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E-Bayesian Estimation of Parameters of Inverse Weibull Distribution based on a Unified Hybrid Censoring Scheme

Shahram Yaghoobzadeh Shahrestani[a], Reza Zarei[b], Parviz Malekzadeh[c,*]

[a]*Department of Statistics, Payame Noor University, P. O. Box 19395-4697, Tehran, Iran*
[b]*Department of Statistics, Faculty of Mathematical Sciences, University of Guilan, Rasht, Iran*
[c]*Department of Statistics, Faculty of Mathematics, Statistics and Computer Science, Semnan University, Semnan, Iran*

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Abstract

The combination of generalization Type-I hybrid censoring and generalization Type-II hybrid censoring schemes create a new censoring called a unified hybrid censoring scheme. Therefore, in this study, the E-Bayesian estimation of parameters of the inverse Weibull distribution is obtained under the unified hybrid censoring scheme, and the efficiency of the proposed method was compared with the Bayesian estimator using Monte Carlo simulation and a real data set.

Keywords: E-Bayesian estimation, Unified hybrid censoring scheme, Inverse Weibull distribution, LINEX loss function
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1. Introduction

One of the most popular distribution in modeling and analyzing the life time data is Weibull distribution because of the flexibility of the probability density function (pdf) and failure rate function. For see some applications of the Weibull distribution, readers may refer to [8, 9]. Depending on the value of the shape parameter, its pdf can be decreasing or uni-modal and the failure rate function can be either decreasing or increasing. Therefore, in real data analysis, when the data indicate a monotone hazard function, the Weibull distribution has been extensively used. However, in some

*Corresponding author
Email addresses: yagoubzade@gmail.com (Shahram Yaghoobzadeh Shahrestani), r.zarei@guilan.ac.ir (Reza Zarei), pmalekzadeh@semnan.ac.ir (Parviz Malekzadeh)

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real data sets, the analyzing data indicate a non-monotone hazard function. In such situations, use of Weibull distribution is inappropriate. The inverse Weibull distribution (denoted by IWD) model has been derived as a suitable model for describing the degradation phenomena of mechanical components, such as the dynamic components of diesel engines, see for example [18]. The physical failure process given by [7] also leads to the IW model. They showed that the IW model provides a good fit to survival data such as the times to breakdown of an insulating fluid subject to the action of constant tension, see also [19]. Interpretation of IW distribution in the context of load strength relationship for a component was provided by [2].

In reliability engineering research, IW distribution is often used in statistical analysis of life time and response time data. [15] in their theoretical analysis of IW distribution mention that numerous failure characteristics such aswear out periods and infant mortality can be modeled through IW distribution. They mention about the wide range of areas in reliability analysis where IW distribution model can be used successfully. [20] mention that IW distribution is an appropriate model for situations where hazard function is unimodal. They further mention the distribution as one of the popular distributions in complementary risk problems.

As the authors known, the Bayes estimators of parameters of inverse Weibull distribution firstly was introduced by [16]. They proposed the Bayes estimator based on complete, type I and type II censored samples under general entropy and squared error loss functions. The hierarchical Bayesian prior distribution was primarily introduced by [17]. Then, it was examined by [10] and expected Bayesian (denoted by E-Bayesian) and hierarchical Bayesian (denoted by H-Bayesian) approaches were introduced. Recently, E-Bayesian and H-Bayesian methods have been used by [11, 12] to estimate the parameter of exponential distribution and reliability of the binomial distribution. In this context, [14] estimate the reliability of the Type 12 distribution based on Type II progressive censoring samples. Estimation of parameters in Pascal distribution was studied by [24] and [25]. [26] using these approaches in order to estimate the scale parameter of Gompertz distribution under type II censoring schemes based on fuzzy data. Also, [13] gives the property of E-Bayesian estimation and H-Bayesian estimation of the system reliability parameter.

In Bayesian estimation approach, the loss functions have a key role. Linear-Exponential loss function (known as the LINEX) is a useful asymmetric loss function was introduced by [22]. This function rises approximately exponentially on one side of zero and approximately linearly on the other side. Note that in especial case, the squared-error loss function can be obtained as a particular member of the LINEX loss function for a specific choice of the loss function parameter. Since the estimators under asymmetric loss function involve integral expressions, which are not analytically solvable, the Bayesian estimation under this type of loss function is not frequently discussed. Therefore, one has to use the numerical quadrature techniques or certain approximation methods for the solutions. Lindley's approximation technique is one of the methods suitable for solving such problems.

In the present work, the E-Bayesian estimation parameters of IW distribution is studies based on the unified hybrid censored samples under square error and LINEX loss functions. This paper is organized as follows: First, in Section 2, we recall the concept of IW distribution and then formulated the problem. Then, in Section 3, we investigated the E-Bayesian estimation of parameters in IW distribution. Simulation results and an numerical example are prsented in Section 4. Finally, a brief conclusion presented in Section 5.

