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A generalization of Darbo's theorem with application to the solvability of systems of integral-Differential equations in Sobolev spaces

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Abstract

In this article, we introduce the notion of (α,β)-generalized Meir-Keeler condensing operator in a Banach space, a characterization using strictly L-functions and provide an extension of Darbo's fixed point theorem associated with measures of noncompactness. Then, we establish some results on the existence of coupled fixed points for a class of condensing operators in Banach spaces. As an application, we study the problem of existence of entire solutions for a general system of nonlinear integral-differential equations in a Sobolev space. Further, an example is presented to verify the effectiveness and applicability of our main results.

Keywords: Coupled fixed points, Measure of noncompactness, Meir-Keleer condensing operator, Sobolev space, System of integral equations.
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1. Introduction

The theory of systems of differential and integral equations play an important role in nonlinear analysis and is applicable to numerous problems of the other branches of sciences. There have recently been many papers regarding the existence solutions of systems of integral equations on some spaces. For example, Aghajani and Allahyari [2] , Aghajani and Jalilian [3], Aghajani and Sabzali [6] obtained some interesting results of the existence solutions for systems of nonlinear integral equations

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in Banach spaces. We recall that the existence results of these literatures were formulated with the help of measures of noncompactness.

The concept of measure of noncompactness was initiated by Kuratowski [14]. Banaś *et al.* [8] proposed a generalization of this notion which is more convenient in the applications. The tool of measure of noncompactness has been used in the theory of operator equations in Banach spaces. They are frequently used in the theory of functional equations, including ordinary differential equations, equations with partial derivatives, integral and integro-differential equations, optimal control theory, etc. In particular, the fixed point theorems derived from them have many applications. The principal application of measures of noncompactness in the fixed point theory is contained in the Darbo's fixed point theorem [11]. The technique of measures of noncompactness in conjunction with it turned into a tool to investigate the existence and behavior of solutions of many classes of integral equations such as Volterra, Fredholm and Uryson type integral equations.

In 1969, Meir and Keeler [16] introduced the concept of Meir-Keeler contractive mapping and proved some fixed point theorems for this kind of mappings. Thereafter, Aghajani *et al.*, [4] generalized some fixed point and coupled fixed point theorems for Meir-Keeler condensing operators via measures of noncompactness.

On the other hand, Sobolev spaces [9], *i.e.,* the class of functions with derivatives in L^p, play an outstanding role in the modern analysis. In the last decades, there has been increasing attempts to study of these spaces. Their importance comes from the fact solutions of partial differential equations are naturally found in Sobolev spaces. They also highlighted in approximation theory, calculus of variation, differential geometry, spectral theory, *etc.*

In this paper, we introduce the notion of (α, β)-generalized Meir-Keeler condensing operator in a Banach space, and give an extension of Darbo's fixed point theorem associated with measures of noncompactness. Then, we establish an existence result of coupled fixed points for a class of condensing operators in Banach spaces. As an application, we study the problem of existence of solutions for the following system of nonlinear integral-differential equations in a Sobolev space.

$$\begin{cases} u(x) = f(x, u(x), v(x), \int g(y, u(y), \frac{\partial u}{\partial x_1}(y), \dots, \frac{\partial u}{\partial x_n}(y), v(y))dy) \\ v(x) = f(x, v(x), u(x), \int g(y, v(y), \frac{\partial v}{\partial x_1}(y), \dots, \frac{\partial v}{\partial x_n}(y), u(y))dy) \end{cases} \tag{1.1}$$

Further, an example is presented to verify the effectiveness and applicability of our main results.

2. Preliminaries

In this section, we provide some notations, definitions and preliminary facts which will be needed further on. Denote by $\mathbb{R}$ the set of real numbers and put $\mathbb{R}_+ = [0, +\infty)$. For the Lebesgue measurable subset D of $\mathbb{R}^n$ $(n \in \mathbb{N})$, let $m(D)$ be the lebesgue measure of D and let $L^1(D)$ be the space of all Lebesgue integrable functions f on D equipped with the standard norm $\|f\|_{L^1(D)} = \int_D |f(x)|dx$.

Let $(E, \|\cdot\|)$ be a real Banach space with zero element 0. The symbol $\overline{B(x,r)}$ stands for the closed ball centered at x with radius r and put $\overline{B_r} = \overline{B(0,r)}$. For a nonempty subset X of E, the symbols $\overline{X}$ and Conv$X$ will denote the closure and closed convex hull of X, respectively. Moreover, let $\mathfrak{M}_E$ indicate the family of nonempty and bounded subsets of E and $\mathfrak{N}_E$ indicate the family of all nonempty and relatively compact subsets of E.

Definition 2.1. *[8] A mapping $\mu : \mathfrak{M}_E \to \mathbb{R}_+$ is said to be a measure of noncompactness in E if it satisfies the following conditions:*

1° *The family* $\ker\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ *is nonempty and* $\ker\mu \subset \mathfrak{N}_E$.

2° $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.

3° $\mu(\overline{X}) = \mu(X)$.

4° $\mu(ConvX = \mu(X)$.

5° $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ *for* $\lambda \in [0,1]$.

6° *If* $\{X_n\}$ *is a sequence of closed chains of* $\mathfrak{M}_E$ *such that* $X_{n+1} \subset X_n$ *for* $n = 1, 2, \ldots$ *and if* $\lim_{n\to\infty} \mu(X_n) = 0$, *then the set* $X_\infty = \bigcap_{n=1}^{\infty} X_n \neq \emptyset$.

Definition 2.2. *[7] Suppose that* E_1 *and* E_2 *are two Banach spaces and* μ_1 *and* μ_2 *are arbitrary measures of noncompactness on* E_1 *and* E_2, *respectively. Also, suppose* $T : E_1 \to E_2$ *is a continuous operator satisfies the following condition:*

$$\mu_2(T(\Omega)) < \mu_1(\Omega)$$

for every bounded noncompact set $\Omega \subset E_1$. *Then* T *is called a* (μ_1, μ_2)*-condensing operator.*

Theorem 2.3. *(Darbo [8]) Let* C *be a nonempty, bounded, closed and convex subset of a Banach space* E *and let* $T : C \to C$ *be a continuous mapping. Assume that a constant* $k \in [0, 1)$ *exists such that*

$$\mu(T(X)) \leq k\mu(X)$$

for any nonempty subset X *of* C, *where* μ *is a measure of noncompactness defined in* E. *Then* T *has a fixed point in the set* C.

Theorem 2.4. *(Tychonoff fixed point theorem [1]) Let* E *be a Hausdorff locally convex linear topological space,* C *a convex subset of* E *and* $T : C \to E$ *a continuous mapping such that*

$$T(C) \subseteq A \subseteq C,$$

with A *compact. Then* T *has at least one fixed point.*

Definition 2.5. *[4] Let* C *be a nonempty subset of a Banach space* E *and* μ *an arbitrary measure of noncompactness on* E. *An operator* $T : C \to C$ *is called a Mier-keeler condensing operator if for any* $\varepsilon > 0$, $\delta > 0$ *exists such that*

$$\varepsilon \leq \mu(X) < \varepsilon + \delta \quad \textit{implies} \quad \mu(T(X)) < \varepsilon$$

for any bounded subset X *of* C.

Definition 2.6. *[10] Let* X *be a nonempty set. An element* $(x, y) \in X \times X$ *is called a coupled fixed point of a mapping* $G : X \times X \to X$ *if* $G(x, y) = x$ *and* $G(y, x) = y$.

