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Existence Theory for Higher-Order Nonlinear Ordinary Differential Equations with Nonlocal Stieltjes Boundary Conditions

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Abstract

In this paper, we develop the existence theory for some boundary value problems of nonlinear *nth*-order ordinary differential equations supplemented with nonlocal Stieltjes boundary conditions. Our results are based on some standard theorems of fixed point theory and are well illustrated with the aid of examples.

Keywords: higher-order differential equations; Stieltjes; nonlocal boundary conditions; fixed point.
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1. Introduction

The study of nonlinear boundary value problems of differential equations is of central importance in mathematics in view of their extensive applications in applied sciences such as fluid mechanics, geophysics, mathematical physics, etc. The recent development of the subject includes several kinds of nonlocal and integral boundary conditions. The nonlocal conditions take care of peculiarities of physical, chemical or other processes occurring at some intermediate positions of the given domain while the integral conditions provide an alternative for the assumption of 'circular cross-section' throughout the vessels in the study of fluid flow problems. For examples and details of nonlocal

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problems, we refer the reader to a series of papers ([1]-[12]), whereas the works involving integral boundary conditions can be found in ([13]-[21]) and the references cited therein.

In this paper, we consider nonlocal multi-point and strip type Riemann-Stieltjes integral boundary conditions and establish the existence theory for boundary value problems of nth-order ordinary differential equations supplemented with these conditions. The concept of Stieltjes conditions provides a unified approach for dealing with a variety of boundary conditions such as multipoint and integral boundary conditions. For details on Riemann-Stieltjes integral conditions, we refer the reviews by Whyburn [22] and Conti [23]. Recently, Webb [24, 25] and Webb and Infante [26] studied the higher order problems with Stieltjes integral boundary conditions via fixed point index. Some more works on Riemann-Stieltjes integral conditions can be found in a series of papers [27, 28, 29, 30] and the references cited therein.

The rest of the paper is organized as follows. In Section 2, we formulate and solve a model problem. The existence results for this problem are obtained via contraction mapping principle and Schauder's fixed point and are explained with the help of some examples. In Section 3, we discuss some more problems involving Stieltjes conditions.

2. Model Problem

We consider a nonlocal Stieltjes type boundary value problem involving an nth-order nonlinear ordinary differential equation given by

$$\begin{cases} u^{(n)}(t) = f(t, u(t)), \quad t \in [0,1], \\ u(0) = \delta u(\xi), \quad u'(0) = 0, \quad u''(0) = 0, \dots, \quad u^{(n-2)}(0) = 0, \\ \alpha u(1) + \beta u'(1) = \displaystyle\int_0^1 u(s) d\mu(s), \quad 0 < \xi < 1, \end{cases} \tag{2.1}$$

where $f : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a given continuous function, μ is function of bounded variation and $\alpha,\ \beta,\ \delta,\ \xi$ are real constants satisfying the relation:

$$\Lambda = \Big(\alpha - \int_0^1 d\mu(s)\Big)\Big(\delta\xi^{n-1}\Big) + \Big(\alpha + \beta(n-1) - \int_0^1 s^{n-1} d\mu(s)\Big)\Big(1-\delta\Big) \neq 0. \tag{2.2}$$

In the following lemma, we solve a linear variant of problem (2.1).

2.1. Basic result

Lemma 2.1. *For any $y \in C([0,1], \mathbb{R})$, the linear differential equation*

$$u^{(n)}(t) = y(t), \quad t \in [0,1], \tag{2.3}$$

supplemented with boundary conditions of problem (2.1) *is equivalent to the integral equation:*

$$\begin{aligned} u(t) \ &= \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} y(s) ds + \sigma_1(t) \Big[\int_0^1 \Big(\int_0^s \frac{(s-p)^{n-1}}{(n-1)!} y(p) dp \Big) d\mu(s) \\ &\quad - \int_0^1 \frac{(1-s)^{n-2}[\beta(n-1) + \alpha(1-s)]}{(n-1)!} y(s) ds \Big] \\ &\quad + \sigma_2(t) \int_0^\xi \frac{(\xi - s)^{n-1}}{(n-1)!} y(s) ds, \end{aligned} \tag{2.4}$$

where

$$\sigma_1(t) = \frac{1}{\Lambda}[\delta\xi^{n-1} + t^{n-1}(1-\delta)], \tag{2.5}$$

$$\sigma_2(t) = \frac{\delta}{\Lambda}\Big[\alpha + \beta(n-1) - \int_0^1 s^{n-1}d\mu(s) - t^{n-1}\Big(\alpha - \int_0^1 d\mu(s)\Big)\Big], \tag{2.6}$$

and Λ is given by (2.2).