2. Mathematical formulation

Suppose that the random variable Y has a Weibull distribution with the *pdf*

$$f\left(y;\alpha,\lambda\right)=\alpha\lambda y^{\alpha-1}e^{-\lambda y^{\alpha}},\quad y>0.$$

Then, the random variable $X = \frac{1}{Y}$ has an Inverse Weibull distribution with the *pdf*

$$f(x; \alpha, \lambda) = \alpha\lambda x^{-(\alpha+1)} e^{-\lambda x^{-\alpha}}, \quad x > 0. \tag{2.1}$$

The quantities $\alpha > 0$ and $\lambda > 0$ are the shape and scale parameters, respectively. From now on it will be denoted by IW(α, λ). If X follow IW(α, λ), then the distribution function of X is given by

$$F(x; \alpha, \lambda) = e^{-\lambda x^{-\alpha}}, \quad x > 0. \tag{2.2}$$

Consider a lifetime test with n units. Suppose that the units have independent and identically lifetime with the probability density function $f(x;\ \theta)$ and the cumulative distribution function $F(x;\ \theta)$, and $Y_{1:n} < \cdots < Y_{n:n}$ are the lifetime of the units until their failure. For the first time, [6] investigated a scheme in a survival experiment in which the experiment ended at time $T^* = \min(Y_{r:n}\ ,\ T)$ and the values of T and r were pre-determined. [4] called this Type-I hybrid censoring. In this scheme, there may be very few failures up to time T. They investigated a scheme in which the experiment ended at time $T^* = \max(Y_{r:n}\ ,\ T)$. This scheme was called the Type-II hybrid censoring scheme. Obviously, this scheme does not have the problem of the previous scheme. Even before time T, all units can failure, but the time to test is not predictable. [3] introduced two Types of generalization hybrid censoring of Type I and II, so that the problem has somewhat improved the previous two schemes (not having the minimum failure in the Type-I hybrid censoring scheme and prolonging the test time in the Type-II hybrid censoring scheme).

In generalization Type-I hybrid censoring scheme, suppose $T \in (0,\ \infty)$ and the values of k and r such that $k < r$ are predetermined. If the k^{th} failure occurs before time T, the experiment at $\min(Y_{r:n}\ , T)$ and if, after time T, the experiment ends at $Y_{k:n}$. Therefore, this scheme guarantees at least k failures. In general Type-II hybrid censoring scheme, assume that r and $T_1,\ T_2 \in (0,\ \infty)$, so that $T_1 < T_2$, are constant and predetermined values. If the r^{th} failure occurs before time T_1, the experiment at time T_1, if between T_1 and T_2, occurs at time $Y_{r:n}$, and if after T_2, the experiment ends at T_2. Therefore, this scheme guarantees that the experiment ends up at time T_2.

The combination of the above scheme creates a new censoring called a unified hybrid censoring scheme. This scheme was first introduced by [1]. In this scheme, the values T_1, T_2, r, and k, so that $T_1 < T_2$ and $k < r$, are predetermined before the experiment begins. If the k^{th} failure occurs before time T_1, the experiment at time $\min(\max(Y_{r:n},\ T_1)\ ,\ T_2)$, if between T_1 and T_2, occurs at time $\min(Y_{r:n}\ , T_2)$, and if after T_2, the experiment ends at $Y_{k:n}$. In this censoring, one of the following six occurrences occurs. Suppose that for j = 1,2, d_j the number of failures is up to T_j. In this case, we have six typs of observations.

1. If $0 < Y_{k:n} < Y_{r:n} < T_1 < T_2$, the experiment ends at time T_1 with D failures.
2. If $0 < Y_{k:n} < T_1 < Y_{r:n} < T_2$, the experiment ends with the failure of r^{th}.
3. If $0 < Y_{k:n} < T_1 < T_2 < Y_{r:n}$, the experiment ends at time T_2 with d_2 failures.
4. If $0 < T_1 < Y_{k:n} < Y_{r:n} < T_2$, the experiment ends at time $Y_{r:n}$.
5. If $0 < T_1 < Y_{k:n} < T_2 < Y_{r:n}$, experiment ends at time T_2 with d_2 failures.
6. If $0 < T_1 < T_2 < Y_{k:n} < Y_{r:n}$, The experiment ends with the failure of k^{th}.

Note that in the first case, $d_1 = d_2 = D$, $T_1 < Y_{(D+1):n}$ and $r \le D$, so that the experiment $(D+1)^{\text{th}}$ does not occur before T_1, and in the third and fifth cases, $T_2 < Y_{(d_2+1):n}$ and $k \le d_2$ are such that the $(d_1+1)^{\text{th}}$ experiment does not occur before T_2. If c is the stopping point and d is the number of failures until time c, the likelihood function of this hybrid censored observed sample is

$$L(\theta \mid \boldsymbol{y}) = \frac{n!}{(n-d)!} \prod_{i=1}^{d} f(y_{i:n};\ \theta)[1 - F(c)]^{n-d} \tag{2.3}$$

where $\boldsymbol{y} = (y_{1:n},\ \ldots,\ y_{d:n})$, $d\epsilon\{D,\ d_1,\ d_2,\ k,\ r\}$, and $c\epsilon\{T_1,\ T_2,\ Y_{r:n},\ Y_{k:n}\ \}$.

In this paper, we aim to proposed the E-Bayesian estimation for parameters of IW distribution based on these unified censoring schems.

3. The Bayes and E-Bayes Estimation of the Parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\lambda}$

In this section, we have obtained the Bayes and E-Bayes estimators of the parameters α and λ under the symmetric and asymmetric loss functions. Squared error loss (SEL) is the common used loss function defined as

$$L(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2, \tag{3.1}$$

where $\hat{\theta}$ is the estimator of the parameter θ. It is well known that the Bayes estimator under the above symmetric loss function is the posterior mean.

The LINEX loss function with parameter a and k is defined by

$$L(\hat{\theta}, \theta) = k\{e^{a(\hat{\theta}-\theta)} - a(\hat{\theta} - \theta) - 1\}, \tag{3.2}$$

where $\hat{\theta}$ is the estimator of the parameter θ. The Bayes estimator under LINEX loss function is given by

$$\hat{\theta}_{BL} = -\frac{1}{k} lnE_{\theta}(e^{k\theta}), \tag{3.3}$$

where E_θ stands for posterior expectation.

