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On certain properties for new subclass of meromorphic starlike functions

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Abstract

In this paper we studying some properties of starlike function of order λ which satisfy in the condition

$$\Re(\frac{zf'(z)}{f(z)} + \alpha\frac{z^2 f''(z)}{f(z)}) < 1 - \lambda + \alpha$$

for all $z \in \mathbb{U} = \{z : |z| < 1\}$, where $f(z) = 1 + \sum_{k=1}^{\infty} a_k z^k$ is analytic in $\mathbb{U}$, $0 \leqslant \alpha < 2$ and $0 \leqslant \lambda < 1$. Our results extend previos results given by Aghalary et al. (2009) and Wang et al.(2014).

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1. Introduction

Let Σ denote the class of functions f of the form

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the punctured open unit disk

$$\mathbb{U}^* = \{z : 0 < |z| < 1\} = \mathbb{U} - \{0\}.$$

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A function $f \in \Sigma$ is said to be in the class $\mathcal{MS}^*(\alpha)$ of meromorphically starlike functions of order α if it satisfies the inequality

$$\Re(\frac{zf'(z)}{f(z)}) < -\alpha \qquad (z \in \mathbb{U}; 0 \leqslant \alpha < 1).$$

Let $\mathcal{P}$ denote the class of functions p given by

$$p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k \qquad (z \in \mathbb{U}) \tag{1.2}$$

which are analytic in $\mathbb{U}$ and satisfy the condition

$$\Re(p(z)) > 0 \qquad (z \in \mathbb{U}).$$

Many authors have studied analytic starlike functions. For some recent investigation, see, for example [1, 2, 8, 12, 14, 15, 18, 19, 20, 23] and the references therein.
Wang et al. [23] introduced a new class of starlike analytic functions on $\mathbb{U}^*$ as follows:

$$\mathcal{H}(\beta,\lambda) = \left\{ f \in \Sigma : \Re\left(\frac{zf'(z)}{f(z)} + \beta\frac{z^2 f''(z)}{f(z)}\right) < \beta\lambda(\lambda + \frac{1}{2}) + \frac{\beta}{2} - \lambda,\ (z \in U^*) \right\},$$

where $\beta \geqslant 0$ and $\frac{1}{2} \leqslant \lambda < 1$.
In [23] Wang et al. had proved that $\mathcal{H}(\beta,\lambda)$ is a subclass of $\mathcal{MS}^*(\lambda)$. Also Wang et al. [22] introduced the following subclass of $\mathcal{H}(\beta,\lambda)$.
Let $\mathcal{H}^+(\beta,\lambda)$ denote the subset of $\mathcal{H}(\beta,\lambda)$ such that all functions $f \in \mathcal{H}(\beta,\lambda)$ having the following form:

$$f(z) = \frac{1}{z} - \sum_{k=1}^{\infty} a_k z^k \qquad (a_k \geqslant 0).$$

The following two lemmas can be derived from ([6], Theorem 1) (see also [7]).

Lemma 1.1. *Let*

$$1 + \beta\lambda(\lambda + \frac{1}{2}) - \lambda - \frac{3}{2}\beta > 0. \tag{1.3}$$

Suppose also that $f \in \Sigma$ is given by (1.1)*. If*

$$\sum_{k=1}^{\infty}[k + \beta k(k-1) + \gamma]|a_k| \leqslant 1 - \gamma - 2\beta.$$

where (and throughout this paper unless otherwise mentioned) the parameter γ is constrained as follows:

$$\gamma = \lambda - \beta\lambda(\lambda + \frac{1}{2}) - \frac{\beta}{2}, \tag{1.4}$$

then $f \in \mathcal{H}(\beta,\lambda)$.

Lemma 1.2. *Let $f \in \Sigma$ be given by* (1.1)*. Suppose also that γ is defined by* (1.4) *and the condition* (1.3) *holds. Then $f \in \mathcal{H}^+(\beta,\lambda)$ if and only if*

$$\sum_{k=1}^{\infty}[k + \beta k(k-1) + \gamma]a_k \leqslant 1 - \gamma - 2\beta.$$

Recently Wang et al. [23]proved some coefficient inequalities, neighborhoods, partial sums and inclusion relationships for two classes $\mathcal{H}(\beta,\lambda)$ and $\mathcal{H}^+(\beta,\lambda)$.

In Section 2, we introduced a new class of analytic starlike function. In Section 3 and Section 4, we prove some coefficient inequalities, neighborhoods and partial sums. Our results extend previous results given by Aghalary et al. [1] as well as by Wang et al. [23].

