,v) \in X \times X$ $\max\{x(t),u(t)\}$, $\max\{y(t),v(t)\}$, in $X \times X$ for all $t \in [0,T]$ is comparable to (x,y) and (u,v).
Define $F : X \times X \to X$ by

$$F(x,y)(t) = \frac{f(t,x(t),y(t))}{\Gamma(\alpha)} \int_0^t \frac{h'(s)g(s,x(s),y(s))}{(h(t)-h(s))^{1-\alpha}}ds, \text{ for all } t \in [0,T].$$

Since f is nondecreasing in the second and third of its variables then F is nondecreasing in each of its variables.
Now, for $x \geq u$, $y \geq v$, that is, $x(t) \geq u(t)$, $y(t) \geq v(t)$ for all $t \in [0,T]$. we have

$$\begin{aligned}
&d(F(x,y),F(u,v)) = \sup\nolimits_{t\in[0,T]} |F(x,y)(t) - F(u,v)(t)| \\
&= \sup\nolimits_{t\in[0,T]}\{\tfrac{f(t,x(t),y(t))}{\Gamma(\alpha)} \int_0^t \tfrac{h'(s)g(s,x(s),y(s))}{(h(t)-h(s))^{1-\alpha}}ds - \tfrac{f(t,u(t),v(t))}{\Gamma(\alpha)} \int_0^t \tfrac{h'(s)g(s,u(s),v(s))}{(h(t)-h(s))^{1-\alpha}}ds\} \\
&\leq \sup\nolimits_{t\in[0,T]}\{\tfrac{F_1}{\Gamma(\alpha)} \int_0^t \tfrac{h'(s)}{(h(t)-h(s))^{1-\alpha}}(g(s,x(s),y(s)) - g(s,u(s),v(s))ds\} \\
&\leq \sup\nolimits_{t\in[0,T]}\{\tfrac{F_0}{\Gamma(\alpha)} \times \tfrac{\Gamma(\alpha+1)}{F_0(h(t)-h(s))^\alpha}(\psi_1(x-u)+\psi_2(y-v)) \int_0^t \tfrac{h'(s)}{(h(t)-h(s))^{1-\alpha}}ds\} \\
&\leq \sup\nolimits_{t\in[0,T]}\{\tfrac{F_0}{\Gamma(\alpha)} \times \tfrac{\Gamma(\alpha+1)}{F_1(h(t)-h(s))^\alpha}(\psi_1(x-u)+\psi_2(y-v))\tfrac{(h(t)-h(0))^\alpha}{\alpha}\} \\
&\leq \sup\nolimits_{t\in[0,T]}\{\tfrac{F_0}{\Gamma(\alpha)} \times \tfrac{\Gamma(\alpha+1)}{F_1(h(t)-h(s))^\alpha} \times \tfrac{(h(t)-h(0))^\alpha}{\alpha}(\psi_1(x-u)+\psi_2(y-v))\} \\
&\leq \psi_1(d(x,u)) + \psi_2(d(y,v)).
\end{aligned} \tag{6.5}$$

Thus F satisfies the condition of Theorem (5.3). Now, let (x^*, y^*) be a coupled lower solution of the fractional integral equation problem (6.1) then we have $x^* \leq F(x^*,y^*)$ and $y^* \leq F(y^*,x^*)$. Then, Theorem (5.3) gives that F has a unique coupled fixed point (x^*,y^*) with $x^* = y^*$. Then $x^*(t)$ is the solution of the integral equation (6.1). □

7. Open problem

Definition 7.1. *[12] Let $X \neq \emptyset$ and $\perp \subseteq X \times X$ be a binary relation. If $\perp$ satisfies the following condition*

$$\exists x_0 \in X; (\forall y \in X, y \perp x_0) \vee (\forall y \in X, x_0 \perp Y),$$

it is called an orthogonal set (shortly O-set). And $(X, \perp)$ is called O-set. And the element x_0 is called an orthogonal element.

Definition 7.2. *[12] Let $(X, \perp)$ be an orthogonal set (O-set). Any two elements $x, y \in X$ are said to be orthogonally relation if $x \perp y$.*

Definition 7.3. *[12] A sequence x_n is called orthogonal sequence (Shortly O-sequence)if*

$$(\forall n \in N; x_n \perp x_{n+1}) \vee (\forall n \in N; x_{n+1} \perp x_n)$$

Let $(X, d, \perp)$ be an orthogonal metric space ($(X, \perp)$ is an O-set and (X, d) is a metric space). Now, we consider following definitions.

Definition 7.4. *[12] The space X is orthogonally complete (briefly O-complete)if every cauchy O-sequence is convergent.*

Definition 7.5. *[12] A map $d : X \times X \to [0, \infty]$ is called a generalized metric on the orthogonal set $(X, \perp)$. If the following condition are satisfied:*
1. *$d(x, y) = d(y, x)$ for any points $x, y \in X$ such that $x \perp y$ and $y \perp x$;*
2. *$d(x, y) = 0 \Leftrightarrow x = y$ for any points $x, y \in X$ such that $x \perp y$ and $y \perp x$;*
3. *$d(x, z) \leq d(x, y) + d(y, z)$ for any points $x, y, z \in X$ such that $x \perp y$, $y \perp z$ and $x \perp z$ considering that if $d(x, y) = \infty$ or $d(y, z) = \infty$ then $d(x, y) + d(y, z) = \infty$.*

In this case the orthogonal set X called generalized orthogonal metric space and is denoted by $(X, d, \perp)$.

Definition 7.6. *[16] Suppose (X, d) is a metric space and R is a relation on X. Then the triple (X, d, R) or in brief X is called R-metric space.*

Open problem (I)- Some fixed point theorems for α_*-ψ-common rational type mappings on generalized orthogonal metric spaces
Open problem (II)- Some fixed point theorems for α_*-ψ-common rational type mappings on generalized R-metric spaces

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