Suppose that $Y_{1:n}, ..., Y_{n:n}$ be a random sample based on unified hybrid censored schems and are identical to the probability density function 2.1. In order to using the Bayesian approach, we need prior distribution for the parameters α and λ. It is assumed that the parameters α and λ have independent prior distributions $Gamma(a_1, b_1)$ and $Gamma(a_2, b_2)$. Based on above assumptions, we have that

$$\pi_1 (\alpha | a_1, b_1) = \frac{b_1^{a_1}}{\Gamma(a_1)} \alpha^{a_1 - 1} e^{-b_1 \alpha}, \quad \alpha > 0,$$
$$\pi_2 (\lambda | a_2, b_2) = \frac{b_2^{a_2}}{\Gamma(a_2)} \lambda^{a_2 - 1} e^{-b_2 \lambda}, \quad \lambda > 0.$$

where a_1, b_1, a_2 and b_2 are positive and known values.

Remark 3.1. *According to Han (1997), a_1 and b_1 should be chosen to guarantee that $\pi(\alpha \mid a_1, b_1)$ is a decreasing function of α. The derivative of $\pi(\alpha \mid a_1, b_1)$ with respect to α is*

$$\frac{d\pi(\alpha \mid a_1, b_1)}{d\alpha} = \frac{b_1^{a_1} \alpha^{a_1-2} e^{-b_1\alpha}}{\Gamma(a_1)} \left((a_1 - 1) - b_1 \alpha\right).$$

Thus, $b_1 > 0$ and $0 < a_1 < 1$. Given $a_1 = 1$, and the larger the value of b_1, the thinner the tail of the density function is. Berger (1985) showed that the thinner tailed prior distribution often reduces the robustness of the Bayesian estimation. Consequently, the hyperparameter b_1 should be chosen under the restriction $0 < b_1 < c_1$, where c_1 is a given upper bound (c_1 is a positive constant).

In this study, we only consider the case when $a_1 = 1$. In this case, the density function $\pi(\alpha \mid a_1, b_1)$ becomes

$$\pi(\alpha \mid b_1) = b_1 e^{-b_1\alpha}, \quad \alpha > 0.$$

Also, we consider the perior distribution b_1 as $\pi(b_1) = \frac{1}{c_1}$, $0 < b_1 < c_1$.
As the same way, $\pi(\lambda \mid a_2, b_2)$ becomes

$$\pi(\lambda \mid b_2) = b_2 e^{-b_2\lambda}, \quad \lambda > 0,$$

and $\pi(b_2) = \frac{1}{c_2}$, $0 < b_2 < c_2$.
Let $h(\alpha, \lambda)$ be a function of α and λ. Therefore, the Bayesian estimator under square error loss function is given by

$$\hat{h} = E(h(\alpha,\lambda)|\boldsymbol{y}) = \frac{\int_0^\infty \int_0^\infty h(\alpha,\lambda) M(\alpha,\lambda|\boldsymbol{y}) d\alpha d\lambda}{\int_0^\infty \int_0^\infty M(\alpha,\lambda|\boldsymbol{y}) d\alpha d\lambda}, \tag{3.4}$$

where

$$M(\alpha, \lambda \mid \boldsymbol{y}) \propto \alpha^d \left(\prod_{i=1}^d y_{i:n}\right)^{-(\alpha+1)} e^{-b_1\alpha} \sum_{j=0}^{n-d} \frac{(-1)^j}{j!\,(n-d-j)!} f_{\lambda|\alpha}(a_1^*, b_1^*(j)),$$

and $f_{\lambda|\alpha}(a_1^*, b_1^*(j))$ is the density of Gamma distribution with shape parameter $a_1^* = d+1$ and scale parameter

$$b_1^*(j) = b_2 + \sum_{i=1}^d y_{i:n}^{-\alpha} + jc^{-\alpha}.$$

Now, suppose that $h(\alpha, \lambda) = \alpha$ in Equation 3.4. Thus, the Bayesian estimations for the α under square loss function, denoted by $\widehat{\alpha}_{BS}(b_1, b_2)$, is given by

$$\widehat{\alpha}_{BS}(b_1,b_2) = \frac{\sum_{j=0}^{n-d} \frac{(-1)^j}{j!(n-d-j)!} \int_0^\infty \left(\frac{\alpha}{b_1^*(j)}\right)^{d+1} \left(\prod_{i=1}^d y_{i:n}\right)^{-(\alpha+1)} e^{-b_1\alpha} d\alpha}{d! \sum_{j=0}^{n-d} \frac{(-1)^j}{j!(n-d-j)!} \int_0^\infty \frac{\alpha^d}{b_1^*(j)^{d+1}} \left(\prod_{i=1}^d y_{i:n}\right)^{-(\alpha+1)} e^{-b_1\alpha} d\alpha}, \tag{3.5}$$

Also, the Bayesian estimation of parameter λ under square loss function, $\widehat{\lambda}_{BS}(b_1, b_2)$, is given by the following equation by replacing $h(\alpha, \lambda) = \lambda$ in Equation 3.4

$$\widehat{\lambda}_{BS}(b_1,b_2) = \frac{(d+1) \sum_{j=0}^{n-d} \frac{(-1)^j}{j!(n-d-j)!} \int_0^\infty \frac{\alpha^d}{b_1^*(j)^{d+2}} \left(\prod_{i=1}^d y_{i:n}\right)^{-(\alpha+1)} e^{-b_1\alpha} d\alpha}{\sum_{j=0}^{n-d} \frac{(-1)^j}{j!(n-d-j)!} \int_0^\infty \frac{\alpha^d}{b_1^*(j)^{d+1}} \left(\prod_{i=1}^d y_{i:n}\right)^{-(\alpha+1)} e^{-b_1\alpha} d\alpha}. \tag{3.6}$$