2. Preliminaries

In this section we introduce the notation $\mathcal{A}$ for the class of all functions f of the form

$$f(z)=1+\sum_{k=1}^{\infty}a_kz^k \tag{2.1}$$

which are analytic in the open unit disk $\mathbb{U}$. Let $0\leqslant\alpha<2$ and $0\leqslant\lambda<1$ and $\Lambda(\alpha,\lambda)$ denotes the class of functions $f\in\mathcal{A}$ and satisfies the condition

$$\Re\left(\frac{zf'(z)}{f(z)}+\alpha\frac{z^2f''(z)}{f(z)}\right)<1-\lambda+\alpha. \tag{2.2}$$

A function $f\in\mathcal{A}$ is said to be in the class $\mathcal{V}(\alpha,\lambda)$ if it satisfies the condition

$$\Re\left(\frac{zf'(z)}{f(z)}+\alpha\frac{z^2f''(z)}{f(z)}\right)<1-\lambda+\alpha(\frac{1}{2}+\lambda^2-\frac{3}{2}\lambda), \tag{2.3}$$

such that $0\leqslant\alpha<2$ and $0\leqslant\lambda<1$. Obviously $\mathcal{V}(\alpha,\lambda)\subseteq\Lambda(\alpha,\lambda)$. Also suppose that $\Lambda^*(\lambda)$ denotes the class of functions $f\in\mathcal{A}$ such that satisfies the following condition

$$\Re\left(\frac{zf'(z)}{f(z)}\right)<1-\lambda. \tag{2.4}$$

Obviously $\Lambda(0,\lambda)=\mathcal{V}(0,\lambda)=\Lambda^*(\lambda)$.

Given two functions $f,g\in\mathcal{A}$ where f is given by (2.1) and g is given by

$$g(z)=1+\sum_{k=1}^{\infty}b_kz^k.$$

The Hadamard product (or convolution) $f*g$ is defined by

$$(f*g)(z)=1+\sum_{k=1}^{\infty}a_kb_kz^k:=(g*f)(z).$$

At first we prove the following lemma.

Lemma 2.1. *Let $0\leqslant\alpha<2$, $0\leqslant\lambda<1$ and $f\in\mathcal{A}$. Then $f\in\mathcal{V}(\alpha,\lambda)$ if and only if $\frac{1}{z}f\in\mathcal{H}(\frac{\alpha}{1+2\alpha},\lambda)$.*

Proof . Let $f \in \mathcal{A}$. Then $\frac{1}{z}f \in \mathcal{H}(\frac{\alpha}{1+2\alpha}, \lambda)$ if and only if

$$\Re\left(\frac{z(\frac{1}{z}f(z))'}{(\frac{1}{z}f(z))} + \frac{\alpha}{1+2\alpha}\frac{z^2(\frac{1}{z}f(z))''}{(\frac{1}{z}f(z))}\right) < \frac{\alpha}{1+2\alpha}\lambda(\lambda+\frac{1}{2}) + \frac{\alpha}{2(1+2\alpha)} - \lambda.$$

Which is equivalent to

$$\Re\left((1-\frac{2\alpha}{1+2\alpha})\frac{zf'(z)}{f(z)} + \frac{\alpha}{1+2\alpha}\frac{z^2f''(z)}{f(z)}\right) < \frac{\alpha}{1+2\alpha}\lambda(\lambda+\frac{1}{2}) + \frac{\alpha}{2(1+2\alpha)} - \lambda + 1 - \frac{2\alpha}{1+2\alpha}.$$

Hence $\frac{1}{z}f \in \mathcal{H}(\frac{\alpha}{1+2\alpha}, \lambda)$ if and only if

$$\Re\left(\frac{zf'(z)}{f(z)} + \alpha\frac{z^2f''(z)}{f(z)}\right) < 1 - \lambda + \alpha + \alpha(\frac{1}{2} + \lambda^2 - \frac{3}{2}\lambda).$$

This ineqvality is equivalent to $f \in \mathcal{V}(\alpha, \lambda)$, and completes the proof. □

Remark 2.2. *If $0 \leqslant \lambda < 1$ and $0 \leqslant \beta < \frac{2}{5}$ and $\alpha = \frac{\beta}{1-2\beta}$ then by similar method in proof of Lemma 2.1 we can prove that $h \in \mathcal{H}(\beta, \lambda)$ if and only if $zh \in \mathcal{V}(\frac{\beta}{1-2\beta}, \lambda)$.*

In order to prove our main results, we need the following useful lemma.

Lemma 2.3. *(See[11]) If the function $p \in \mathcal{P}$ is given by (1.2), then*

$$|p_k| \leqslant 2 \qquad (k \in \mathbb{N}).$$

A function $f \in \Lambda(\alpha, \lambda)$ of the from

$$f(z) = 1 - \sum_{k=1}^{\infty} a_k z^k \ (a_k \geqslant 0) \tag{2.5}$$

is said to be in the class $\Lambda^+(\alpha, \lambda)$.

3. Main Results

We start this section by the following lemmas.