Here we quote a useful theorem in [7] concerning the construction of a measure of noncompactness on a finite product space.

Theorem 2.7. *Let $\mu_1, \mu_2, \ldots, \mu_n$ be measures of noncompactness in Banach spaces $E_1, E_2, \ldots, E_n$, respectively. Moreover, suppose that the function $F : [0, \infty)^n \to [0, \infty)$ is convex and $F(x_1, x_2, \ldots, x_n) = 0$ if and only if $x_i = 0$ for $i = 1, 2, \ldots, n$. Then*

$$\widetilde{\mu}(X) = F(\mu_1(X_1), \mu_2(X_2), \ldots, \mu_n(X_n)),$$

defines a measure of noncompactness in $E_1 \times E_2 \times, \ldots \times E_n$, where X_i denotes the natural projection of X into E_i, for $i = 1, 2, \ldots, n$.

As a result from Theorem 2.7 above, we have the following example which is presented in [6].

Example 2.8. *Let μ be a measure of noncompactness on a Banach space E. Take $F(x, y) = x + y$ for any $(x, y) \in \mathbb{R}^2_+$. Then all the conditions of Theorem 2.7 are satisfied. Therefore, $\widetilde{\mu}(X) = \mu(X_1) + \mu(X_2)$ defines a measure of noncompactness in the space $E \times E$ where X_i, $i = 1, 2$ denote the natural projections of X.*

Denote with Ψ the family of increasing functions $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ continuous in $t = 0$ such that
• $\psi(t) = 0$ if and only if $t = 0$,
• $\psi(t + s) \leq \psi(t) + \psi(s)$.

3. Fixed point results for (α, β)-generalized Meir-Keeler condensing operators

In this section, we define the notion of an (α, β)-generalized Meir-Keeler condensing operator on a Banach space and describe some fixed point results.

Definition 3.1. *Let C be a nonempty subset of a Banach space E and μ be an arbitrary measure of noncompactness on E. Also, suppose $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing mapping such that $\psi(t) = 0$ if and only if $t = 0$. We say that an operator $T : C \to C$ is an (α, β)-generalized Meir-Keeler condensing operator if for any $\varepsilon > 0$, $\delta > 0$ exists such that*

$$\varepsilon \leq \beta(\mu(X))\psi(\mu(X)) < \varepsilon + \delta \quad \text{implies} \quad \alpha(\mu(T(X)))\psi(\mu(T(X))) < \varepsilon \tag{3.1}$$

for any bounded subset X of C, where $\alpha : \mathbb{R}_+ \to [1, +\infty)$ and $\beta : \mathbb{R}_+ \to (0, 1]$ are mappings.

Theorem 3.2. *Let C be a nonempty, bounded, closed, and convex subset of a Banach space E and let μ be an arbitrary measure of noncompactness on E. If $T : C \to C$ is a continuous and (α, β)-generalized Meir-Keeler condensing operator, then T has at least one fixed point in the set C and the set of all fixed points of T in C is compact.*

Proof . By induction, we define a sequence $\{C_n\}$ such that $C_0 = C$ and $C_n = \mathrm{Conv}T(C_{n-1})$, $n \geq 1$. If $\mu(C_N) = 0$ for some integer $N \geq 0$, then C_N is compact. Thus, Theorem 2.4 implies that T has a fixed point. Now, assume $\mu(C_n) > 0$ for any $n \geq 0$. Take $\varepsilon_n = \beta(\mu(C_n))\psi(\mu(C_n)) > 0$ and consider $\delta_n = \delta(\varepsilon_n) > 0$ such that (3.1) holds. Therefore, by (3.1), we obtain

$$\alpha(\mu(T(C_n)))\psi(\mu(T(C_n))) < \beta(\mu(C_n))\psi(\mu(C_n)) \tag{3.2}$$

for each integer $n \geq 0$. By using (3.2), we derive that

$$\begin{aligned} \varepsilon_{n+1} &= \beta(\mu(C_{n+1}))\psi(\mu(C_{n+1})) \\ &\leq \psi(\mu(C_{n+1})) \\ &= \psi(\mu(\mathrm{Conv}T(C_n))) \\ &\leq \alpha(\mu(T(C_n)))\psi(\mu(T(C_n))) \\ &< \beta(\mu(C_n))\psi(\mu(C_n)) = \varepsilon_n, \end{aligned}$$

which implies that $\{\varepsilon_n\}$ is a strictly decreasing sequence of positive real numbers. Thus, there is an $r \geq 0$ so that $\varepsilon_n \to r$ as $n \to \infty$. We will show that $r = 0$. If $r > 0$, then by hypothesis, a $\delta(r) > 0$ exists such that (3.1) holds and so $N_0 > 0$ exists such that

$$r \leq \varepsilon_n = \beta(\mu(C_n))\psi(\mu(C_n)) < r + \delta(r),$$

for any $n \geq N_0$. By the definition of (α, β)-generalized Meir-Keeler condensing operator, we get $\alpha(\mu(T(C_n)))\psi(\mu(T(C_n))) < r$ for each $n \geq N_0$. Then it can be concluded that $\varepsilon_{n+1} < r$ for any $n \geq N_0$ which gives us a contradiction, so $r = 0$.

It can be shown that $\lim\limits_{n\to\infty} \psi(\mu(C_n)) = 0$, too. For, let $\varrho > 0$ be given. Then $N_1 > 0$ exists such that for each $n \geq N_1$, $0 < \varepsilon_n = \beta(\mu(C_n))\psi(\mu(C_n)) < \varrho$. Regarding to (3.2), we can write

$$\begin{aligned}
-\varrho &< \varepsilon_{n+1} = \beta(\mu(C_{n+1}))\psi(\mu(C_{n+1})) \\
&\leq \psi(\mu(C_{n+1})) \\
&\leq \alpha(\mu(C_{n+1}))\psi(\mu(C_{n+1})) \\
&= \alpha(\mu(T(C_n)))\psi(\mu(T(C_n))) \\
&< \varepsilon_n = \beta(\mu(C_n))\psi(\mu(C_n)) \\
&< \varrho
\end{aligned}$$

for all $n \geq N_1$. It follows that $\psi(\mu(C_{n+1})) \to 0$ and so $\psi(\mu(C_n)) \to 0$ as $n \to \infty$. Next, we claim that $\lim\limits_{n\to\infty} \mu(C_n) = 0$. To support the claim, let it be untrue. Thus, there is an $\varepsilon > 0$ such that for each positive integer N, $n_N \geq N$ exists in which $\mu(C_{n_N}) \geq \varepsilon$. By increasing of ψ we have $\psi(\mu(C_{n_N})) \geq \psi(\varepsilon)$, which is a contradiction with $\lim\limits_{n\to\infty} \psi(\mu(C_n)) = 0$. Hence, we deduce that, $\mu(C_n) \to 0$ as $n \to \infty$.

Using this fact and since the sequence $\{C_n\}$ is nested, in view of part 6° of Definition 2.1, it can be concluded that the set $C_\infty = \bigcap\limits_{n=1}^{\infty} C_n$ is nonempty, closed, and convex subset of the C. Furthermore, the set C_∞ is invariant under T, and $C_\infty \in \ker \mu$. Thus, applying Tychonoff fixed point theorem, we find that the operator T has a fixed point. Now, suppose that $F_T = \{x \in C : T(x) = x\}$. We are going to show that $\mu(F_T) = 0$. Suppose to the contrary, that $\mu(F_T) > 0$. Take $\varepsilon_0 = \beta(\mu(F_T))\psi(\mu(F_T))$, then by (3.2) and $T(F_T) = F_T$, we infer that $\psi(\mu(F_T)) < \psi(\mu(F_T))$, which leads to a contradiction. Then $\mu(F_T) = 0$, which means that F_T is relatively compact. As T is a continuous function, thus F_T is compact in C. □

Below, we recall from [15] the notion of a strictly L-function and then we establish an extension of Darbo's fixed point theorem using strictly L-functions.