In similar way, the Bayesian estimations for the α and λ, under LINEX loss function, denoted by $\hat{\alpha}_{BL}$ and $\hat{\lambda}_{BL}$, respectively, are as follows

$$\hat{\alpha}_{BL}(b_1,b_2) = -\frac{1}{k} \log \left[\frac{\sum_{j=0}^{n-d} \frac{(-1)^j}{j!(n-d-j)!} \int_0^\infty \left(\frac{\alpha}{b_1^*(j)}\right)^{d+1} \left(\prod_{i=1}^d y_{i:n}\right)^{-(\alpha+1)} e^{-b_1\alpha} d\alpha}{d! \sum_{j=0}^{n-d} \frac{(-1)^j}{j!(n-d-j)!} \int_0^\infty \frac{\alpha^d}{b_1^*(j)^{d+1}} \left(\prod_{i=1}^d y_{i:n}\right)^{-(\alpha+1)} e^{-b_1\alpha} d\alpha} \right], \tag{3.7}$$

$$\hat{\lambda}_{BL}(b_1,b_2)=$$
$$-\frac{1}{k}\log\left[\frac{(d+1)\sum_{j=0}^{n-d}\frac{(-1)^j}{j!(n-d-j)!}\int_0^\infty\frac{\alpha^d}{b_1^*(j)^{d+2}}\left(\prod_{i=1}^d y_{i:n}\right)^{-(\alpha+1)}e^{-b_1\alpha}d\alpha}{\sum_{j=0}^{n-d}\frac{(-1)^j}{j!(n-d-j)!}\int_0^\infty\frac{\alpha^d}{b_1^*(j)^{d+1}}\left(\prod_{i=1}^d y_{i:n}\right)^{-(\alpha+1)}e^{-b_1\alpha}d\alpha}\right]. \quad (3.8)$$

The following definition was originally proposed by [11] for E-Bayesian estimation.

Definition 3.2. *With $\widehat{\theta}_B\,(b_1,b_2)$ being continuous*

$$\widehat{\theta}_{EB}=\int\int_D\widehat{\theta}_B\,(b_1,b_2)\,\pi\,(b_1,b_2)db_1db_2$$

is called the E-Bayesian estimation of θ which is assumed to be finite, where D is the domain of b_1 and b_2, $\widehat{\theta}_B\,(b_1,b_2)$ is the Bayesian estimation of θ with hyperparameters b_1 and b_2, and $\pi\,(b_1,b_2)$ is the density function of b_1 and b_2 over D.

Definition 3.2 indicates that the E-Bayesian estimation of θ is just the expectation of the Bayesian estimation of θ for all the hyperparameters.

Therefore, with respect to Equations 3.5 and 3.6 and Definition 3.2, the E-Bayesian estimation of parameters α and λ under square loss function are obtained respectively as follows

$$\widehat{\alpha}_{EBS}=$$
$$\frac{1}{c_1c_2}\int_0^{c_1}\int_0^{c_2}\left(\frac{\sum_{j=0}^{n-d}\frac{(-1)^j}{j!(n-d-j)!}\int_0^\infty\left(\frac{\alpha}{b_1^*(j)}\right)^{d+1}\left(\prod_{i=1}^d y_{i:n}\right)^{-(\alpha+1)}e^{-b_1\alpha}d\alpha}{d!\sum_{j=0}^{n-d}\frac{(-1)^j}{j!(n-d-j)!}\int_0^\infty\frac{\alpha^d}{b_1^*(j)^{d+1}}\left(\prod_{i=1}^d y_{i:n}\right)^{-(\alpha+1)}e^{-b_1\alpha}d\alpha}\right)db_1db_2, \quad (3.9)$$

$$\widehat{\lambda}_{EBS}=$$
$$\frac{1}{c_1c_2}\int_0^{c_1}\int_0^{c_2}\left(\frac{(d+1)\sum_{j=0}^{n-d}\frac{(-1)^j}{j!(n-d-j)!}\int_0^\infty\frac{\alpha^d}{b_1^*(j)^{d+2}}\left(\prod_{i=1}^d y_{i:n}\right)^{-(\alpha+1)}e^{-b_1\alpha}d\alpha}{\sum_{j=0}^{n-d}\frac{(-1)^j}{j!(n-d-j)!}\int_0^\infty\frac{\alpha^d}{b_1^*(j)^{d+1}}\left(\prod_{i=1}^d y_{i:n}\right)^{-(\alpha+1)}e^{-b_1\alpha}d\alpha}\right)db_1db_2. \quad (3.10)$$

Also, with respect to 3.7 and 3.8 and Definition 3.2, the E-Bayesian estimation of parameters α and λ under LINEX loss function are obtained respectively as follows

$$\widehat{\alpha}_{EBL}=$$
$$-\frac{1}{kc_1c_2}\int_0^{c_1}\int_0^{c_2}\left(\log\left[\frac{\sum_{j=0}^{n-d}\frac{(-1)^j}{j!(n-d-j)!}\int_0^\infty\left(\frac{\alpha}{b_1^*(j)}\right)^{d+1}\left(\prod_{i=1}^d y_{i:n}\right)^{-(\alpha+1)}e^{-b_1\alpha}d\alpha}{d!\sum_{j=0}^{n-d}\frac{(-1)^j}{j!(n-d-j)!}\int_0^\infty\frac{\alpha^d}{b_1^*(j)^{d+1}}\left(\prod_{i=1}^d y_{i:n}\right)^{-(\alpha+1)}e^{-b_1\alpha}d\alpha}\right]\right)db_1db_2, \quad (3.11)$$