Lemma 3.1. *Let $0 \leqslant \alpha < 2$, $0 \leqslant \lambda < 1$ and $f \in \mathcal{A}$ is given by (2.1). If*

$$\sum_{k=1}^{\infty}[\alpha k^2 - \alpha k + k - \gamma]|a_k| < \gamma \qquad (\gamma = 1 - \lambda + \alpha) \tag{3.1}$$

then $f \in \Lambda(\alpha, \lambda)$.

Proof . For $z = re^{i\theta}, 0 \leqslant r < 1$ and $0 \leqslant \theta < 2\pi$, from (3.1), we get

$$\Re\left(\frac{\sum_{k=1}^{\infty}[\alpha k^2 - \alpha k + k]a_k z^k}{1 + \sum_{k=1}^{\infty} a_k z^k}\right) \leqslant \frac{\sum_{k=1}^{\infty}[\alpha k^2 - \alpha k + k]|a_k| r^k}{1 + \sum_{k=1}^{\infty} |a_k| r^k} = \frac{\sum_{k=1}^{\infty}[\alpha k^2 - \alpha k + k]|a_k|}{1 + \sum_{k=1}^{\infty} |a_k|} < \gamma, \qquad (r \to 1).$$

The above inequalities show that $f \in \Lambda(\alpha, \lambda)$. □

Lemma 3.2. *Let* $0 \leqslant \alpha < 2$ *and* $0 \leqslant \lambda < 1$ *and* $f \in \mathcal{A}$ *is given by* (2.5). *Then*

$$\sum_{k=1}^{\infty}[\alpha k^2 - \alpha k + k - \gamma]a_k < \gamma \qquad \gamma = (1 - \lambda + \alpha)$$

if and only if $f \in \Lambda^+(\alpha, \lambda)$.

Proof . In view of Lemma 3.1, we need only show that $f \in \Lambda^+(\alpha, \lambda)$ satisfies the coefficient condition. We give $f \in \Lambda^+(\alpha, \lambda)$, so

$$\Re\left(\frac{-\sum_{k=1}^{\infty}[\alpha k^2 - \alpha k + k]a_k z^k}{1 - \sum_{k=1}^{\infty} a_k z^k}\right) < \gamma,$$

for $z = re^{i\theta}, 0 \leqslant r < 1$ and $0 \leqslant \theta < 2\pi$, we have

$$\frac{-\sum_{k=1}^{\infty}[\alpha k^2 - \alpha k + k]a_k r^k}{1 - \sum_{k=1}^{\infty} a_k r^k} < \gamma.$$

The result follows upon letting $r \to 1$.

$$\frac{\sum_{k=1}^{\infty}[\alpha k^2 - \alpha k + k]a_k}{1 + \sum_{k=1}^{\infty} a_k} \leqslant \frac{\sum_{k=1}^{\infty}[\alpha k^2 - \alpha k + k]a_k}{-1 + \sum_{k=1}^{\infty} a_k} < \gamma.$$

□

Lemma 3.3. *Let* $0 \leqslant \alpha < 2$ *and* $0 \leqslant \lambda < 1$. *Suppose also that the sequence* $\{B_k\}_{k=1}^{\infty}$ *is defined by*

$$B_1 = 2(1 - \lambda + \alpha) \quad \textit{and} \quad B_{k+1} = \frac{2(1 - \lambda + \alpha)}{k + 1 + \alpha k(k+1)}\left(1 + \sum_{i=1}^{k} B_i\right) \quad (k \in \mathbb{N}). \tag{3.2}$$

Then

$$B_k = 2(1 - \lambda + \alpha)\prod_{j=1}^{k-1}\frac{j + \alpha j(j-1) + 2(1 - \lambda + \alpha)}{j + 1 + \alpha j(j+1)} \quad (k \in \mathbb{N}). \tag{3.3}$$

Proof . By virtue of (3.2), we easily get

$$[k+1+\alpha k(k+1)]B_{k+1} = 2(1-\lambda+\alpha)\left(1+\sum_{i=1}^{k} B_i\right)$$

and

$$[k+\alpha k(k-1)]B_k = 2(1-\lambda+\alpha)\left(1+\sum_{i=1}^{k-1} B_i\right).$$

We obtain that

$$\frac{B_{k+1}}{B_k} = \frac{k+\alpha k(k-1)+2(1-\lambda+\alpha)}{k+1+\alpha k(k+1)}$$

Thus, for $k \geqslant 2$, so we give

$$B_k = \frac{B_k}{B_{k-1}}\cdot\frac{B_{k-1}}{B_{k-2}}\cdot\ldots\cdot\frac{B_2}{B_1}\cdot B_1 = 2(1-\lambda+\alpha)\prod_{j=1}^{k-1}\frac{j+\alpha j(j-1)+2(1-\lambda+\alpha)}{j+1+\alpha j(j+1)}$$

and this evidently completes the proof. □

By using induction and (3.2) we conclude the following proposition.