Definition 3.3. *A function $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ is called a strictly L-function if $\theta(0) = 0$, $\theta(s) > 0$ for $s \in (0, +\infty)$, and for any $s > 0$, $\delta > 0$ exists such that $\theta(t) < s$, for all $t \in [s, s + \delta]$.*

Theorem 3.4. *Let α, β, and ψ be as Definition 3.1, C be a nonempty, bounded, closed and convex subset of a Banach space E, and let $T : C \to C$ be a continuous operator such that*

$$\alpha(\mu(T(X)))\psi(\mu(T(X))) \leq \theta\Big(\beta(\mu(X))\psi(\mu(X))\Big)$$

for any $X \subseteq C$, where μ is an arbitrary measure of noncompactness on E and θ is a strictly L-function. Then, T has at least one fixed point.

Proof . We are going to show that T is an (α, β)-generalized Meir-Keeler condensing operator. For this purpose, let $\varepsilon > 0$ be arbitrary. By the hypothesis, $\delta > 0$ exists such that

$$\varepsilon \leq t \leq \varepsilon + \delta \quad \text{implies} \quad \theta(t) < \varepsilon. \tag{3.3}$$

If X is a subset of C such that $\varepsilon \leq \beta(\mu(X))\psi(\mu(X)) < \varepsilon+\delta$, then using (3.3) $\theta(\beta(\mu(X))\psi(\mu(X))) < \varepsilon$ and by considering our assumptions, we have

$$\alpha(\mu(T(X)))\psi(\mu(T(X))) \leq \theta\Big(\beta(\mu(X))\psi(\mu(X))\Big) < \varepsilon.$$

Hence, by making appeal to Theorem 3.2, we conclude that T has a fixed point. □
Now, we present a coupled fixed point theorem using strictly L-functions.

Theorem 3.5. *Let E, C, β, θ and μ be as Theorem 3.4 and let $\alpha : \mathbb{R}_+ \to [1, +\infty)$ be an increasing map. Also, suppose that $\psi \in \Psi$ and $G : C \times C \to C$, is a continuous mapping satisfying*

$$\begin{aligned}\alpha\Big(\mu(G(X_1 \times X_2)) + \mu(G(X_2 \times X_1))\Big)\psi(\mu(G(X_1 \times X_2)))& \\ \leq \frac{1}{2}\theta\Big(\beta(\frac{\mu(X_1) + \mu(X_2)}{2})\psi(\frac{\mu(X_1) + \mu(X_2)}{2})\Big)&\end{aligned} \tag{3.4}$$

for all subsets X_1,X_2 of C. Then G has at least a coupled fixed point.

Proof . We first note that Example 2.8 implies that $\widetilde{\mu}(X) = \mu(X_1) + \mu(X_2)$ is a measure of noncompactness in the space $E \times E$, where X_i, $i = 1, 2$ are the natural projections of X. Define the mapping $\widetilde{G} : C\times C \to C\times C$ by $\widetilde{G}(x, y) = (G(x, y), G(y, x))$. Clearly $\widetilde{G}$ is continuous. We claim that $\widetilde{G}$ satisfies all the conditions of Theorem 3.4. To prove this fact, let us choose a nonempty subset X of $C \times C$. By properties of the mappings α and ψ and using (3.4) we have

$$\begin{aligned}
&\alpha\Big(\widetilde{\mu}(\widetilde{G}(X_1 \times X_2))\Big)\psi(\widetilde{\mu}(\widetilde{G}(X_1 \times X_2)))\\
&\quad\leq \alpha\Big(\widetilde{\mu}(G(X_1 \times X_2) \times G(X_2 \times X_1))\Big)\psi(\widetilde{\mu}(G(X_1 \times X_2) \times G(X_2 \times X_1)))\\
&\quad\leq \alpha\Big(\mu(G(X_1 \times X_2)) + \mu(G(X_2 \times X_1))\Big)\psi(\mu(G(X_1 \times X_2)))\\
&\qquad +\alpha\Big(\mu(G(X_2 \times X_1)) + \mu(G(X_1 \times X_2))\Big)\psi(\mu(G(X_2 \times X_1)))\\
&\quad\leq \theta\Big(\beta(\frac{\mu(X_1) + \mu(X_2)}{2})\psi(\frac{\mu(X_1) + \mu(X_2)}{2})\Big)\\
&\quad= \theta\Big(\beta(\frac{\widetilde{\mu}(X_1 \times X_2)}{2})\psi(\frac{\widetilde{\mu}(X_1 \times X_2)}{2})\Big).
\end{aligned}$$

Therefore,

$$\alpha\Big(\frac{1}{2}\widetilde{\mu}(\widetilde{G}(X_1 \times X_2))\Big)\psi(\frac{1}{2}\widetilde{\mu}(\widetilde{G}(X_1 \times X_2))) \leq \theta\Big(\beta(\frac{1}{2}\widetilde{\mu}(X_1 \times X_2))\psi(\frac{1}{2}\widetilde{\mu}(X_1 \times X_2))\Big)$$

and taking $\widehat{\mu} = \frac{1}{2}\widetilde{\mu}$, we obtain

$$\alpha\Big(\widehat{\mu}(\widetilde{G}(X_1 \times X_2))\Big)\psi(\widehat{\mu}(\widetilde{G}(X_1 \times X_2))) \leq \theta\Big(\beta(\widehat{\mu}(X_1 \times X_2))\psi(\widehat{\mu}(X_1 \times X_2))\Big).$$

Since $\widehat{\mu}$, is also a measure of noncompactness, so Theorem 3.4 guarantees that $\widetilde{G}$ has a fixed point, or equivalently G has a coupled fixed point. □
The next result is a special case of the theorem above, which will be used in Section 4.

Corollary 3.6. *Let C be a nonempty, bounded, closed and convex subset of a Banach space E and $G : C \times C \to C$ be a continuous mapping satisfying*

$$\psi(\mu(G(X_1 \times X_2))) \leq \frac{1}{2}\theta\Big(\psi(\frac{\mu(X_1)+\mu(X_2)}{2})\Big) \tag{3.5}$$

for all subsets X_1,X_2 of C, where μ is an arbitrary measure of noncompactness in the space E, θ is a strictly L-function, and $\psi \in \Psi$. Then G has at least a coupled fixed point.

4. Application

In this section, we study the existence of solutions for a system of nonlinear integral-differential equations. We also provide an illustrative example to verify the effectiveness and applicability of our results.

We start with some preliminaries which we need in subsequent.