$$\widehat{\lambda}_{EBL}=$$
$$-\frac{1}{kc_1c_2}\int_0^{c_1}\int_0^{c_2}\left(\log\left[\frac{(d+1)\sum_{j=0}^{n-d}\frac{(-1)^j}{j!(n-d-j)!}\int_0^\infty\frac{\alpha^d}{b_1^*(j)^{d+2}}\left(\prod_{i=1}^d y_{i:n}\right)^{-(\alpha+1)}e^{-b_1\alpha}d\alpha}{\sum_{j=0}^{n-d}\frac{(-1)^j}{j!(n-d-j)!}\int_0^\infty\frac{\alpha^d}{b_1^*(j)^{d+1}}\left(\prod_{i=1}^d y_{i:n}\right)^{-(\alpha+1)}e^{-b_1\alpha}d\alpha}\right]\right)db_1db_2.$$

4. Numerical Experiments

In this section, a Monte Carlo simulation and a numerical example are presented to illustrate all the estimation methods described in the section 3.

4.1. Simulation Study

In thissection, we present some results using Monte Carlo simulations to compare the performance of the different methods based on unified hybrid censored schemes. To this end, the Bayesian and E-Bayesian estimation of parameters are compute and compare based on the mean square error (MSE) criterion. For this purpose, we generate 50 numbers from IW distribution with $\alpha = 2.5$ and $\lambda = 0.05$. Then, the Bayesian and E-Bayesian estimations of α and λ based on square and LINEX loss functions were computed using Equations 3.5 to 3.12. The performance of all estimates have been compared numerically of the MSE value. This process have been iterated 1000 times and the average all estimates and their MSEs were computed and reported in Tables 1 to 4. The simulation is conducted by R software.

Drawn upon the simulation results, we found out that

1. According to reported results in Tables 1 and 2, in both the cases $(b_1,\ b_2)$ and for fixed values of r, k and T_2 , when T_1 is increased, the performance of the E-Bayesian estimation of the parameters α and λ is better than their Bayesian estimations. Also, the MSE of all estimators decreases with increasing T_1 and the numerical value of the estimators approaches the real values of the parameters by increasing T_1.

2. According to Tables 3 and 4, in both the cases $(b_1,\ b_2)$, for fixed r, k and T_1, when T_2 is increased, the performance of the Bayesian estimation of the parameters α and λ is better than their E-Bayesian estimations. Also, the MSE of all estimators decreases with increasing T_2 and the numerical value of the estimators approaches the real values of the parameters by increasing T_2.

4.2. Application to real data set

In this subsection, a real data set is used to analyze the estimation methods proposed for parameters α and λ. The data set represent repair times (in hour) for an airborne communication transceiver. They were first analyzed by [23]. The data set is presented in Table 5. Before analyzing the data, we fit the IW model to this data set. We used the Kolmogorov-Smirnov (K-S) distance between the fitted the empirical distribution functions, and corresponding p-values. It is observed that for this data, the K-S test statistic and its p-value are 0.08069 and 0.9255, respectively. In addition, the values of Anderson-Darling and the Cramer Von-Mises statistic were obtained 11.001 and 0.0509, respectively. Hence, we conclude that the IW model fit quite well to this data set.

Moreover, to compute the Bayesian and E-Bayesian estimations, we prefer to use the non-informative prior, because we do not have any prior information. On the other hand, the non-informative prior provides prior distributions which are not proper, we adopt the suggestion of Congdon [5] by choosing $b_1 = b_2 = 0.01$. Therefore, for $c_1 = c_2 = 1$, for these data, six unified hybrid censored schemes are considered under the following conditions

Scheme 1: $K = 14$, $r = 30$, T_1 =3.5, $T_2 = 4.5$.
Scheme 2: $K = 19$, $r = 25$, $T_1 = 1.5$, $T_2 = 3$.
Scheme 3: $K = 19$, $r = 30$, $T_1 = 1.5$, $T_2 = 2$.