Proposition 3.4. *Let $0 \leqslant \alpha < 2$, $0 \leqslant \lambda < 1$ and the sequence $\{B_k\}$ is given by (3.2). Then*

$$B_k \leqslant \frac{k+1}{3}(1+B_1) \quad (k \geqslant 2). \tag{3.4}$$

Theorem 3.5. *Let $0 \leqslant \alpha < 2$ and $0 \leqslant \lambda < 1$. If $f \in \Lambda(\alpha,\lambda)$, then*

$$|a_1| \leqslant 2(1-\lambda+\alpha) \tag{3.5}$$

and

$$|a_k| \leqslant 2(1-\lambda+\alpha)\prod_{j=1}^{k-1}\frac{j+\alpha j(j-1)+2(1-\lambda+\alpha)}{j+1+j(j+1)} \quad (k \geqslant 2). \tag{3.6}$$

Proof . Suppose that

$$q(z) = -\frac{zf'(z)}{f(z)} - \alpha\frac{z^2 f''(z)}{f(z)} + 1 - \lambda + \alpha$$

Then, by $f \in \Lambda(\alpha,\lambda)$, we know that q is analytic in $\mathbb{U}$, $q(0) = 1-\lambda+\alpha > 0$ and $\Re[q(z)] > 0$. Hence

$$h(z) = \frac{q(z)}{q(0)} = \frac{q(z)}{1-\lambda+\alpha} \in \mathcal{P}.$$

If we put

$$q(z) = c_0 + \sum_{k=1}^{\infty} c_k z^k \quad (c_0 = 1-\lambda+\alpha),$$

then by Lemma 2.3,

$$|c_k| \leqslant 2(1-\lambda+\alpha) \qquad (k \in \mathbb{N}).$$

Also

$$q(z)f(z) = -zf'(z) - \alpha z^2 f''(z) + (1-\lambda+\alpha)f(z)$$

and so

$$\left(c_0 + \sum_{k=1}^{\infty} c_k z^k\right)\left(1 + \sum_{k=1}^{\infty} a_k z^k\right) = -\sum_{k=1}^{\infty} k a_k z^k - \alpha \sum_{k=1}^{\infty} k(k-1)a_k z^k + (1-\lambda+\alpha)\left(1 + \sum_{k=1}^{\infty} a_k z^k\right).$$

Thus

$$c_0 a_1 + c_1 = -a_1 + (1-\lambda+\alpha)a_1$$

and

$$c_{k+1} + c_0 a_{k+1} + \sum_{i=1}^{k} a_i c_{k+1-i} = -(k+1)a_{k+1} - \alpha k(k+1)a_{k+1} + (1-\lambda+\alpha)a_{k+1} \quad (k \in \mathbb{N}).$$

Therefore

$$|a_1| \leqslant 2(1-\lambda+\alpha)$$

and

$$|a_{k+1}| \leqslant \frac{2(1-\lambda+\alpha)}{k+1+\alpha k(k+1)}\left(1 + \sum_{i=1}^{k} a_i\right) \qquad (k \in \mathbb{N}).$$

Next, we define the sequence $\{B_k\}$ as follows:

$$B_1 = 2(1-\lambda+\alpha) \quad \textit{and} \quad B_{k+1} = \frac{2(1-\lambda+\alpha)}{k+1+\alpha k(k+1)}\left(1 + \sum_{i=1}^{k} B_i\right) \qquad (k \in \mathbb{N}).$$

Hence, by the principle of mathematical induction, we easily have

$$|a_k| \leqslant B_k \, (k \in \mathbb{N}).$$

By using Lemma 3.3, the conditions (3.5) and (3.6) are hold and this completes the proof. □

By using above theorem and Proposition 3.4 we can conclude the following corollaries.

Corollary 3.6. *Let* $0 \leqslant \alpha < 2$ *and* $0 \leqslant \lambda < 1$*. If* $f \in \Lambda(\alpha, \lambda)$*,then*

$$|a_k| \leqslant \frac{k+1}{3}(1+B_1) \qquad (k \geqslant 2)$$

where $B_1 = 2(1-\lambda+\alpha)$*.*

Corollary 3.7. *Let* $0 \leqslant \lambda < 1$ *and* $f \in \Lambda^*(\lambda)$*. Then*

$$|a_1| \leqslant 2(1-\lambda) \quad \textit{and} \quad |a_k| \leqslant \frac{k+1}{3}(3-2\lambda) \qquad (k \geqslant 2).$$