Lemma 4.1. *[12] Let Ω be a Lebesgue measurable subset of $\mathbb{R}^n$ and $1 \leq p \leq \infty$. If $\{f_k\}$ is convergent to f in the L^p-norm, then there is a subsequence $\{f_{k_m}\}$ which converges to f a.e., and there is $g \in L^p(\Omega)$, $g \geq 0$, such that*

$$|f_{k_m}(x)| \leq g(x) \quad \text{for a.e. } x \in \Omega.$$

Definition 4.2. *[5] We say that a function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ satisfies the Carathéodory conditions if the function $f(., u)$ is measurable for each $u \in \mathbb{R}^m$ and the function $f(x, .)$ is continuous for almost all $x \in \mathbb{R}^n$.*

Let Ω be a subset of $\mathbb{R}^n$ and $k \in \mathbb{N}$, we denote by $W^{k,1}(\Omega)$ the space of functions f which, together with all their distributional derivatives $D^\alpha f$ of order $|\alpha| \leq k$, belong to $L^1(\Omega)$. Here $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index, *i.e.*, each α_j is a nonnegative integer, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, and

$$D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}.$$

Then, $W^{k,1}(\Omega)$ is equipped with the complete norm

$$\|f\|_{k,1}^{\Omega} = \max_{0\leq|\alpha|\leq k} \|D^\alpha f\|_{L^1(\Omega)}.$$

Now, we are ready to define a measure of noncompactness on the spaces $W^{k,1}(\Omega)$.

Theorem 4.3. *Suppose that $1 \leq k < \infty$ and U is a bounded subset of $W^{k,1}(\Omega)$. For $u \in U$, $\varepsilon > 0$ and $0 \leq |\alpha| \leq k$, let*

$$\begin{aligned}
\omega^T(u,\varepsilon) &= \sup\{\|\mathcal{T}_h D^\alpha u - D^\alpha u\|_{L^1(B_T)} : h \in \Omega,\ \|h\|_{\mathbb{R}^n} < \varepsilon, 0 \leq |\alpha| \leq k\},\\
\omega^T(U,\varepsilon) &= \sup\{\omega^T(u,\varepsilon) : u \in U\},\\
\omega^T(U) &= \lim_{\varepsilon\to 0}\omega^T(U,\varepsilon),\\
\omega(U) &= \lim_{T\to\infty}\omega^T(U),\\
&\textit{and}\\
d(U) &= \lim_{T\to\infty}\sup\{\|D^\alpha u\|_{L^1(\Omega\setminus B_T)} : u \in U, 0 \leq |\alpha| \leq k\},
\end{aligned}$$

where $B_T = \{a \in \Omega : \|a\|_{\mathbb{R}^n} \leq T\}$ and $\mathcal{T}_h u(t) = u(t+h)$.
Then $\omega_0 : \mathfrak{M}_{W^{k,1}(\Omega)} \to \mathbb{R}$ given by

$$\omega_0(U) = \omega(U) + d(U)$$

defines a measure of noncompactness on $W^{k,1}(\Omega)$.

Proof . Here, we give a sketch of proof.
1° follows from the definition of ω_0 and applying [13, Theorem 5].
2° is obvious by the definition of ω_0.
3° is a straightforward consequence of the definition of ω_0 and part 2°.
4° follows directly from $D^\alpha[\mathrm{Conv}(U)] = \mathrm{Conv}(D^\alpha U)$ $(U \in \mathfrak{M}_{W^{k,1}(\Omega)})$.
5° can be obtained by using the equality

$$D^\alpha(\lambda u_1 + (1-\lambda)u_2) = \lambda D^\alpha u_1 + (1-\lambda)D^\alpha u_2$$

for all $\lambda \in [0,1]$, $u_1 \in X$ and $u_2 \in Y$.
To verify 6°, suppose that $\{U_n\}$ is a sequence of closed and nonempty sets of $\mathfrak{M}_{W^{k,1}(\Omega)}$ such that $U_{n+1} \subset U_n$ for $n = 1, 2, \ldots$, and $\lim\limits_{n\to\infty} \omega_0(U_n) = 0$. Now, for any $n \in \mathbb{N}$, take $u_n \in U_n$ and set $\mathcal{G} = \overline{\{u_n\}}$. We show that $\mathcal{G}$ is a compact set in $W^{k,1}(\Omega)$. For, let $\varepsilon > 0$ be fixed. Since $\lim\limits_{n\to\infty} \omega_0(U_n) = 0$, there exists sufficiently large $m_1 \in \mathbb{N}$ such that $\omega_0(U_{m_1}) < \varepsilon$. Hence, there is small enough $\delta_1 > 0$ and large enough $T_1 > 0$ such that $\omega^{T_1}(U_{m_1}, \delta_1) < \varepsilon$ and $d(U_{m_1}) < \varepsilon$. Therefore,

$$\|\mathcal{T}_h D^\alpha u_n - D^\alpha u_n\|_{L^1(B_{T_1})} < \varepsilon$$

and

$$\|D^\alpha u_n\|_{L^1(\Omega \setminus B_{T_1})} < \varepsilon$$

for all $n > m_1$, $0 \le |\alpha| \le k$ and $h \in \Omega$ such that $\|h\|_{\mathbb{R}^n} < \delta_1$. Then, we obtain

$$\begin{aligned}
&\|\mathcal{T}_h D^\alpha u_n - D^\alpha u_n\|_{L^1(\Omega)} \\
&\quad \le \|\mathcal{T}_h D^\alpha u_n - D^\alpha u_n\|_{L^1(B_{T_1})} + \|\mathcal{T}_h D^\alpha u_n - D^\alpha u_n\|_{L^1(\Omega\setminus B_{T_1})} \\
&\quad \le \|\mathcal{T}_h D^\alpha u_n - D^\alpha u_n\|_{L^1(B_{T_1})} + \|\mathcal{T}_h D^\alpha u_n\|_{L^1(\Omega\setminus B_{T_1})} + \|D^\alpha u_n\|_{L^1(\Omega\setminus B_{T_1})} \\
&\quad < 3\varepsilon.
\end{aligned}$$

On the other hand, we know that the set $\{u_1, u_2, \ldots, u_{m_1}\}$ is compact, hence $\delta_2 > 0$ and $T_2 > 0$ exist such that

$$\|\mathcal{T}_h D^\alpha u_n - D^\alpha u_n\|_{L^1(B_{T_2})} < \varepsilon$$

for all $n = 1, 2, \ldots, m_1$, $0 \le |\alpha| \le k$ and $h \in \Omega$ with $\|h\|_{\mathbb{R}^n} < \delta_2$.
Furthermore,

$$\|D^\alpha u_n\|_{L^1(\Omega\setminus B_{T_2})} < \varepsilon,$$

which implies that

$$\|\mathcal{T}_h D^\alpha u_n - D^\alpha u_n\|_{L^1(\Omega)} < 3\varepsilon$$

for all $n = 1, 2, \ldots, m_1$.
Thus,

$$\|\mathcal{T}_h D^\alpha u_n - D^\alpha u_n\|_{L^1(\Omega)} < 3\varepsilon$$

and

$$\|D^\alpha u_n\|_{L^1(\Omega\setminus B_T)} < \varepsilon < 3\varepsilon$$

for all $n \in \mathbb{N}$, $\|h\|_{\mathbb{R}^n} < \min\{\delta_1, \delta_2\}$ and $T = \max\{T_1, T_2\}$. By making use of [13, Theorem 5] we find that $\mathcal{G}$ is a compact set.

Using compactness of $\mathcal{G}$, a subsequence $\{u_{n_j}\}$ and $u_0 \in W^{k,1}(\Omega)$ exist such that $u_{n_j} \to u_0$. Since $u_n \in U_n$, $U_{n+1} \subset U_n$ and U_n is closed for all $n \in \mathbb{N}$, we yield

$$u_0 \in \bigcap_{n=1}^{\infty} U_n = U_\infty,$$

that finishes the proof of 6°.□

In the sequel, to demonstrate the applicability of our results, we study the existence of solutions for the system of integral-differential equations (1.1) in the Sobolev space $W^{1,1}(\Omega) \times W^{1,1}(\Omega)$ under the following general assumptions.