Table 1: Estimation the parameter α for with $T_2 = 100$

			SEL		*LINEX*	
(b_1, b_2)	(k, r)	T_1	$\hat{\alpha}_B(MSE)$	$\hat{\alpha}_{EB}(MSE)$	$\hat{\alpha}_B(MSE)$	$\hat{\alpha}_{EB}(MSE)$
(1.5, 2)	(11, 20)	80	1.0084(0.9722)	1.3116(0.9582)	1.9123(0.8765)	2.1230(0.8012)
		85	1.6295(0.9291)	1.8872(0.8110)	2.1561(0.7865)	2.3478(0.7231)
		95	2.2896(0.8350)	2.5735(0.7265)	2.4561(0.7012)	2.5011(0.6987)
	(15, 20)	80	0.9002(1.2943)	1.4889(1.0106)	2.1345(0.9878)	2.3421(0.9216)
		85	2.0657(1.0263)	2.4462(0.9644)	2.2786(0.8906)	2.3565(0.8542)
		95	2.3750(0.9364)	2.6011(0.9012)	2.4678(0.8123)	2.5102(0.7996)
	(18, 20)	80	0.9603(0.7655)	1.2968(0.7446)	2.2998(0.7011)	2.4987(0.6977)
		85	1.0417(0.6260)	2.1999(0.5859)	2.3224(0.6056)	2.4078(0.5786)
		95	1.8491(0.5438)	2.5619(0.4673)	2.4776(0.5244)	2.5017(0.5009)
	(12, 25)	80	3.9689(1.7479)	2.3963(1.0443)	2.1236(0.8789)	2.1787(0.8081)
		85	2.8977(0.9601)	2.4516(0.8931)	2.1921(0.8177)	2.4980(0.7998)
		95	2.7533(0.8093)	2.5567(0.7713)	2.4256(0.7665)	2.5111(0.7234)
	(12, 35)	80	3.1183(1.0693)	2.2918(0.9778)	2.3702(0.7876)	2.4421(0.7543)
		85	2.9658(1.0433)	2.3658(0.8797)	2.4098(0.7034)	2.4587(0.6971)
		95	2.7705(0.8388)	2.5194(0.6179)	2.4642(0.6897)	2.5098(0.6542)
(2.5, 3)	(11, 20)	80	1.2644(1.2980)	1.6612(1.2388)	2.1122(0.9876)	2.2341(0.8675)
		85	1.5360(1.1631)	2.2058(1.1217)	2.1343(0.8765)	2.3564(0.8432)
		95	2.2346(0.9901)	2.5497(0.9096)	2.3571(0.8431)	2.5139(0.8156)
	(18, 20)	80	1.8603(1.6419)	2.1271(1.5753)	2.5812(1.1235)	2.5431(1.0971)
		85	2.0782(0.9715)	2.2608(0.8473)	2.5165(0.9042)	2.4981(0.8234)
		95	2.9904(0.7014)	2.5826(0.6473)	2.7654(0.7012)	2.5511(0.6879)
	(12, 25)	80	1.7583(1.4209)	2.1329(1.0292)	2.6211(1.2341)	2.5987(0.9765)
		85	2.0836(1.0031)	2.2943(0.9421)	2.6170(0.9564)	2.5632(0.8567)
		95	2.9015(0.9064)	2.5290(0.8547)	2.5879(0.9011)	2.5231(0.8241)
	(12, 35)	80	1.0192(1.1086)	2.0194(1.0041)	2.7167(0.9786)	2.6981(0.9775)
		85	1.8166(0.9388)	2.2617(0.8259)	2.7090(0.9064)	2.6017(0.9012)
		95	2.1086(0.8166)	2.5130(0.7741)	2.6098(0.8025)	2.5045(0.7981)

Table 2: Estimate, the SEL and LINEX loss functions for λ with $T_2 = 100$

			SEL		*LINEX*	
(b_1, b_2)	(k, r)	T_1	$\hat{\lambda}_B(MSE)$	$\hat{\lambda}_{EB}(MSE)$	$\hat{\lambda}_B(MSE)$	$\hat{\lambda}_{EB}(MSE)$
(1.5, 2)	(11, 20)	80	0.01001(0.3658)	0.0202(0.1516)	0.0347(0.3076)	0.0567(0.1412)
		85	0.0138(0.2454)	0.0208(0.1109)	0.0398(0.2211)	0.0528(0.1397)
		95	0.0755(0.1328)	0.0569(0.0990)	0.0687(0.1308)	0.0505(0.1280)
	(15, 20)	80	0.0854(0.1541)	0.0782(0.1192)	0.0776(0.1488)	0.0691(0.1117)
		85	0.0832(0.0865)	0.0721(0.0745)	0.0655(0.0815)	0.0580(0.0711)
		95	0.0728(0.0791)	0.0516(0.0665)	0.0702(0.0752)	0.0512(0.0625)
	(18, 20)	80	0.0129(0.1819)	0.0215(0.1126)	0.0342(0.1751)	0.0497(0.1089)
		85	0.0269(0.1121)	0.0387(0.0946)	0.0378(0.1076)	0.0510(0.1025)
		95	0.0393(0.1011)	0.0509(0.0897)	0.0416(0.0981)	0.0502(0.0801)
	(12, 25)	80	0.0902(0.0998)	0.0219(0.0813)	0.0765(0.0906)	0.0621(0.0761)
		85	0.0825(0.0743)	0.0325(0.0663)	0.0692(0.0705)	0.0608(0.0612)
		95	0.0363(0.0641)	0.0629(0.0547)	0.0582(0.0613)	0.0521(0.0517)
	(12, 35)	80	0.0125(1.0147)	0.0282(0.9998)	0.0349(0.9128)	0.0652(0.8349)
		85	0.0129(0.8813)	0.0392(0.7843)	0.0412(0.8012)	0.0492(0.7121)
		95	0.0219(0.7663)	0.0563(0.6940)	0.0432(0.0715)	0.0511(0.6015)
(2.5, 3)	(11, 20)	80	0.0121(0.7215)	0.0298(0.6643)	0.0299(0.6918)	0.0398(0.6137)
		85	0.0204(0.5213)	0.0359(0.4037)	0.0312(0.4911)	0.0511(0.3998)
		95	0.0396(0.3039)	0.0588(0.2820)	0.0423(0.2912)	0.0506(0.2180)
	(18, 20)	80	0.0171(0.6818)	0.0224(0.5188)	0.0265(0.6231)	0.0409(0.4912)
		85	0.0271(0.4224)	0.0394(0.3748)	0.0356(0.4019)	0.0521(0.3562)
		95	0.0317(0.2715)	0.0495(0.1178)	0.4109(0.2137)	0.0511(0.1018)
	(12, 25)	80	0.0115(0.6393)	0.0239(0.5137)	0.0285(0.6012)	0.0617(0.4981)
		85	0.0284(0.4340)	0.0312(0.3915)	0.0350(0.4008)	0.0592(0.3128)
		95	0.0317(0.2715)	0.0511(0.2705)	0.0391(0.2016)	0.0513(0.1919)
	(12, 35)	80	0.0163(0.2107)	0.0317(0.1092)	0.0287(0.2085)	0.0712(0.0986)
		85	0.0327(0.1629)	0.0414(0.0831)	0.0399(0.1521)	0.0655(0.0877)
		95	0.0406(0.1204)	0.0517(0.0772)	0.0433(0.1194)	0.0510(0.0712)