Proof . Let $\alpha = 0$ and apply Corollary 3.6. □

4. Neighborhoods

We can see the earlier works (based upon the familiar concept of neighborhood of analytic functions) by Goodman [10] and Ruscheweyh [19], and (more recently) by Altintaş et al. [3], Cataş [4], Cho et al. [5], Liu and Srivastava [15], Frasin [9], Keerthi et al. [13], Srivastava et al. [21] and Wang et al. [24]. Let $0 \leqslant \alpha < 2$ and $0 \leqslant \lambda < 1$ and $f \in \mathcal{A}$ of the from (2.1). For $\delta > 0$, we denote the $\delta -$ *neighborhood* of f by the notation $\mathcal{N}(f, \delta)$ and the following definition:

$$\mathcal{N}(f,\delta) = \left\{ g \in \mathcal{A} : g(z) = 1 + \sum_{k=1}^{\infty} d_k z^k \; and \; \sum_{k=1}^{\infty} \frac{\alpha k^2 - \alpha k + k - 1 + \lambda - \alpha}{\eta} |d_k - a_k| < \delta \right\} \tag{4.1}$$

where

$$\eta = \begin{cases} 1, & \lambda - \alpha < 0; \\ 1 - \lambda + \alpha, & \lambda - \alpha \geqslant 0. \end{cases} \tag{4.2}$$

By above definition, we now prove the following useful theorem.

Theorem 4.1. *Let $0 \leqslant \alpha < 2$ and $0 \leqslant \lambda < 1$. If $f \in \mathcal{A}$ is given by (2.1) and satisfies the following condition*

$$\frac{f + \epsilon}{1 + \epsilon} \in \Lambda(\alpha, \lambda) \quad (\epsilon \in \mathcal{C} : |\epsilon| < \delta; \delta > 0)$$

then

$$\mathcal{N}(f, \delta) \subset \Lambda(\alpha, \lambda).$$

Proof . Suppose that

$$g(z) = 1 + \sum_{k=1}^{\infty} b_k z^k \in \Lambda(\alpha, \lambda).$$

Hence

$$\Re \left(\frac{zg'(z)}{g(z)} + \alpha \frac{z^2 g''(z)}{g(z)} - 1 + \lambda - \alpha \right) < 0. \tag{4.3}$$

The condition (4.3) can be written as

$$\left| \frac{\frac{zg'(z)}{g(z)} + \alpha \frac{z^2 g''(z)}{g(z)} - 1 + \lambda - \alpha + 1}{\frac{zg'(z)}{g(z)} + \alpha \frac{z^2 g''(z)}{g(z)} - 1 + \lambda - \alpha - 1} \right| < 1 \qquad (z \in \mathbb{U}),$$

which is equivalent to

$$\left| \frac{zg'(z) + \alpha z^2 g''(z) + (\lambda - \alpha) g(z)}{zg'(z) + \alpha z^2 g''(z) + (\lambda - \alpha - 2) g(z)} \right| < 1 \qquad (z \in \mathbb{U}). \tag{4.4}$$

We easily find from (4.4) that $g \in \Lambda(\alpha, \lambda)$ if and only if

$$\frac{zg'(z) + \alpha z^2 g''(z) + (\lambda - \alpha) g(z)}{zg'(z) + \alpha z^2 g''(z) + (\lambda - \alpha - 2) g(z)} \neq \sigma, \quad (z \in \mathbb{U}, \sigma \in \mathcal{C}, |\sigma| = 1)$$

or equivalently

$$1 + \sum_{k=1}^{\infty} \frac{k + \alpha k(k-1) + \lambda - \alpha - \sigma[k + \alpha k(k-1) + \lambda - \alpha - 2]}{\lambda - \alpha - \sigma(\lambda - \alpha - 2)} b_k z^k \neq 0$$

which is equivalent to $(g * h)(z) \neq 0$ where

$$h(z) = 1 + \sum_{k=1}^{\infty} c_k z^k, \quad c_k = \frac{k + \alpha k(k-1) + \lambda - \alpha - \sigma[k + \alpha k(k-1) + \lambda - \alpha - 2]}{\lambda - \alpha - \sigma(\lambda - \alpha - 2)}. \tag{4.5}$$

It follows fram (4.5) that

$$\begin{aligned} |c_k| &= \left| \frac{k + \alpha k(k-1) + \lambda - \alpha - \sigma[k + \alpha k(k-1)] + \lambda - \alpha - 2}{\lambda - \alpha - \sigma(\lambda - \alpha - 2)} \right| \\ &\leqslant \frac{k + \alpha k(k-1) + \lambda - \alpha + |\sigma|[k + \alpha k(k-1) + \lambda - \alpha - 2]}{|\sigma|(\lambda - \alpha - 2) - |\lambda - \alpha|} \\ &= \frac{k + \alpha k(k-1) - 1 + \lambda - \alpha}{\eta} \quad (|\sigma| = 1). \end{aligned}$$

If $f \in \mathcal{A}$ and $g(z) = \dfrac{f + \epsilon}{1 + \epsilon} \in \Lambda(\alpha, \lambda)$, we deduce from $(g * h)(z) \neq 0$ that

$$(f * h)(z) \neq -\epsilon \quad (\epsilon \in \mathcal{C} : |\epsilon| < \delta; \delta > 0)$$

or equivalently,

$$|(f * h)(z)| \geqslant \delta \qquad (z \in \mathbb{U}; \delta > 0). \tag{4.6}$$