Proof . It is well known that the solution of the differential equation (2.3) can be written as

$$u(t) = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}y(s)ds + c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-2}t^{n-2} + c_{n-1}t^{n-1}, \tag{2.7}$$

where c_i $(i = 0,\ 1,\ \ldots n-1)$ are arbitrary real constants. Using the boundary conditions: $u'(0) = 0,\ u''(0) = 0,\ \ldots,\ u^{(n-2)}(0) = 0$ in (2.7), we find that $c_1 = c_2 = \ldots = c_{n-2} = 0$, and consequently, (2.7) takes the form:

$$u(t) = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}y(s)ds + c_0 + c_{n-1}t^{n-1}. \tag{2.8}$$

Now using the remaining boundary conditions:

$$u(0) = \delta u(\xi), \quad \alpha u(1) + \beta u'(1) = \int_0^1 u(s)d\mu(s),$$

in (2.8), we get

$$A_1c_0 - A_2c_{n-1} = A_3 \quad \textit{and} \quad E_1c_0 + E_2c_{n-1} = E_3, \tag{2.9}$$

where

$$A_1 = 1-\delta,\ A_2 = \delta\xi^{n-1},\ A_3 = \delta\int_0^\xi \frac{(\xi-s)^{n-1}}{(n-1)!}y(s)ds,$$

$$E_1 = \alpha - \int_0^1 d\mu(s),\ E_2 = \alpha + (n-1)\beta - \int_0^1 s^{n-1}d\mu(s),\ E_3 = B_3 - B_2 - B_1,$$

$$B_1 = \alpha\int_0^1 \frac{(1-s)^{n-1}}{(n-1)!}y(s)ds,\ B_2 = \beta\int_0^1 \frac{(1-s)^{n-2}}{(n-2)!}y(s)ds,$$

$$B_3 = \int_0^1 \Big(\int_0^s \frac{(s-p)^{n-1}}{(n-1)!}y(p)dp\Big)d\mu(s).$$

Solving the system (2.9), we obtain

$$c_0 = \frac{E_3A_2 + E_2A_3}{E_1A_2 + A_1E_2},\ c_{n-1} = \frac{E_3A_1 - A_3E_1}{A_2E_1 + A_1E_2},$$

where $A_2E_1 + A_1E_2 \neq 0$. Substituting the values of $c_0,\ c_{n-1}$ in (2.8), we get the solution (4.3). □

3. Existence results

In view of Lemma 2.1, we define a fixed point problem related to problem (2.1) as $Fu = u$, where $F : C([0,1],\mathbb{R}) \longrightarrow C([0,1],\mathbb{R})$ is defined by

$$\begin{aligned}(Fu)(t) &= \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s,u(s))ds + \sigma_1(t)\Big[\int_0^1 \Big(\int_0^s \frac{(s-g)^{n-1}}{(n-1)!} f(g,u(g))dg\Big)d\mu(s) \\ &- \int_0^1 \frac{(1-s)^{n-2}[\beta(n-1)+\alpha(1-s)]}{(n-1)!} f(s,u(s))ds\Big] \\ &+\sigma_2(t)\int_0^\xi \frac{(\xi-s)^{n-1}}{(n-1)!} f(s,u(s))ds,\end{aligned} \tag{3.1}$$

For the subsequent analysis, we define $\|u\| = \sup_{t\in[0,1]} |u(t)|$, and

$$\vartheta = \Big\{\frac{1}{n!} + h_1\Big[\int_0^1 \frac{s^n}{n!}d\mu(s) + \frac{(\beta n+\alpha)}{n!}\Big] + h_2\frac{\xi^n}{n!}\Big\}, \tag{3.2}$$

where $\max_{t\in[0,1]} |\sigma_1(t)| = h_1,\ \ \max_{t\in[0,1]} |\sigma_2(t)| = h_2$.

Theorem 3.1. *Assume that $f : [0,1]\times\mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function satisfying the Lipschitz condition: $|f(t,u) - f(t,v)| \le \ell|u-v|,\ \ell > 0,\ \forall\, u,v \in \mathbb{R}, t \in [0,1]$. Then the boundary value problem (2.1) has a unique solution if $\ell\vartheta < 1$, where ϑ is given by* (3.2)*.*