Table 3: Estimate, the SEL and LINEX loss functions for α with $T_1 = 45$

			SEL		*LINEX*	
(b_1, b_2)	(k, r)	T_2	$\hat{\alpha}_B(MSE)$	$\hat{\alpha}_{EB}(MSE)$	$\hat{\alpha}_B(MSE)$	$\hat{\alpha}_{EB}(MSE)$
(1.5, 2)	(11, 20)	90	0.99575(1.0621)	0.63391(1.2444)	1.76211(0.9128)	1.56411(1.1121)
		110	2.31118(0.8306)	1.15315(0.9192)	2.43125(0.8127)	2.12341(0.9071)
		150	2.63391(0.7622)	1.79575(0.7944)	2.51010(0.7123)	2.34321(0.7876)
	(15, 20)	90	1.70658(1.0376)	0.89371(1.9544)	2.11231(0.9765)	1.14564(1.7896)
		110	2.01852(0.9089)	1.05520(1.4909)	2.32111(0.8675)	2.12897(1.3678)
		150	2.46112(0.8360)	1.44802(0.9594)	2.47652(0.8095)	2.34765(0.9018)
	(18, 20)	90	1.13082(0.8311)	0.81276(0.9279)	2.15234(0.7986)	1.91234(0.8867)
		110	1.93099(0.7749)	1.00862(0.8242)	2.27865(0.7098)	2.18970(0.8623)
		150	2.61224(0.6582)	1.88063(0.7499)	2.49876(0.6128)	2.37865(0.7021)
	(12, 25)	90	1.18227(0.8584)	0.9285(0.9079)	2.34121(0.7789)	1.98765(0.8765)
		110	1.98892(0.7596)	1.64098(0.8117)	2.41112(0.7214)	2.21112(0.8054)
		150	2.43747(0.6094)	1.85008(0.7384)	2.51106(0.5987)	2.39876(0.7011)
	(12, 35)	90	1.57187(1.3895)	1.10972(1.4073)	2.10456(1.2454)	1.67543(1.3786)
		110	1.98079(1.2069)	1.64098(1.2506)	2.38976(1.1007)	2.21211(1.2070)
		150	2.61495(0.8416)	2.05601(0.9255)	2.50107(0.8097)	2.42134(0.9011)
(2.5, 3)	(11, 20)	90	1.82853(1.3840)	1.42421(1.5504)	2.29087(1.2211)	2.11098(1.4712)
		110	2.18507(1.1109)	1.88956(1.4645)	2.24098(1.1024)	2.17099(1.3411)
		150	2.70777(0.9614)	2.16277(1.0019)	2.60211(0.9211)	2.34211(0.9765)
	(18, 20)	90	1.10786(0.7908)	0.83458(1.0080)	2.21345(0.7564)	1.98011(0.9542)
		110	2.11391(0.7324)	1.10092(0.8379)	2.34256(0.7019)	2.27778(0.9459)
		150	2.45542(0.6246)	2.08759(0.7161)	2.52134(0.6012)	2.35622(0.6985)
	(12, 25)	90	0.99264(1.0783)	0.89101(1.8420)	1.87652(0.97678)	1.13123(1.5467)
		110	1.12965(0.9910)	1.78883(1.6249)	2.32145(0.9437)	2.27123(1.1076)
		150	2.42852(0.8991)	2.13112(1.1214)	2.51127(0.8113)	2.34212(0.9765)
	(12, 35)	90	1.92323(1.7885)	1.25342(1.8667)	2.09876(1.6542)	1.98760(1.7655)
		110	2.26944(1.6021)	1.79186(1.7326)	2.37421(1.6227)	2.08976(1.7065)
		150	2.51287(0.9408)	2.16811(1.1667)	2.50611(1.4532)	2.32451(1.5432)