Now suppose that

$$q(z) = 1 + \sum_{k=1}^{\infty} d_k z^k \in \mathcal{N}(f, \delta).$$

It follows from (4.1) that

$$|(q - f) * h(z)| = \left| \sum_{k=1}^{\infty} (d_k - a_k) c_k z^k \right| \leqslant |z| \sum_{k=1}^{\infty} \frac{k + \alpha k(k-1) - 1 + \lambda - \alpha}{\eta} |d_k - a_k| < \delta. \tag{4.7}$$

By combining (4.6) and (4.7), we can find that

$$|(q * h)(z)| = |([f + (q - f)] * h)(z)| \geqslant |(f * h)(z)| - |([q - f] * h)(z)| > 0,$$

which implies that

$$(q * h)(z) \neq 0 \qquad (z \in \mathbb{U}).$$

Hence

$$q(z) \in \Lambda(\alpha, \lambda).$$

Therefore $\mathcal{N}(f, \delta) \subset \Lambda(\alpha, \lambda)$ and this completes the proof. □

By taking $\alpha = 0$ we can conclude the following corollary.

Corollary 4.2. *Let* $0 \leqslant \lambda < 1$. *If* $f \in \mathcal{A}$ *is given by* (2.1) *satisfies the following condition*

$$\frac{f + \epsilon}{1 + \epsilon} \in \Lambda^*(\lambda) \quad (\epsilon \in \mathcal{C} : |\epsilon| < \delta; \delta > 0)$$

then

$$\mathcal{N}(f, \delta) \subset \Lambda^*(\lambda).$$

Theorem 4.3. *Let $f \in \mathcal{A}$ be given by (2.1) and define the partial sums $f_n(z)$ of f by*

$$f_n(z) = 1 + \sum_{k=1}^{n} a_k z^k \qquad (n \in \mathbb{N}). \tag{4.8}$$

If

$$\sum_{k=1}^{\infty} \frac{k + \alpha k(k-1) - \gamma}{\eta} |a_k| \leqslant 1, \tag{4.9}$$

where $\gamma = 1 - \lambda + \alpha$ and η is given by (4.2),then

1. $f \in \Lambda(\alpha, \lambda)$;
2. $$\Re\left(\frac{f(z)}{f_n(z)}\right) \leqslant \frac{n+1+\alpha n(n+1)-\gamma-\eta}{n+1+\alpha n(n+1)-\gamma} \tag{4.10}$$

   and

   $$\Re\left(\frac{f_n(z)}{f(z)}\right) \leqslant \frac{n+1+\alpha n(n+1)-\gamma}{n+1+\alpha n(n+1)-\gamma+\eta} \tag{4.11}$$

   Also the bounds in (4.10) and (4.11) are sharp.

Proof . First of all, we suppose that $f_0(z) = 1$. We know that

$$\frac{f_0 + \epsilon}{1 + \epsilon} = 1 \in \Lambda(\alpha, \lambda).$$

From (4.9), we easily find that

$$\sum_{k=1}^{\infty} \frac{k + \alpha k(k-1) - \gamma}{\eta} |a_k - 0| \leqslant 1,$$

which implies that $f \in \mathcal{N}(f_0, 1) \subset \Lambda(\alpha, \lambda)$ (by virtue of Theorem 4.1).
Next, it is easy to see that

$$\frac{n+1+\alpha n(n+1)-\gamma}{\eta} > \frac{n+\alpha n(n-1)-\gamma}{\eta} > 1 \quad (n \in \mathbb{N}).$$

Therefore, we have

$$\sum_{k=1}^{n} |a_k| + \frac{n+1+\alpha n(n+1)-\gamma}{\eta} \sum_{k=n+1}^{\infty} |a_k| \leqslant \sum_{k=1}^{\infty} \frac{k+\alpha k(k-1)-\gamma}{\eta} |a_k| \leqslant 1 \tag{4.12}$$

We now suppose that

$$\begin{aligned} h(z) &= \frac{n+1+\alpha n(n+1)-\gamma}{\eta} \left(\frac{f(z)}{f_n(z)} - \frac{n+1+\alpha n(n+1)-\gamma-\eta}{n+1+\alpha n(n+1)-\gamma} \right) \\ &= 1 + \frac{\frac{n+1+\alpha n(n+1)-\gamma}{\eta} \sum_{k=n+1}^{\infty} a_k z^k}{1 + \sum_{k=1}^{n} a_k z^k} \end{aligned} \tag{4.13}$$