(1) $f : \Omega \times \mathbb{R}^3 \to \mathbb{R}$ and $\frac{\partial f}{\partial x_i}$, $i = 1, 2, \ldots, n$ satisfy the Caratéodory conditions, and constants $b > 0$ and $\lambda \in [0, \frac{1}{6})$ and $a \in L^1(\Omega)$ exist such that

(i) $|f(x, u(x), v(x), w)| \le a(x) + \lambda \max\Big\{|D^\alpha u(x)| : \alpha = 0, 1\Big\} + \lambda \max\Big\{|D^\alpha v(x)| : \alpha = 0, 1\Big\} + b|w|,$

(ii) $|\frac{\partial f}{\partial x_i}(x, u(x), v(x), w)| \le a(x) + \lambda \max\Big\{|D^\alpha u(x)| : \alpha = 0, 1\Big\} + \lambda \max\Big\{|D^\alpha v(x)| : \alpha = 0, 1\Big\} + b|w|,$

(iii) $|\frac{\partial f}{\partial u}(x, u(x), v(x), w)\frac{\partial u}{\partial x_i}(x)| \le a(x) + \lambda \max\Big\{|D^\alpha u(x)| : \alpha = 0, 1\Big\} + \lambda \max\Big\{|D^\alpha v(x)| : \alpha = 0, 1\Big\} + b|w|,$

(iv) $|\frac{\partial f}{\partial v}(x, u(x), v(x), w)\frac{\partial v}{\partial x_i}(x)| \le a(x) + \lambda \max\Big\{|D^\alpha u(x)| : \alpha = 0, 1\Big\} + \lambda \max\Big\{|D^\alpha v(x)| : \alpha = 0, 1\Big\} + b|w|,$

for any $u, v \in W^{1,1}(\Omega)$ and $x \in \Omega$, where

$$w = w(u, v) = \int_\Omega g(y, u(y), \frac{\partial u}{\partial x_1}(y), \frac{\partial u}{\partial x_2}(y), \ldots, \frac{\partial u}{\partial x_n}(y), v(y))dy.$$

(2) Suppose $h \in \mathbb{R}^n$, with $\|h\|_{\mathbb{R}^n}$ small enough, $T_h : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be the transformation mapping *i.e.*, $T_h(x) = x + h$, and let $\psi \in \Psi$ and $\Lambda \subseteq \Omega$ be arbitrary so that

(i) $\psi(\int_\Lambda |f(T_h(x), u_1(x), v_1(x), w) - f(x, u(x), v(x), w)|dx)$
$\le \frac{1}{6}\theta\Big(\frac{1}{4}\psi(\max\Big\{\|D^\alpha(u_1 - u)\|_{L^1(\Lambda)} : \alpha = 0, 1\Big\} + \max\Big\{\|D^\alpha(v_1 - v)\|_{L^1(\Lambda)} : \alpha = 0, 1\Big\})\Big),$

(ii) $\psi(\int_\Lambda |\frac{\partial f}{\partial x_i}(T_h(x), u_1(x), v_1(x), w) - \frac{\partial f}{\partial x_i}(x, u(x), v(x), w)|dx)$
$\le \frac{1}{6}\theta\Big(\frac{1}{4}\psi(\max\Big\{\|D^\alpha(u_1 - u)\|_{L^1(\Lambda)} : \alpha = 0, 1\Big\} + \max\Big\{\|D^\alpha(v_1 - v)\|_{L^1(\Lambda)} : \alpha = 0, 1\Big\})\Big),$

(iii) $\psi(\int_\Lambda |\frac{\partial f}{\partial u_1}(T_h(x), u_1(x), v_1(x), w)\frac{\partial u_1}{\partial x_i}(x + h) - \frac{\partial f}{\partial u}(x, u(x), v(x), w)\frac{\partial u}{\partial x_i}(x)|dx)$
$\le \frac{1}{6}\theta\Big(\frac{1}{4}\psi(\max\Big\{\|D^\alpha(u_1 - u)\|_{L^1(\Lambda)} : \alpha = 0, 1\Big\} + \max\Big\{\|D^\alpha(v_1 - v)\|_{L^1(\Lambda)} : \alpha = 0, 1\Big\})\Big),$

(iv) $\psi(\int_\Lambda |\frac{\partial f}{\partial v_1}(T_h(x), u_1(x), v_1(x), w)\frac{\partial v_1}{\partial x_i}(x + h) - \frac{\partial f}{\partial v}(x, u(x), v(x), w)\frac{\partial v}{\partial x_i}(x)|dx)$
$\le \frac{1}{6}\theta\Big(\frac{1}{4}\psi(\max\Big\{\|D^\alpha(u_1 - u)\|_{L^1(\Lambda)} : \alpha = 0, 1\Big\} + \max\Big\{\|D^\alpha(v_1 - v)\|_{L^1(\Lambda)} : \alpha = 0, 1\Big\})\Big),$

where θ is a continuous strictly L-function such that $\theta(a + b) \ge \theta(a) + \theta(b)$ $(a, b \in \mathbb{R}_+)$.

(3) $g : \Omega \times \mathbb{R}^{n+2} \to \mathbb{R}$ satisfies the Caratéodory conditions and there exists a bounded continuous function $a_1 : \Omega \to \mathbb{R}_+$ such that $|a_1(x)| \le M$ for all $x \in \Omega$ and some $M > 0$, and a concave increasing lower semi-continuous function $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ so that

$$|g(x, u_0, u_1, \ldots, u_{n+1})| \le a_1(x)\xi(\max_{0 \le i \le n+1}|u_i|).$$

(4) There exists a positive solution r_0 of the inequality

$$3\Big(\|a\|_{L^1(\Omega)} + 2\lambda r + bMm(\Omega)^2\xi(\frac{1}{m(\Omega)}r)\Big) \le r. \tag{4.1}$$

Theorem 4.4. *Under the assumptions* (1)-(4)*, the system of integral-differential equations (1.1) has at least one solution in the space* $W^{1,1}(\Omega)\times W^{1,1}(\Omega)$.

Proof . First, we define the operator $G: W^{1,1}(\Omega)\times W^{1,1}(\Omega)\to W^{1,1}(\Omega)$ by

$$G(u,v)(x)=f(x,u(x),v(x),\int_\Omega g(y,u(y),\frac{\partial u}{\partial x_1}(y),\dots,\frac{\partial u}{\partial x_n}(y),v(y))dy).$$

Notice that, the space $W^{1,1}(\Omega)\times W^{1,1}(\Omega)$ is equipped with the norm $\|(u,v)\|_{1,1}=\|u\|^\Omega_{1,1}+\|v\|^\Omega_{1,1}$ for each $(u,v)\in W^{1,1}(\Omega)\times W^{1,1}(\Omega)$. Now, by using of conditions (1), (3) and Jensen's inequality we have

$$\begin{aligned}
|G(u,v)(x)| &= \Big|f(x,u(x),v(x),\int_\Omega g(y,u(y),\frac{\partial u}{\partial x_1}(y),\dots,\frac{\partial u}{\partial x_n}(y),v(y))dy)\Big|\\
&\le a(x)+\lambda\max\Big\{|D^\alpha u(x)|:\alpha=0,1\Big\}+\lambda\max\Big\{|D^\alpha v(x)|:\alpha=0,1\Big\}\\
&\quad +b\int_\Omega a_1(y)\xi\Big(\max\Big\{|u(y)|,|v(y)|,|\frac{\partial u}{\partial x_i}(y)|:\ i=1,2,\dots,n\Big\}\Big)dy\\
&\le a(x)+\lambda\max\Big\{|D^\alpha u(x)|:\alpha=0,1\Big\}+\lambda\max\Big\{|D^\alpha v(x)|:\alpha=0,1\Big\}\\
&\quad + bMm(\Omega)\xi\Big(\frac{1}{m(\Omega)}\max\Big\{\|u\|^\Omega_{1,1},\|v\|^\Omega_{1,1}\Big\}\Big).
\end{aligned}$$