Proof . In the first step, we show that $FB_a \subset B_a$, where F is the operator defined by (3.1) and $B_a = \{u \in C([0,1],\mathbb{R}) : \|u\| \le a\}$, with $a \ge \dfrac{b\vartheta}{1-\ell\vartheta}$ and $b = \sup_{t\in[0,1]} |f(t,0)|$. Using $|f(t,u(t))| = |f(t,u(t)) - f(t,0) + f(t,0)| \le |f(t,u(t)) - f(t,0)| + |f(t,0)| \le \ell\|u\| + b \le \ell a + b$ for any $u \in B_a, t \in [0,1]$, we obtain

$$\begin{aligned}|(Fu)(t)| \quad &\le \sup_{t\in[0,1]}\Big\{\int_0^t \frac{(t-s)^{n-1}}{(n-1)!}|f(s,u(s))|ds \\ &+|\sigma_1(t)|\Big[\int_0^1 \Big(\int_0^s \frac{(s-g)^{n-1}}{(n-1)!}|f(g,u(g))|dg\Big)d\mu(s) \\ &+\int_0^1 \frac{(1-s)^{n-2}[\beta(n-1)+\alpha(1-s)]}{(n-1)!}|f(s,u(s))|ds\Big] \\ &+|\sigma_2(t)|\int_0^\xi \frac{(\xi-s)^{n-1}}{(n-1)!}|f(s,u(s))|ds\Big\}. \\ &\le (\ell a+b)\sup_{t\in[0,1]}\Big\{\int_0^t \frac{(t-s)^{n-1}}{(n-1)!}ds + |\sigma_1(t)|\Big[\int_0^1 \Big(\int_0^s \frac{(s-g)^{n-1}}{(n-1)!}dg\Big)d\mu(s) \\ &+\int_0^1 \frac{(1-s)^{n-2}[\beta(n-1)+\alpha(1-s)]}{(n-1)!}ds\Big] + |\sigma_2(t)|\int_0^\xi \frac{(\xi-s)^{n-1}}{(n-1)!}ds\Big\} \\ &\le \ (\ell a+b)\vartheta \le a,\end{aligned}$$

which implies that $\|Fu\| \le a$. In consequence, it follows that $FB_a \subset B_a$. Next, for $u,v \in C([0,1],\mathbb{R})$

and for each $t \in [0,1]$, we have that

$$
\begin{aligned}
&|(Fu)(t)-(Fv)(t)|\\
&\leq \sup_{t\in[0,1]}\Big\{\int_0^t \frac{(t-s)^{n-1}}{(n-1)!}|f(s,u(s))-f(s,v(s))|ds\\
&+|\sigma_1(t)|\Big[\int_0^1\Big(\int_0^s \frac{(s-g)^{n-1}}{(n-1)!}|f(g,u(g))-f(g,v(g))|dg\Big)d\mu(s)\\
&+\int_0^1 \frac{(1-s)^{n-2}[\beta(n-1)+\alpha(1-s)]}{(n-1)!}|f(s,u(s))-f(s,v(s))|ds\Big]\\
&+|\sigma_2(t)|\int_0^\xi \frac{(\xi-s)^{n-1}}{(n-1)!}|f(s,u(s))-f(s,v(s))|ds\Big\}\\
&\leq \ell\|u-v\|\sup_{t\in[0,1]}\Big\{\int_0^t \frac{(t-s)^{n-1}}{(n-1)!}ds\\
&+|\sigma_1(t)|\Big[\int_0^1\Big(\int_0^s \frac{(s-g)^{n-1}}{(n-1)!}dg\Big)d\mu(s)+\int_0^1 \frac{(1-s)^{n-2}[\beta(n-1)+\alpha(1-s)]}{(n-1)!}ds\Big]\\
&+|\sigma_2(t)|\int_0^\xi \frac{(\xi-s)^{n-1}}{(n-1)!}ds\Big\}.
\end{aligned}
$$

Taking maximum over the interval $[0,1]$, we get $\|(Fu)-(Fv)\| \leq \ell\vartheta\|u-v\|$, where ϑ is given by (3.2). By the assumption: $\ell\vartheta < 1$, we deduce that F is a contraction. Hence, by the contraction mapping principle, problem (2.1) has a unique solution. □

Example 3.2. *Consider the following boundary value problem*

$$
\begin{cases}
u'''(t) = \dfrac{3}{8}\Big(\dfrac{|u|}{1+|u|}+u+\dfrac{1}{2}\Big),\ t\in[0,1],\\
u(0)=\dfrac{1}{2}u(1/4),\quad u'(0)=0,\ u(1)+u'(1)=\displaystyle\int_0^1 u(s)d\mu(s).
\end{cases}
\tag{3.3}
$$

Here $n=3$, $\alpha=1$, $\beta=1$, $\delta=1/2$, $\xi=1/4$, *and* $f(t,u)=(3/8)\Big(\dfrac{|u|}{1+|u|}+u+\dfrac{1}{2}\Big)$ *and* $\mu(s)=s^2/2$. *Using the given data, we find that* $\ell=3/4$ *as* $|f(t,u)-f(t,v)| \leq (3/4)\|u-v\|$, $\vartheta \simeq 0.436189$, $\Lambda \simeq 1.390625$. *Obviously* $\ell\vartheta \simeq 0.327141854 < 1$. *Thus, all the conditions of Theorem 3.1 are satisfied. Hence, by the conclusion of Theorem* (3.1)*, problem* (3.3) *has a unique solution on* $[0,1]$.