Table 4: Estimate, the SEL and LINEX loss functions for λ with $T_1 = 45$

			SEL		*LINEX*	
(b_1, b_2)	(k, r)	T_2	$\hat{\lambda}_B(MSE)$	$\hat{\lambda}_{EB}(MSE)$	$\hat{\lambda}_B(MSE)$	$\hat{\lambda}_{EB}(MSE)$
(1.5, 2)	(11, 20)	90	0.03067(0.0899)	0.01754(0.3560)	0.03987(0.0754)	0.02876(0.3112)
		110	0.04394(0.0708)	0.02913(0.2360)	0.49017(0.0697)	0.03456(0.2715)
		150	0.05997(0.0515)	0.03434(0.1644)	0.05112(0.0508)	0.04567(0.1546)
	(15, 20)	90	0.02808(0.1615)	0.02031(0.2092)	0.03145(0.1534)	0.02756(0.1497)
		110	0.03180(0.1279)	0.02436(0.1784)	0.03912(0.1190)	0.03113(0.1220)
		150	0.04985(0.1021)	0.03367(0.1548)	0.05011(0.0987)	0.04123(0.1154)
	(18, 20)	90	0.02733(0.1154)	0.01501(0.1379)	0.03247(0.1081)	0.02256(0.1101)
		110	0.04519(0.0828)	0.02207(0.1065)	0.04917(0.0798))	0.03245(0.1013)
		150	0.05386(0.0684)	0.03622(0.0856)	0.05117(0.0611)	0.04125(0.0988)
	(12, 25)	90	0.03324(0.0871)	0.02761(0.1054)	0.03987(0.0765)	0.03011(0.0806)
		110	0.04360(0.0802)	0.03335(0.0910)	0.04811(0.0709	0.03812(0.0756)
		150	0.05222(0.0792)	0.04173(0.0810)	0.05012(0.0654)	0.04213((0.0719)
	(12, 35)	90	0.02557(0.1393)	0.01339(0.1972)	0.03156(0.1280)	0.03098(0.1765)
		110	0.03144(0.1076)	0.02542(0.1748)	0.04254(0.0987)	0.03879(0.1628)
		150	0.04888(0.0813)	0.03173(0.1519)	0.05109(0.0745)	0.04099(0.1435)
(2.5, 3)	(11, 20)	90	0.02766(0.1429)	0.01985(0.1841)	0.03011(0.1399)	0.02118(0.1705)
		110	0.03932(0.1283)	0.02065(0.1522)	0.04102(0.1212)	0.02876(0.1254)
		150	0.05599(0.0882)	0.04142(0.1248)	0.04454(0.0765)	0.03657(0.1175)
	(15, 20)	90	0.01655(0.2212)	0.01043(0.2697)	0.02542(0.2016)	0.01968(0.2453)
		110	0.03960(0.1367)	0.03177(0.1826)	0.04139(0.1210)	0.03890(0.1721)
		150	0.04705(0.0917)	0.03644(0.1591)	0.05497(0.0887)	0.04338(0.1327)
	(18, 20)	90	0.02150(0.0937)	0.01415(0.1245)	0.03566(0.0912)	0.01879(0.1119)
		110	0.03133(0.0854)	0.02014(0.1218)	0.035(0.08127)	0.02211(0.1092)
		150	0.04753(0.0625)	0.03622(0.1078)	0.05124(0.5809)	0.02978(0.0965)
	(12, 25)	90	0.01674(0.0901)	0.01173(0.1533)	0.02563(0.0865)	0.02011(0.1434)
		110	0.03900(0.0881)	0.02676(0.1346)	0.04098(0.0774)	0.03115(0.1228)
		150	0.05142(0.0601)	0.03914(0.1155)	0.05102(0.0594)	0.04231(0.1093)
	(12, 35)	90	0.01886(0.1118)	0.01283(0.2455)	0.02131(0.1010)	0.01789(0.2129)
		110	0.03178(0.0758)	0.02982(0.1902)	0.03908(0.0712)	0.03654(0.1897)
		150	0.04825(0.0616)	0.03501(0.1564)	0.05053(0.0598)	0.04565(0.1452)

Table 5: Repair times (in h) for an airborne communication transceiver.

0.2	0.3	0.5	0.5	0.5	0.5	0.6	0.6	0.7	0.7
0.7	0.8	0.8	1.0	1.0	1.0	1.0	1.1	1.3	1.5
1.5	1.5	1.5	2.0	2.0	2.2	2.5	2.7	3.0	3.0
3.3	3.3	4.0	4.0	4.5	4.7	5.0	5.4	5.4	7.0
7.5	8.8	9.0	10.3	22	24.5				

Scheme 4: $K = 30$, $r = 32$, $T_1 = 2$, $T_2 = 4$.
Scheme 5: $K = 30$, $r = 32$, $T_1 = 1$, $T_2 = 3$.
Scheme 6: $K = 18$, $r = 20$, $T_1 = 1$, $T_2 = 3$.

In all shemes, the Bayesian and E-Bayesian estimation of parameters α and λ have been computed using Equations 3.5-3.12 and presented in Table 6. Moreover, the maximum likelihood estimation for parameters α and λ were obtained based on complete uncensored data set as 1.011941 and 1.125229, respectively. Based on reported results in Table 6, we can conclude that the E-Bayesian estimation of the parameters are closer to their estimated value in the complete sample in all schemes. Therefore, we can say that the E-Bayesian method for parameters estimation has a better than other methods.

Table 6: Bayesian and E-Bayesian estimations of parameters α and λ, under SEL and LINEX loss functions

	SEL				*LINEX*			
Scheme	$\hat{\alpha}_B$	$\hat{\alpha}_{EB}$	$\hat{\lambda}_B$	$\hat{\lambda}_{EB}$	$\hat{\alpha}_B$	$\hat{\alpha}_{EB}$	$\hat{\lambda}_B$	$\hat{\lambda}_{EB}$
1	0.075438	0.094656	0.827177	1.144378	0.865123	0.998675	1.109780	1.132334
2	0.940139	1.009714	1.022434	1.108393	1.011342	1.012321	1.111332	1.129987
3	0.983522	1.013935	1.772656	1.031561	1.001987	1.010453	1.145987	1.134567
4	1.263046	1.030439	1.795299	1.176216	1.178921	1.012432	1.142564	1.121342
5	0.925065	1.013107	2.348933	1.025538	1.111232	1.010707	1.898765	1.191231
6	2.006398	1.094094	2.575507	1.500216	1.675432	1.098765	2.132453	1.232132

5. Conclusion

In this study, the Bayesian and E-Bayesian estimations of the inverse Weibull distribution parameters were obtained under the unified hybrid censored scheme with squared error and LINEX loss functions. Six unified hybrid censored schemes were considered and using a real data set. Also, using Monte Carlo simulation the conditions of superiority of the estimator were obtained with respect to each other. The obtained results show that in all schemes the E-Bayesian estimation approach is better than Bayesian and maximum likelihood estimations.
Imprecise Bayes and E-Bayes estimation under the unified censored scheme may be a suitable topic for future research [27]. In addition, further research is required to investigate some important topics in reliability analysis such as mean time to failure, failure rate, and mean residual life, under vague environments especially based on imprecise Weibull lifetime data [21, 26].

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