By integrating over Ω we obtain

$$\int_\Omega|G(u,v)(x)|dx\le\|a\|_{L^1(\Omega)}+\lambda\|u\|^\Omega_{1,1}+\lambda\|v\|^\Omega_{1,1}+bMm(\Omega)^2\xi\Big(\frac{1}{m(\Omega)}\max\Big\{\|u\|^\Omega_{1,1},\|v\|^\Omega_{1,1}\Big\}\Big).\tag{4.2}$$

Using chain rule and with a similar argument as above we have

$$\begin{aligned}
\Big|\frac{\partial G(u,v)(x)}{\partial x_i}\Big| &= \Big|\frac{\partial}{\partial x_i}f(x,u(x),v(x),w)\Big|\\
&\le \Big|\frac{\partial f}{\partial x_i}(x,u(x),v(x),w)\Big|+\Big|\frac{\partial f}{\partial u}(x,u(x),v(x),w)\frac{\partial u}{\partial x_i}(x)\Big|\\
&\quad +\Big|\frac{\partial f}{\partial v}(x,u(x),v(x),w)\frac{\partial v}{\partial x_i}(x)\Big|+\Big|\frac{\partial f}{\partial w}(x,u(x),v(x),w)\frac{\partial w}{\partial x_i}\Big|\\
&\le 3\Big(a(x)+\lambda\max\Big\{|D^\alpha u(x)|:\alpha=0,1\Big\}+\lambda\max\Big\{|D^\alpha v(x)|:\alpha=0,1\Big\}\\
&\quad +bMm(\Omega)\xi\Big(\frac{1}{m(\Omega)}\max\Big\{\|u\|^\Omega_{1,1},\|v\|^\Omega_{1,1}\Big\}\Big)\Big).
\end{aligned}$$

By integrating over Ω, we deduce

$$\int_\Omega\Big|\frac{\partial G(u,v)(x)}{\partial x_i}\Big|\le 3\Big(\|a\|_{L^1(\Omega)}+\lambda\|u\|^\Omega_{1,1}+\lambda\|v\|^\Omega_{1,1}+bMm(\Omega)^2\xi\Big(\frac{1}{m(\Omega)}\max\Big\{\|u\|^\Omega_{1,1},\|v\|^\Omega_{1,1}\Big\}\Big)\Big).\tag{4.3}$$

Due to inequalities (4.2) and (4.3), we infer that G is well defined and

$$G(\overline{B_{r_0}}\times\overline{B_{r_0}})\subseteq\overline{B_{r_0}},$$

where r_0 is any solution of (4.1).

Now, we show that the map $G:\overline{B_{r_0}}\times\overline{B_{r_0}}\to\overline{B_{r_0}}$ is continuous. For, let $\{(u_n,v_n)\}$ be an arbitrary sequence in $\overline{B_{r_0}}\times\overline{B_{r_0}}$ which converges to $(u,v)\in\overline{B_{r_0}}\times\overline{B_{r_0}}$. By Lemma 4.1 there are subsequences $\{u_{n_k}\}$ and $\{v_{n_k}\}$ which converge to u and v *a.e.*, respectively, and for $i=1,\dots,n$ $\{\frac{\partial u_{n_k}}{\partial x_i}\}$, $\{\frac{\partial v_{n_k}}{\partial x_i}\}$ converge to $\{\frac{\partial u}{\partial x_i}\}$, $\{\frac{\partial v}{\partial x_i}\}$ *a.e.*, respectively and there is $h\in L^1(\Omega)$, $h\ge0$ such that

$$\max\Big\{|u_{n_k}(y)|,\Big|\frac{\partial u_{n_k}}{\partial x_1}(y)\Big|,\dots,\Big|\frac{\partial u_{n_k}}{\partial x_n}(y)\Big|,|v_{n_k}(y)|\Big\}\le h(y)\quad\text{for}\quad a.e.\quad y\in\Omega.$$

From condition (3) we have

$$\begin{aligned} g(y,u_{n_k}(y),\frac{\partial u_{n_k}}{\partial x_1}(y),\ldots,\frac{\partial u_{n_k}}{\partial x_n}(y),v_{n_k}(y)) &\leq a_1(y)\xi\Big(\max\Big\{|u_{n_k}(y)|,|v_{n_k}(y)|,\Big|\frac{\partial u_{n_k}}{\partial x_i}(y)\Big|: i=1,\ldots,n\Big\}\Big) \\ &\leq a_1(y)\xi(h(y)). \end{aligned}$$

Since g satisfies the Carat$\acute{e}$odory conditions, it follow that

$$g(y,u_{n_k}(y),\frac{\partial u_{n_k}}{\partial x_1}(y),\ldots,\frac{\partial u_{n_k}}{\partial x_n}(y),v_{n_k}(y))\to g(y,u(y),\frac{\partial u}{\partial x_1}(y),\ldots,\frac{\partial u}{\partial x_n}(y),v(y))$$

as $k\to\infty$. As a consequence of Lebesgue's Dominated Convergence Theorem, it yields that

$$\int g(y,u_{n_k}(y),\frac{\partial u_{n_k}}{\partial x_1}(y),\ldots,\frac{\partial u_{n_k}}{\partial x_n}(y),v_{n_k}(y))dy\to\int g(y,u(y),\frac{\partial u}{\partial x_1}(y),\ldots,\frac{\partial u}{\partial x_n}(y),v(y))dy$$

as $k\to\infty$. We know that f satisfies Carat$\acute{e}$odory conditions, then

$$G(u_{n_k},v_{n_k})=f(x,u_{n_k}(x),v_{n_k}(x),\int_\Omega g(y,u_{n_k}(y),\frac{\partial u_{n_k}}{\partial x_1}(y),\ldots,\frac{\partial u_{n_k}}{\partial x_n}(y),v_{n_k}(y))dy)$$

is convergent to

$$G(u,v)=f(x,u(x),v(x),\int_\Omega g(y,u(y),\frac{\partial u}{\partial x_1}(y),\ldots,\frac{\partial u}{\partial x_n}(y),v(y))dy).$$