Our next existence result is based on Schauder's fixed point theorem.

Theorem 3.3. *Assume that*

(A_1) $f:[0,1]\times\mathbb{R}\longrightarrow\mathbb{R}$ *is continuous;*

(A_2) *there exists a positive constant* M *such that* $|f(t,u)|\leq M$ *for each* $t\in[0,1]$ *and all* $u\in\mathbb{R}$.

Then the problem (2.1) has at least one solution on $[0,1]$.

Proof . We show that the operator F defined by (3.1) satisfies the hypotheses of Schauder's fixed point theorem. This will be done in several steps.
Step 1: F is continuous.
Let $\{u_k\}$ be a sequence such that $u_k \longrightarrow u$ in $C([0,1],\mathbb{R})$. Then, for each $t\in[0,1]$, we have

$$\begin{aligned}
& |F(u_k)(t)-F(u)(t)| \\
\leq\ & \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}|f(s,u_k(s))-f(s,u(s))|ds \\
& +|\sigma_1(t)|\Big[\int_0^1 (\int_0^s \frac{(s-g)^{n-1}}{(n-1)!}|f(g,u_k(g))-f(g,u(g))|dg)d\mu(s) \\
& +\int_0^1 \frac{(1-s)^{n-2}[\beta(n-1)+\alpha(1-s)]}{(n-1)!}|f(s,u_k(s))-f(s,u(s))|ds\Big] \\
& +|\sigma_2(t)|\int_0^\xi \frac{(\xi-s)^{n-1}}{(n-1)!}|f(s,u_k(s))-f(s,u(s))|ds \\
\leq\ & \|f(.,u_k(.))-f(.,u(.))\| \sup_{t\in[0,1]}\Big\{\int_0^t \frac{(t-s)^{n-1}}{(n-1)!}ds+|\sigma_1(t)|\Big[\int_0^1(\int_0^s \frac{(s-g)^{n-1}}{(n-1)!}dg)d\mu(s) \\
& +\int_0^1 \frac{(1-s)^{n-2}[\beta(n-1)+\alpha(1-s)]}{(n-1)!}ds\Big]+|\sigma_2(t)|\int_0^\xi \frac{(\xi-s)^{n-1}}{(n-1)!}ds\Big\} \\
\leq\ & \vartheta\|f(.,u_k(.))-f(.,u(.))\| \longrightarrow 0 \ as\ k\longrightarrow\infty,
\end{aligned}$$

in view of continuity of f (ϑ is given by (3.2).
Step 2: *F maps bounded sets into bounded sets in $C([0,1],\mathbb{R})$.*
Indeed, it is enough to show that for any $\eta^*>0$, there exists a positive constant L such that for each $u\in B_{\eta^*}=\{u\in C([0,1],\mathbb{R}): \|u\|\leq\eta^*\}$, we have $\|F(u)\|\leq L$. For each $t\in[0,1]$, by the condition (A_2), we have that

$$\begin{aligned}
|F(u)(t)| \leq\ & \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}|f(s,u(s))|ds \\
& +|\sigma_1(t)|\Big[\int_0^1 (\int_0^s \frac{(s-g)^{n-1}}{(n-1)!}|f(g,u(g))|dg)d\mu(s) \\
& +\int_0^1 \frac{(1-s)^{n-2}[\beta(n-1)+\alpha(1-s)]}{(n-1)!}|f(s,u(s))|ds\Big] \\
& +|\sigma_2(t)|\int_0^\xi \frac{(\xi-s)^{n-1}}{(n-1)!}|f(s,u(s))|ds.
\end{aligned}$$

Taking the norm for $t\in[0,1]$, the above inequality yields $\|F(u)\|\leq\vartheta L$, where ϑ is given by (3.2).
Step 3: *F maps bounded sets into equicontinuous sets of $C([0,1],\mathbb{R})$.*
Let $t_1,\ t_2\in(0,1),\ t_1<t_2$, and B_{η^*} be a bounded set in $C([0,1],\mathbb{R})$ as in Step 2. Then, for $u\in B_{\eta^*}$,