It follows from (4.12) and (4.13) that

$$\left|\frac{h(z)-1}{h(z)+1}\right| \leqslant \frac{\dfrac{n+1+\alpha n(n+1)-\gamma}{\eta}\displaystyle\sum_{k=n+1}^{\infty}|a_k|}{2-2\displaystyle\sum_{k=1}^{n}|a_k|-\dfrac{n+1+\alpha n(n+1)-\gamma}{\eta}\displaystyle\sum_{k=n+1}^{\infty}|a_k|} \leqslant 1,$$

which implies that $\Re(h(z))>0$.
Therefor, we deduce that the assertion (4.10) holds true. Furthermore, if we put

$$f(z)=1-\frac{\eta}{n+1+\alpha n(n+1)-\gamma}z^{n+1}, \tag{4.14}$$

then

$$\frac{f(z)}{f_n(z)}=1-\frac{\eta}{n+1+\alpha n(n+1)-\gamma}z^{n+1}\longrightarrow\frac{n+1+\alpha n(n+1)-\gamma-\eta}{n+1+\alpha n(n+1)-\gamma}\,(|z|\to 1^-),$$

which implies that the bound in (4.10) is the best possible for each $n\in\mathbb{N}$.
Similarly, we suppose that

$$h(z)=\frac{n+1+\alpha n(n+1)-\gamma+\eta}{\eta}\left(\frac{f_n(z)}{f(z)}-\frac{n+1+\alpha n(n+1)-\gamma}{n+1+\alpha n(n+1)-\gamma+\eta}\right),$$

we readily get the assertion (4.10) of Theorem 4.3. The bound in (4.10) is sharp with the extremal function f given by (4.14).We thus complete the proof of Theorem. □

The proof of the following theorem is similar to that of Theorem 4.3, we here choose to omit the analogous details.

Theorem 4.4. *Let $f\in\mathcal{A}$ be given by* (2.1) *and define the partial sums $f_n(z)$ of f by* (4.8).*If the conditions* (4.9) *hold, where $\gamma=1-\lambda+\alpha$ and η is given by* (4.2), *then*

$$\Re\left(\frac{f'(z)}{f_n'(z)}\right)\leqslant\frac{n+1+\alpha n(n+1)-\gamma-(n+1)\eta}{n+1+\alpha n(n+1)-\gamma} \tag{4.15}$$

and

$$\Re\left(\frac{f_n'(z)}{f'(z)}\right)\leqslant\frac{n+1+\alpha n(n+1)-\gamma}{n+1+\alpha n(n+1)-\gamma+(n+1)\eta}. \tag{4.16}$$

The bounds in (4.15) and (4.16) are sharp with the extremal function given by (4.14). Finally, we prove the following inclusion relationship for the function class $\Lambda(\alpha,\lambda)$.

Theorem 4.5. *If $0\leqslant\alpha_2<\alpha_1<2$ and $0\leqslant\lambda_2<\lambda_1<1$,then*

$$\Lambda(\alpha_1,\lambda_1)\subset\Lambda(\alpha_2,\lambda_2).$$

Proof . Let $f\in\Lambda(\alpha_1,\lambda_1)$. Then

$$\Re\left(\frac{zf'(z)}{f(z)}+\alpha_1\frac{z^2f''(z)}{f(z)}\right)<1-\lambda_1+\alpha_1<1-\lambda_2+\alpha_1,$$

which shows that $f \in \Lambda(\alpha_1, \lambda_2)$, and subsequently, we see that $f \in \Lambda^*(\lambda_2)$, that is

$$\Re\left(\frac{zf'(z)}{f(z)}\right) < 1 - \lambda_2.$$

Now, by setting $\mu = \dfrac{\alpha_2}{\alpha_1}$, so that $0 < \mu < 1$. Therefore, we have

$$\Re\left(\frac{zf'(z)}{f(z)} + \alpha_2 \frac{z^2 f''(z)}{f(z)}\right) - 1 + \lambda_2 - \alpha_2 =$$

$$\mu\left[\Re\left(\frac{zf'(z)}{f(z)} + \alpha_1 \frac{z^2 f''(z)}{f(z)}\right) - 1 + \lambda_2 - \alpha_1\right] + (1-\mu)\left[\Re\left(\frac{zf'(z)}{f(z)}\right) + \lambda_2 - 1\right] < 0,$$

that is, $f \in \Lambda(\alpha_2, \lambda_2)$. □

From Theorem 4.5 and the definition of the function class $\Lambda^+(\alpha_1, \lambda_1)$, we easily get the following inclusion relationship.

Corollary 4.6. *If* $0 \leqslant \alpha_2 < \alpha_1 < 2$ *and* $0 \leqslant \lambda_2 < \lambda_1 < 1$*, then*

$$\Lambda^+(\alpha_1, \lambda_1) \subset \Lambda^+(\alpha_2, \lambda_2) \subset \Lambda^*(\lambda_2).$$

By virtue of Lemma 3.2, we obtain the following result.