Similarly, $\frac{\partial G(u_{n_k},v_{n_k})}{\partial x_i}$ converges to $\frac{\partial G(u,v)}{\partial x_i}(x)$ as $k\to\infty$. These give us

$$\|G(u_{n_k},v_{n_k})-G(u,v)\|_{1,1}\to 0$$

as $k\to\infty$. Therefore G is a continuous function from $\overline{B_{r_0}}\times\overline{B_{r_0}}$ into $\overline{B_{r_0}}$. To finish the proof we have to verify that condition (3.5) is satisfied. For this, let $T>0$ and $\varepsilon>0$ be arbitrary constants and let $U\times V$ be a nonempty and bounded subset of $\overline{B_{r_0}}\times\overline{B_{r_0}}$. Choose $(u,v)\in U\times V$ and $x,h\in B_T$ with $\|h\|_{\mathbb{R}^n}\leq\varepsilon$, then from condition (2) we have

$$\begin{aligned} &\psi(\|\tau_h G(u\times v)-G(u\times v)\|_{L^1(B_T)}) \\ &= \psi(\int_{B_T}|f(T_h(x),\tau_h u(x),\tau_h v(x),w)-f(x,u(x),v(x),w)|dx) \\ &\leq \frac{1}{6}\theta\Big(\frac{1}{4}\psi(\max\Big\{\|\tau_h D^\alpha u-D^\alpha u\|_{L^1(B_T)}:\alpha=0,1\Big\} \\ &\quad+\max\Big\{\|\tau_h D^\alpha v-D^\alpha v\|_{L^1(B_T)}:\alpha=0,1\Big\}\Big)\Big) \\ &\leq \frac{1}{2}\theta\Big(\frac{1}{4}\psi(\omega^T(u,\varepsilon)+\omega^T(v,\varepsilon))\Big) \\ &\leq \frac{1}{2}\theta\Big(\frac{1}{4}\psi(\omega^T(U,\varepsilon)+\omega^T(V,\varepsilon))\Big). \end{aligned}$$

Thus, by using continuity of ψ and θ, we obtain

$$\begin{aligned} &\lim_{\varepsilon\to0}\sup\Big\{\psi(\|\tau_h G(u\times v)-G(u\times v)\|_{L^1(B_T)}):h\in\Omega,\ \|h\|_{\mathbb{R}^n}<\varepsilon,\ u\in U,v\in V\Big\} \\ &\leq\frac{1}{2}\theta\Big(\frac{1}{4}\psi(\omega^T(U)+\omega^T(V))\Big). \end{aligned}$$

and taking $T\to\infty$ we deduce

$$\lim_{T\to\infty}\lim_{\varepsilon\to0}\sup\Big\{\psi(\|\tau_h G(u\times v)-G(u\times v)\|_{L^1(B_T)}):h\in\Omega,\ \|h\|_{\mathbb{R}^n}<\varepsilon,\ u\in U,v\in V\Big\}\leq\frac{1}{2}\theta\Big(\frac{1}{4}\psi(\omega(U)+\omega(V))\Big). \quad (4.4)$$

By the same reasoning as above we have

$$\begin{aligned}
&\psi(\|\tau_h DG(u\times v)-DG(u\times v)\|_{L^1(B_T)})\\
&\quad= \psi(\int_{B_T}\Big|Df(T_h(x),\tau_h u(x),\tau_h v(x),w)-Df(x,u(x),v(x),w)\Big|dx)\\
&\quad\leq \psi(\int_{B_T}\Big|\frac{\partial f}{\partial x_i}(x+h,\tau_h u(x),\tau_h v(x),w)-\frac{\partial f}{\partial x_i}(x,u(x),v(x),w)\Big|dx\\
&\qquad+\int_{B_T}\Big|\frac{\partial f}{\partial \tau_h u}(x+h,\tau_h u(x),\tau_h v(x),w)\frac{\partial \tau_h u}{\partial x_i}(x+h)-\frac{\partial f}{\partial u}(x,u(x),v(x),w)\frac{\partial u}{\partial x_i}(x)\Big|dx\\
&\qquad+\int_{B_T}\Big|\frac{\partial f}{\partial \tau_h v}(x+h,\tau_h u(x),\tau_h v(x),w)\frac{\partial \tau_h v}{\partial x_i}(x+h)-\frac{\partial f}{\partial v}(x,u(x),v(x),w)\frac{\partial v}{\partial x_i}(x)\Big|dx\\
&\quad\leq \frac{1}{2}\theta\Big(\frac{1}{4}\psi(\max\Big\{\|\tau_h D^\alpha u-D^\alpha u\|_{L^1(B_T)}:\alpha=0,1\Big\}\\
&\quad+ \max\Big\{\|\tau_h D^\alpha v-D^\alpha v\|_{L^1(B_T)}:\alpha=0,1\Big\})\Big)\\
&\quad\leq \frac{1}{2}\theta\Big(\frac{1}{4}\psi(\omega^T(u,\varepsilon)+\omega^T(v,\varepsilon))\Big)\\
&\quad\leq \frac{1}{2}\theta\Big(\frac{1}{4}\psi(\omega^T(U,\varepsilon)+\omega^T(V,\varepsilon)\Big)
\end{aligned}$$

and so

$$\lim_{T\to\infty}\lim_{\varepsilon\to 0}\sup\Big\{\psi(\|\tau_h DG(u\times v)-DG(u\times v)\|_{L^1(B_T)}):h\in\Omega,\quad \|h\|_{\mathbb{R}^n}<\varepsilon,\quad u\in U,\quad v\in V\Big\} \tag{4.5}$$
$$\leq \frac{1}{2}\theta\Big(\frac{1}{4}\psi(\omega(U)+\omega(V))\Big).$$

The relations (4.4) and (4.5) imply that

$$\psi(\omega(G(U\times V))\leq \frac{1}{2}\theta\Big(\frac{1}{4}\psi(\omega(U)+\omega(V))\Big). \tag{4.6}$$

Next, taking into account our hypotheses and Jensen's inequality, for an arbitrary $(u,v)\in U\times V$ and $x\in\Omega\setminus B_T$ we derive that

$$\begin{aligned}
\psi(\|G(u\times v)\|_{L^1(\Omega\setminus B_T)}) &= \psi(\int_{\Omega\setminus B_T}|f(x,u(x),v(x),w|dx)\\
&\leq \psi(\int_{\Omega\setminus B_T}|f(x,u(x),v(x),w)-f(x+h,0,0,w)|dx+\int_{\Omega\setminus B_T}|f(x+h,0,0,w)|dx)\\
&\leq \frac{1}{6}\theta\Big(\frac{1}{4}\psi(\max\Big\{\|D^\alpha u\|_{L^1(\Omega\setminus B_T)}:\alpha=0,1\Big\}+\max\Big\{\|D^\alpha v\|_{L^1(\Omega\setminus B_T)}:\alpha=0,1\Big\})\Big)\\
&\quad+\int_{\Omega\setminus B_T}a(x+h)+b|w|dx\\
&\leq \frac{1}{2}\theta\Big(\frac{1}{4}\psi(\max\Big\{\|D^\alpha u\|_{L^1(\Omega\setminus B_T)}:\alpha=0,1\Big\}+\max\Big\{\|D^\alpha v\|_{L^1(\Omega\setminus B_T)}:\alpha=0,1\Big\})\Big)\\
&\quad+\|a\|_{L^1(\Omega\setminus B_T)}+bMm(\Omega\setminus B_T)m(\Omega)\xi(\frac{1}{m(\Omega)}\max\Big\{\|u\|_{1,1},\|v\|_{1,1}\Big\}).
\end{aligned} \tag{4.7}$$

Similarly,

$$\psi(\|DG(u\times v)\|_{L^1(\Omega\setminus B_T)})\leq \frac{1}{2}\theta\Big(\frac{1}{4}\psi(\max\Big\{\|D^\alpha u\|_{L^1(\Omega\setminus B_T)}:\alpha=0,1\Big\}+\max\Big\{\|D^\alpha v\|_{L^1(\Omega\setminus B_T)}:\alpha=0,1\Big\}\Big)$$
$$+\|a\|_{L^1(\Omega\setminus B_T)}+bMm(\Omega\setminus B_T)m(\Omega)\xi(\frac{1}{m(\Omega)}\max\Big\{\|u\|_{1,1},\|v\|_{1,1}\Big\}). \tag{4.8}$$