we have

$$\begin{aligned}
&|F(u)(t_2) - F(u)(t_1)| \\
\leq\ & \Big|\int_0^{t_1} \frac{[(t_2-s)^{n-1} - (t_1-s)^{n-1}]}{(n-1)!} f(s,u(s))ds + \int_{t_1}^{t_2} \frac{(t_2-s)^{n-1}}{(n-1)!} f(s,u(s))ds\Big| \\
&+|\Lambda(1-\delta)(t_2^{n-1} - t_1^{n-1})|\Big[\int_0^1 (\int_0^s \frac{(s-g)^{n-1}}{(n-1)!} f(g,u(g))dg)d\mu(s) \\
&+\int_0^1 \frac{(1-s)^{n-2}[\beta(n-1)+\alpha(1-s)]}{(n-1)!} f(s,u(s))ds\Big] \\
&+\Big|\Lambda\delta\Big(\alpha - \int_0^1 d\mu(s)\Big)(t_2^{n-1} - t_1^{n-1})\Big| \int_0^\xi \frac{(\xi-s)^{n-1}}{(n-1)!} f(s,u(s))ds. \\
\leq\ & M\Big\{\Big|\int_0^{t_1} \frac{[(t_2-s)^{n-1} - (t_1-s)^{n-1}]}{(n-1)!} ds + \int_{t_1}^{t_2} \frac{(t_2-s)^{n-1}}{(n-1)!} ds\Big| \\
&+|\Lambda(1-\delta)(t_2^{n-1} - t_1^{n-1})|\Big[\int_0^1 (\int_0^s \frac{(s-g)^{n-1}}{(n-1)!} dg)d\mu(s) \\
&+\int_0^1 \frac{(1-s)^{n-2}[\beta(n-1)+\alpha(1-s)]}{(n-1)!} ds\Big] \\
&+\Big|\Lambda\delta\Big(\alpha - \int_0^1 d\mu(s)\Big)(t_2^{n-1} - t_1^{n-1})\Big| \int_0^\xi \frac{(\xi-s)^{n-1}}{(n-1)!} ds\Big\}. \\
\leq\ & \frac{M}{n!}[2(t_2-t_1)^n + |t_2^n - t_1^n|] \\
&+M|\Lambda(1-\delta)(t_2^{n-1} - t_1^{n-1})|\Big[\int_0^1 \frac{s^n}{n!} d\mu(s) + \frac{(\beta n+\alpha)}{n!}\Big] \\
&+\frac{M\xi^n}{n!}\Big|\Lambda\delta\Big(\alpha - \int_0^1 d\mu(s)\Big)(t_2^{n-1} - t_1^{n-1})\Big|.
\end{aligned}$$

Clearly the right hand side of the above inequality tends to zero independent of u as $(t_2 - t_1) \longrightarrow 0$. In view of the above three Steps, the Arzelá-Ascoli theorem applies and consequently the operator $F : C([0,1],\mathbb{R}) \longrightarrow C([0,1],\mathbb{R})$ is continuous and completely continuous.

Step 4: *A priori bounds.*

We show that the set $\varepsilon = \{u \in C([0,1],\mathbb{R}) : u = \lambda F(u)$ for some $0 < \lambda < 1\}$ is bounded. Let $u \in \varepsilon$. Then $u = \lambda F(u)$ for some $0 < \lambda < 1$. Thus, for each $t \in [0,1]$, we have

$$\begin{aligned}
u(t) \quad &= \lambda\Big\{\int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s,u(s))ds + \sigma_1(t)\Big[\int_0^1 (\int_0^s \frac{(s-g)^{n-1}}{(n-1)!} f(g,u(g))dg)d\mu(s) \\
&-\int_0^1 \frac{(1-s)^{n-2}[\beta(n-1)+\alpha(1-s)]}{(n-1)!} f(s,u(s))ds\Big] \\
&+\sigma_2(t)\int_0^\xi \frac{(\xi-s)^{n-1}}{(n-1)!} f(s,u(s))ds\Big\}.
\end{aligned}$$

Using the condition (A_2), it is easy to show that $\|F(u)\| \leq M\vartheta$. This shows that set ε is bounded. Thus, it follows by Schauder's fixed point theorem that the operator F has a fixed point, which is a solution of the problem (2.1). □

Example 3.4. *Consider the following nonlinear boundary value problem*

$$\begin{cases} u^{(4)}(t) = \dfrac{e^{-u^2(t)} + 2\sin(1+3u(t)) + \cos(3+5u^3(t)) + 3t^4}{1+u^2(t)}, \ t \in [0,1], \\ u(0) = u(1/3), \ u'(0) = 0, \ u''(0) = 0, \ \dfrac{2}{3}u(1) + \dfrac{3}{4}u'(1) = \displaystyle\int_0^1 u(s)d\mu(s). \end{cases} \tag{3.4}$$

Here $f(t,u(t)) = \dfrac{e^{-u^2(t)} + 2\sin(1+3u(t)) + \cos(3+5u^3(t)) + 3t^4}{1+u^2(t)}$. *Clearly* $f(t,u(t))$ *is continuous and* $|f(t,u(t))| \le M$ *with* $M = 7$ *for each* $t \in [0,1]$ *and all* $u \in \mathbb{R}$. *Thus the conclusion of Theorem 3.3 applies and the problem* (3.4) *has a solution on* $[0,1]$.