Corollary 4.7. *If* $f \in \Lambda^+(\alpha, \lambda)$*, then*

$$a_k \leqslant \frac{\gamma}{k + \alpha k(k-1) - \gamma} \qquad (\gamma = 1 - \lambda + \alpha).$$

Each of these inequalities is sharp, with the extremal function given by

$$f_k(z) = 1 + \frac{\gamma}{k + \alpha k(k-1) - \gamma} z^k.$$

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References

[1] R. Aghalary, A. Ebadian and M. Eshaghi Gordji, *Subclasses of meromorphic starlike functions connected to multiplier family*, Austr. J. Basic Appl. Sci. 3(4) (2009) 4416–4421.
[2] O.P. Ahuja, J. M. Jahangiri and H. Silverman, *Subclasses of starlike functions related to a multiplier family*, J. Natural Geom. 15 (1999) 65–72.
[3] O. Altinta, *Neighborhoods of certain p-valently analytic functions with negative coefficients*, Appl. Math. Comput. 187(2007) 47–53.
[4] A.C. Ata, *Neighborhoods of a certain class of analytic functions with negative coefficients*, Banach J. Math. Anal. 3 (2009) 111–121.
[5] N.E. Cho, O.S. Kwon and H.M. Srivastava, *Inclusion relationships for certain subclasses of meromorphic functions associated with a family of multiplier transformations*, Integral Transforms Spec. Funct. 16 (2005) 647–658.
[6] J. Dziok, *Classes of meromorphic functions associated with conic regions*, Acta Math. Sci. 32 (2012) 765–774.
[7] J. Dziok, *Classes of multivalent analytic and meromorphic functions with two fixed points*, Fixed Point Theory Appl. 2013 (2013).

[8] R. Fournier and S.T. Ruscheweyh, *Remarks on a multiplier conjecture for univalent functions*, Proc. Amer. Math. Soc. 116 (1992) 35–43.

[9] B.A. Frasin, *Neighborhoods of certain multivalent functions with negative coefficients*, Appl. Math. Comput. 193 (2007) 1–6.

[10] A.W. Goodman, *Univalent functions and nonanalytic curves*, Proc. Amer. Math. Soc. 8 (1957) 598–601.

[11] A.W. Goodman, *Univalent Functions*, vol. 1. Polygonal Publishing House, Washington, New Jersey, 1983.

[12] N.K. Jain and V. Ravichandran, *Product and convolution of certain univalent functions*, Honam Math. J. 38(4) (2016) 701—724.

[13] B.S. Keerthi, A. Gangadharan and H.M. Srivastava, *Neighborhoods of certain subclasses of analytic functions of complex order with negative coefficients*, Math. Comput. Model. 47 (2008) 271–277.

[14] V. Kumar, N.E. Cho, V. Ravichandran and H.M. Srivastava, *Sharp coefficient bounds for starlike functions associated with the Bell numbers*, Math. Slovaca 69(5) (2019) 1053-–1064.

[15] J. L. Liu and H. M. Srivastava, *A linear operator and associated families of meromorphically multivalent functions*, J. Math. Anal. Appl. 259 (2001) 566–581.

[16] M. S. Liu, Y. C. Zhu and H. M. Srivastava, *Properties and characteristics of certain subclasses of starlike functions of order β*, Math.Comput.Model. 48 (2008) 402–419.

[17] M. Nunokawa and OP. Ahuja, *On meromorphic starlike and convex functions*, Indian J. Pure Appl. Math. 32 (2001) 1027–1032.

[18] M.S. Robertson, *Quasi-subordinate function, Mathematical essays dedicated to A*, J. Macintyre, Ohio Univ. Press Athens, (1967) 311–330

[19] S. Ruscheweyh, *Neighborhoods of univalent functions*, Proc. Am. Math. Soc. 81 (1981) 521–527.

[20] K. Sharma, N.E. Cho and V. Ravichandran, *Sufficient conditions for strong starlikeness*, Bull. Iranian Math. Soc. (2020), DOI:10.1007/s41980-020-00452-z.

[21] H. M. Srivastava, S. S. Eker and B. Seker, *Inclusion and neighborhood properties for certain classes of multivalently analytic functions of complex order associated with the convolution structure*, Appl. Math. Comput. 212 (2009) 66–71.

[22] Z.G. Wang, Z.H. Liu and R.G. Xiang, *Some criteria for meromorphic multivalent starlike functions*, Appl. Math. Comput. 218 (2011) 1107–1111.

[23] Z. G. Wang, H. M. Srivastava and S. M. Yuan, *Some basic properties of certain subclasses of meromorphically starlike functions*, J. Inequal. Appl. 2014 (2014) 29.

[24] Z.G. Wang, X.S. Yuan and L. Shi, *Neighborhoods and partial sums of certain subclass of starlike functions*, J. Inequal. Appl. 2013 (2013) 163.