Passing T to infinity in the relations (4.7) and (4.8) it follows that

$$\psi(d(G(U\times V)))\leq \frac{1}{2}\theta\Big(\frac{1}{4}\psi(d(U)+d(V))\Big). \tag{4.9}$$

On combining (4.6) and (4.9) and increasing of mappings θ and ψ we obtain

$$\begin{aligned}\psi(\mu(G(U\times V))) &= \psi(\omega(G(U\times V))+d(G(U\times V)))\\ &\le \frac{1}{2}\Big(\theta(\frac{1}{4}\psi(\omega(U)+\omega(V)))+\theta(\frac{1}{4}\psi(d(U)+d(V)))\Big)\\ &\le \frac{1}{2}\theta\Big(\frac{1}{4}\psi(\omega(U)+\omega(V))+\frac{1}{4}\psi(d(U)+d(V))\Big)\\ &\le \frac{1}{2}\theta\Big(\frac{2}{4}\psi(\omega(U)+d(U)+\omega(V)+d(V))\\ &\le \frac{1}{2}\theta\Big(\psi(\frac{\mu(U)+\mu(V)}{2})\Big).\end{aligned}$$

Now, Corollary 3.6 guarantees that G has a coupled fixed point in $\overline{B_{r_0}}\times\overline{B_{r_0}}$ and thus system of integral-differential equations (1.1) have at least one solution in $W^{1,1}(\Omega)\times W^{1,1}(\Omega)$. $\square$

Example 4.5. *Consider the following system of integral-differential equations*

$$\begin{cases} u(x_1,x_2)=\ln(1+x_1+x_2)+\frac{1}{192}\sin(x_1^2x_2)u(x_1,x_2)+\frac{1}{576}\cos(2x_1+3x_2)v(x_1,x_2)\\ +\frac{1}{8}\int_0^1\int_0^1 e^{y_1+y_2}\tanh(\sqrt[7]{\sin^2 u(y_1,y_2)e^{\frac{\partial u}{\partial x_1}(y_1,y_2)-\frac{\partial u}{\partial x_2}(y_1,y_2)}}+y_1^3\ln|v(y_1,y_2)|)dy_1dy_2,\\ v(x_1,x_2)=\ln(1+x_1+x_2)+\frac{1}{192}\sin(x_1^2x_2)v(x_1,x_2)+\frac{1}{576}\cos(2x_1+3x_2)u(x_1,x_2)\\ +\frac{1}{8}\int_0^1\int_0^1 e^{y_1+y_2}\tanh(\sqrt[7]{\sin^2 v(y_1,y_2)e^{\frac{\partial v}{\partial x_1}(y_1,y_2)-\frac{\partial v}{\partial x_2}(y_1,y_2)}}+y_1^3\ln|u(y_1,y_2)|)dy_1dy_2.\end{cases}\tag{4.10}$$

Observe that system (4.10) is a special case of the system (1.1) with $\Omega=[0,1]^2$,

$$f(x_1,x_2,u(x_1,x_2),v(x_1,x_2),w)=\ln(1+x_1+x_2)+\frac{1}{192}\sin(x_1^2x_2)u(x_1,x_2)+\frac{1}{576}\cos(2x_1+3x_2)v(x_1,x_2)+\frac{1}{8}w$$

and

$$g(x_1,x_2,u(x_1,x_2),\frac{\partial u}{\partial x_1}(x_1,x_2),\frac{\partial u}{\partial x_2}(x_1,x_2))=e^{x_1+x_2}\tanh(\sqrt[7]{\sin^2 u(x_1,x_2)e^{\frac{\partial u}{\partial x_1}(x_1,x_2)-\frac{\partial u}{\partial x_2}(x_1,x_2)}+x_1^3\ln|v(x_1,x_2)|}).$$

In this example, hypothesis (1) of Theorem 4.4 holds if we define $a(x_1,x_2)=\ln 3$, $\lambda=\frac{1}{7}$, $b=\frac{1}{8}$. *Moreover, take* $\psi(t)=\theta(t)=\frac{t}{2}$, *then for each* $h\in\mathbb{R}^n$ *with sufficiently small* $\|h\|_{\mathbb{R}^n}$, *hypothesis (2) is valid, too. Indeed, for arbitrary subset* Λ *of* Ω *we have*

$\int_\Lambda|f(x+h,u_1(x),v_1(x),w)-f(x,u(x),u(x),w)|dx$

$$\begin{aligned}&\le \int_\Lambda|\ln(1+x_1+h_1+x_2+h_2)-\ln(1+x_1+x_2)|dx_1dx_2\\ &\quad+\frac{1}{192}\int_\Lambda|\sin((x_1+h_1)^2(x_2+h_2))u_1(x_1,x_2)-\sin(x_1^2x_2)u(x_1,x_2)|dx_1dx_2\\ &\quad+\frac{1}{576}\int_\Lambda|\cos(2(x_1+h_1)+3(x_2+h_2))v_1(x_1,x_2)-\cos(2x_1+3x_2)v(x_1,x_2)|dx_1dx_2.\end{aligned}\tag{4.11}$$

The first term of right hand side of (4.11) tends to zero as $\|h\|_{\mathbb{R}^n}\to 0$. *On the other hand, we have*

$|\sin((x_1+h_1)^2(x_2+h_2))u_1(x_1,x_2)-\sin(x_1^2x_2)u(x_1,x_2)|$

$$\begin{aligned}&\le |\sin((x_1+h_1)^2(x_2+h_2))||u_1(x_1,x_2)-u(x_1,x_2)|\\ &\quad+|\sin((x_1+h_1)^2(x_2+h_2))-\sin(x_1^2x_2)||u(x_1,x_2)|\\ &\le |u_1(x_1,x_2)-u(x_1,x_2)|+|h_1^2x_2+h_2x_1^2+h_1^2h_2+2h_1x_1x_2+2h_1h_2x_1||u(x_1,x_2)|,\end{aligned}$$

which implies that

$$\int_\Lambda |\sin((x_1+h_1)^2(x_2+h_2))u_1(x_1,x_2)-\sin(x_1^2x_2)u(x_1,x_2)|dx_1dx_2 \leq \|u_1-u\|_{L^1(\Lambda)},$$

where $\|h\|_{\mathbb{R}^n}$ *is small enough.*
Similarly,

$$\int_\Lambda |\cos(2(x_1+h_1)+3(x_2+h_2))v_1(x_1,x_2)-\cos(2x_1+3x_2)v(x_1,x_2)|dx_1dx_2 \leq \|v_1-v\|_{L^1(\Lambda)}.$$

With the help of previous inequalities, part (i) of hypothesis (2) *can be concluded. The other parts of* (2) *are similar and we ignore the details. In addition, property* (3) *holds if we consider* $a(x_1,x_2)=e^2$, $\xi(t)=1$, *and* $M=e^2$. *It can be easily shown that each number* $r \geq 21(\ln 3+\frac{e^2}{8})$ *satisfies the inequality (4.1). Consequently, all the conditions of Theorem 4.4 are satisfied.*
It implies that the system of integral-differential equations (4.10) has at least one solution in the Sobolev space $W^{1,1}(\Omega)\times W^{1,1}(\Omega)$.

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