4. Some related problems

In this section, we study two more *nth*-order boundary value problems involving Stieltjes type integral boundary conditions.

4.1. Problem I

We consider the following *nth*-order ordinary differential equation

$$u^{(n)}(t) = f(t,u(t)), \ \ t \in [0,1], \tag{4.1}$$

supplemented with the Stieltjes integral boundary conditions:

$$\begin{cases} u(0) = \delta \displaystyle\int_0^\xi u(s)d\tau(s), \quad u'(0) = 0, \quad u''(0) = 0, \dots, u^{(n-2)}(0) = 0, \\ \alpha u(1) + \beta u'(1) = \displaystyle\sum_{i=1}^m \gamma_i \int_0^{\beta_i} u(s)d\psi(s), \qquad 0 < \xi < \beta_1 < 1, \end{cases} \tag{4.2}$$

where $f : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a given continuous function, $\alpha, \ \beta, \ \gamma_i, \ \delta, \ \xi, \ \beta_i \ (i = 1, 2, \dots, m)$ are real constants to be chosen appropriately, $\tau(s)$ and $\psi(s)$ are functions of bounded variation.

Observe that the problem (4.1)-(4.2) differs from the problem (2.1) in the sense that it considers the boundary conditions:

$$u(0) = \delta \int_0^\xi u(s)d\tau(s), \ \alpha u(1) + \beta u'(1) = \sum_{i=1}^m \gamma_i \int_0^{\beta_i} u(s)d\psi(s)$$

instead of the following conditions assumed in the problem (2.1):

$$u(0) = \delta u(\xi), \quad \alpha u(1) + \beta u'(1) = \int_0^1 u(s)d\mu(s),$$

whereas the other conditions remain the same.

Lemma 4.1. *The unique solution of problem* (4.1)-(4.2) *is equivalent to the integral equation:*

$$\begin{aligned} u(t) &= \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s,u(s))ds \\ &+\lambda_1(t)\Big[\sum_{i=1}^m \gamma_i \int_0^{\beta_i} \Big(\int_0^s \frac{(s-x)^{n-1}}{(n-1!)} f(x,u(x))dx\Big) d\psi(s) \\ &- \int_0^1 \frac{(1-s)^{n-2}[\beta(n-1)+\alpha(1-s)]}{(n-1)!} f(s,u(s))ds\Big] \\ &+\lambda_2(t)\int_0^{\xi}\Big(\int_0^s \frac{(s-x)^{n-1}}{(n-1!)} f(x,u(x))dx\Big)d\tau(s), \end{aligned} \tag{4.3}$$

where

$$\lambda_1(t) = \frac{1}{\chi}\Big[\delta\int_0^{\xi} s^{n-1}d(\tau(s)) + t^{n-1}\Big(1-\delta\int_0^{\xi}d\tau(s)\Big)\Big], \tag{4.4}$$

$$\begin{aligned} \lambda_2(t) &= \frac{1}{\chi}\delta\Big[\alpha+(n-1)\beta-\sum_{i=1}^m\gamma_i\int_0^{\beta_i}s^{n-1}d\psi(s) \\ &- t^{n-1}\Big(\alpha-\Sigma_{i=1}^m\gamma_i\int_0^{\beta_i}d\psi(s)\Big)\Big], \end{aligned} \tag{4.5}$$

$$\begin{aligned} \chi &= \Big(\alpha-\sum_{i=1}^m\gamma_i\int_0^{\beta_i}d\psi(s)\Big)\Big(\delta\int_0^{\xi}s^{n-1}d\tau(s)\Big) \\ &- \Big(\alpha+(n-1)\beta-\sum_{i=1}^m\gamma_i\int_0^{\beta_i}s^{n-1}d\psi(s)\Big)\Big(1-\delta\int_0^{\xi}d\tau(s)\Big)\neq 0. \end{aligned} \tag{4.6}$$

Proof . The proof is similar to that of Lemma 2.1. So we omit it. □

By Lemma 4.1, we consider a fixed point problem associated with the problem (4.1)-(4.2) as $Gu = u$, where the operator $G : C([0,1],\mathbb{R}) \longrightarrow C([0,1],\mathbb{R})$ is defined by

$$\begin{aligned} (Gu)(t) &= \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s,u(s))ds \\ &+\lambda_1(t)\Big[\sum_{i=1}^m \gamma_i \int_0^{\beta_i} (\int_0^s \frac{(s-x)^{n-1}}{(n-1)!} f(x,u(x))dx) d\psi(s) \\ &- \int_0^1 \frac{(1-s)^{n-2}[\beta(n-1)+\alpha(1-s)]}{(n-1)!} f(s,u(s))ds\Big] \\ &+\lambda_2(t)\int_0^{\xi}(\int_0^s \frac{(s-x)^{n-1}}{(n-1)!} f(x,u(x))dx)d\tau(s). \end{aligned} \tag{4.7}$$

Moreover, we set

$$\vartheta_I = \Big\{\frac{1}{n!} + m_1\Big[\sum_{i=1}^m\gamma_i\int_0^{\beta_i}\frac{s^n}{n!}d\psi(s) + \frac{(\beta n+\alpha)}{n!}\Big] + m_2\int_0^{\xi}\frac{s^n}{n!}d\tau(s)\Big\}, \tag{4.8}$$

where $\max_{t\in[0,1]}|\lambda_1(t)| = m_1$, $\max_{t\in[0,1]}|\lambda_2(t)| = m_2$ (λ_1 and λ_2 are respectively given by (4.4) and (4.5)).

Following the method of proof used for obtaining the two existence results for the problem (2.1) in the previous section, we can establish the similar results for the problem (4.1)-(4.2) with the aid of the operator G defined by (4.7) and the constant ϑ_I given by (4.8). The existence results for the problem (4.1)-(4.2) can be formulated as follows.

Theorem 4.2. *Let $f : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function satisfying the Lipschitz condition: $|f(t,u) - f(t,v)| \leq \ell_1 |u - v|,\ \ell_1 > 0,\ \forall\ u,\ v \in \mathbb{R},\ t \in [0,1]$. Then there exists a unique solution for the problem (4.1)-(4.2) on $[0,1]$ if $\ell_1 \vartheta_I < 1$, where ϑ_I is given by (4.8).*

Theorem 4.3. *Assume that the function $f : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and there exists a positive constant M_1 such that $|f(t,u)| \leq M_1$ for each $t \in [0,1]$ and for all $u \in \mathbb{R}$. Then the problem (4.1)-(4.2) has at least one solution on $[0,1]$.*

Example 4.4. *Consider the third-order boundary value problem given by*

$$\begin{cases} u'''(t) = f(t,u(t)), \quad t \in [0,1], \\ u(0) = \dfrac{1}{2}\displaystyle\int_0^{\xi} u(s)d\tau(s), \quad u'(0) = 0, \quad u(1) + u'(1) = \sum_{i=1}^{4} \gamma_i \int_0^{\beta_i} u(s)d\psi(s). \end{cases} \tag{4.9}$$

Here $n = 3,\ \alpha = 1,\ \beta = 1,\ \delta = 1/2,\ \xi = 1/4,\ m = 4,\ \beta_1 = 1/3,\ \beta_2 = 1/2, \beta_3 = 2/3,\ \beta_4 = 3/4,\ \gamma_1 = 3/4,\ \gamma_2 = 2/3,\ \gamma_3 = 1/2,\ \gamma_4 = 1/3,\ f(t,u) = \dfrac{2|u|}{3(1+|u|)} + \dfrac{u}{3} + e^t,\ \tau(s) = s$, and $\psi(s) = s^2$. Using the given values, it is found that $\chi \simeq 1.083833, \vartheta_I \simeq 0.715664$ and $\ell_1 = 1$ as $|f(t,u) - f(t,v)| \leq \|u - v\|$. Obviously $\ell_1 \vartheta_I \simeq 0.715664 < 1$. Thus, all the conditions of Theorem (4.2) are satisfied. In consequence, by the conclusion of Theorem (4.2), the problem (4.9) has a unique solution on $[0,1]$.

4.2. Problem II

We replace the condition '$\alpha u(1)+\beta u'(1) = \sum_{i=1}^{m} \gamma_i \int_0^{\beta_i} u(s)d\psi(s)$' by '$\alpha u(\eta)+\beta u'(\eta) = \int_\zeta^1 u(s)d\rho(s)$' in (4.2) and consider the following problem:

$$\begin{cases} u^{(n)}(t) = f(t,u(t)), \qquad t \in [0,1], \\ u(0) = \delta \displaystyle\int_0^{\xi} u(s)d\mu(s), \quad u'(0) = 0, \quad u''(0) = 0, \ldots, \quad u^{(n-2)}(0) = 0, \\ \alpha u(\eta) + \beta u'(\eta) = \displaystyle\int_\zeta^1 u(s)d\rho(s), \quad 0 < \xi < \eta < \zeta < 1, \end{cases} \tag{4.10}$$

where $f : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a given continuous function, and $\alpha,\ \beta,\ \delta$ are real constants to be chosen appropriately, $\mu(s)$ and $\rho(s)$ are functions of bounded variation.
As before, associated with the problem (4.10), we define an operator $H : C([0,1],\mathbb{R}) \longrightarrow C([0,1],\mathbb{R})$ as

$$\begin{aligned} (Hu)(t) \ &= \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s,u(s))ds + \nu_1(t) \int_0^{\xi} (\int_0^s \frac{(s-x)^{n-1}}{(n-1)!} f(x,u(x))dx) d\mu(s) \\ &+\nu_2(t) \Big[\int_\zeta^1 \Big(\int_0^s \frac{(s-x)^{n-1}}{(n-1)!} f(x,u(x))dx \Big) d\rho(s) \\ &-\alpha \int_0^{\eta} \frac{(\eta-s)^{n-1}}{(n-1)!} f(s,u(s))ds - \beta \int_0^{\eta} \frac{(\eta-s)^{n-2}}{(n-2)!} f(s,u(s))ds \Big], \end{aligned} \tag{4.11}$$

where

$$\begin{aligned}
\nu_1(t) &= \frac{\delta}{\gamma}\Big[\alpha\eta^{n-1} + \beta(n-1)\eta^{n-2} - \int_\zeta^1 s^{n-1}d\rho(s) - t^{n-1}\Big(\alpha - \int_\zeta^1 d(\rho(s))\Big)\Big] \\
\nu_2(t) &= \frac{1}{\gamma}\Big[\delta\int_0^\xi s^{n-1}d\mu(s) + t^{n-1}\Big(1 - \delta\int_0^\xi d\mu(s)\Big)\Big], \\
\gamma &= \Big(\alpha\eta^{n-1} + \beta(n-1)\eta^{n-2} - \int_\zeta^1 s^{n-1}d\rho(s)\Big)\Big(1 - \delta\int_0^\xi d\mu(s)\Big) \\
&+ \Big(\alpha - \int_\zeta^1 d\rho(s)\Big)\Big(\delta\int_0^\xi s^{n-1}d\mu(s)\Big) \neq 0. \qquad (4.12)
\end{aligned}$$

Notice that the problem (4.10) has solutions only if the operator equation $Hu = u$ has fixed points. In the sequel, we use the notation:

$$Q_{II} = \Big\{\frac{1}{n!} + n_1\int_0^\xi \frac{s^n}{n!}d\mu(s) + n_2\Big[\int_\zeta^1 \frac{s^n}{n!}d(\rho(s)) + \frac{\alpha\eta^n}{n!} + \frac{\beta\eta^{n-1}}{(n-1)!}\Big]\Big\}. \qquad (4.13)$$

where $\max_{t\in[0,1]}|\nu_i(t)| = n_i,\ i = 1, 2.$

Now we present the existence results for the problem (4.10). The method of proof for these results is similar to the one employed in Section 3, so we omit the proofs.

Theorem 4.5. *Assume that the function* $f : [0,1]\times\mathbb{R} \longrightarrow \mathbb{R}$ *is continuous and satisfies the Lipschitz condition:* $|f(t,u) - f(t,v)| \le \ell_2|u - v|,\ \ell_2 > 0,\ \forall\ u,\ v \in \mathbb{R},\ \ell_2 > 0\ t \in [0,1]$. *Then the problem* (4.10) *has a unique solution on* $[0,1]$ *provided that* $\ell_2 Q_{II} < 1$, *where* Q_{II} *is given by* (4.13).

Theorem 4.6. *Assume that the following conditions hold:*

(B_1) *the function* $f : [0,1]\times\mathbb{R} \longrightarrow \mathbb{R}$ *is continuous;*

(B_2) *there exists a positive constant* N *such that* $|f(t,u)| \le N$ *for each* $t \in [0,1]$ *and for all* $u \in \mathbb{R}$.

Then there exists at least one solution for the problem (4.10) *on* $[0,1]$.

Example 4.7. *Let us consider the problem* (4.10) *with* $n = 3,\ \alpha = 1,\ \beta = 1,\ \delta = 1/2,\ \xi = 1/4,\ \zeta = 1/3, \eta = 3/4,$ *and* $f(t,u) = \dfrac{2e^t|u|}{5(1+|u|)} + 3,\ \mu(s) = \dfrac{s^2}{2}$ *and* $\rho(s) = \dfrac{s^3}{3}$.
Using the given data, we find that $\gamma \simeq 1.834539$, $Q_{II} \simeq 0.370316$ *and* $\ell_2 = 2/5$ *as* $|f(t,u) - f(t,v)| \le (2/5)\|u - v\|$. *Clearly*

$$\ell_2 Q_{II} \simeq 0.148126 < 1.$$

Since all the conditions of Theorem 4.5 are satisfied, therefore, the conclusion of Theorem 4.5 applies and the problem (4.10) *with the chosen data has a unique solution on* $[0,1]$.

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