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Solution of a generalized two dimensional fractional integral equation

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Abstract

This paper deals with existence and local attractivity of solution of a quadratic fractional integral equation in two independent variables. The solution space has been considered to be the Banach space of all bounded continuous functions defined on an unbounded interval. The fundamental tool used for the purpose is the notion of noncompactness and the celebrated Schauder fixed point principle. Finally an example has been provided at the end in support of the result.

Keywords: Fractional integral equation, Measure of noncompactness, Solution
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1. Introduction

From the literature(see, for example[1]-[5], [6]-[7], [10]-[13], [15], [17], [19], [20], [21])it has been found that over the recent couple of decades researchers contributed immensely to the theory of integral equations but till date most of the investigations have been limited to integral equations in one independent variable. There are many physical situations notably those of inter-reflections of light among perfectly diffused surfaces, skin effect in electrical conductors where the theory of one independent variable is inadequate and there is urgent need for the theory of integral equations of higher dimension especially of two independent variables.
In this paper the following quadratic fractional integral equation involving two arguments has been considered.

$$u(x,y) = g(x,y) + \frac{\alpha\gamma h(x,y,u(x,y))}{\Gamma(\beta)\Gamma(\delta)} \int_0^x \int_0^y \frac{s^{\alpha-1}t^{\gamma-1} f(x,y,s,t,u(s,t))}{(x^\alpha - s^\alpha)^{1-\beta}(y^\gamma - t^\gamma)^{1-\delta}} dt\, ds, \tag{1.1}$$

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where $(x, y) \in R_+ \times [0, b]$, $R_+ = [0, \infty)$, $(\beta, \delta) \in (0, 1) \times (0, 1)$ is a fixed number, $\alpha, \gamma > 0$ and $\Gamma(\beta), \Gamma(\delta)$ denotes the gamma function.
For $\alpha = 1, \gamma = 1$ the above equation is a mixed Riemann- Liouville type integral equation which was discussed in [18]

2. Preliminaries

Now let us recapitulate some basic ideas regarding measure of noncompactness ([14], [9]).

Here an infinite dimensional Banach space $(E, \|\cdot\|)$ has been considered, with the zero element θ. $\bar{G}$, $ConvG$ symbolizes the closure and convex closure of a subset G of E, respectively. $B(x, \lambda)$ denotes the closed ball centered at x and with radius λ. For our convenience let us denote B_λ for the ball $B(\theta, \lambda)$. Moreover $\mathcal{B}_E$ to be the collection of all nonempty and bounded subsets of E and $\mathcal{N}_E$ to be its subfamily consisting of all nonempty and relatively compact subsets.

Definition 2.1. *A mapping $\mu : \mathcal{B}_E \to R_+$ is a measure of noncompactness in E if*

(i) $Ker\mu = \{G \in \mathcal{B}_E : \mu(G) = 0\}$ is nonempty and $ker\ \mu \subset \mathcal{N}_E$.

(ii) $G \subset H \Rightarrow \mu(G) \le \mu(H)$.

(iii) $\mu(\bar{G}) = \mu(G)$.

(iv) $\mu(Conv\ G) = \mu(G)$.

(v) $\mu(\lambda G + (1-\lambda)H) \le \lambda\mu(G) + (1-\lambda)\mu(H)$ for $\lambda \in [0, 1]$.

(vi) If (G_n) is a sequence of closed sets from $\mathcal{B}_E$ such that $G_{n+1} \subset G_n$ for $n = 1, 2, ...$ and if $\lim_{n\to\infty} \mu(G_n) = 0$, then the intersection $G_\infty = \bigcap_{n=1}^{\infty} G_n$ is nonempty.

$Ker\mu$ defined in (i) is called the kernel of the measure of noncompactness μ.
Remark 1: From the inequality $\mu(G_\infty) \le \mu(G_n)$ for $n = 1, 2, ...$, it concludes that $\mu(G_\infty) = 0$ and thus $G_\infty \in \ker\ \mu$. This property of the intersection set G_∞ has been applied in our study.
The Banach space $B := BC(R_+ \times [0, b])$ with the standard norm

$$\|u\|_{BC} = \sup_{(x,y)\in R_+\times[0,b]} |u(x, y)|.$$

has been taken for the current study. Now to define such measure, a nonempty bounded subset X of the space $BC(R_+ \times [0, b])$ and positive numbers a and b has been fixed. Also from the earlier study the modulus of continuity of the function, u has been denoted by $w^{a,b}(u, \epsilon)$ on $J = [0, a] \times [0, b]$, for $u \in X$ and $\epsilon \ge 0$, i.e.,

$$w^{a,b}(u, \epsilon) = \sup\{|u(x, y) - u(s, t)| : s \le x, t \le y, (x, y) \in [0, a] \times [0, b], |x - s| \le \epsilon, |y - t| \le \epsilon\}.$$

In addition,

$$w^{a,b}(X, \epsilon) = \sup\{w^{a,b}(u, \epsilon) : u \in X\},$$

$$w_0^{a,b}(X) = \lim_{\epsilon\to 0} w^{a,b}(X, \epsilon),$$

and

$$w_0(X) = \lim_{a\to\infty} w_0^{a,b}(X).$$

If (x, y) is fixed number from $R_+ \times [0, b]$, then

$$X(x, y) = \{u(x, y) : u \in X\},$$

and

$$\text{diam } X(x, y) = \sup\{|u(x, y) - v(x, y)| : u, v \in X\}.$$

Lastly, the function μ define on the family $\mathcal{B}_{BC(R_+\times[0,b])}$ has been considered by the formula

$$\mu(X) = w_0(X) + \lim_{x\to\infty} \sup \text{diam } X(x, y), \tag{2.1}$$

for each fixed $y \in [0, b]$. It has been seen from the earlier literature that the capacity μ is the proportion of noncompactness in the space $BC(R_+ \times [0, b])$.

Remark 2.2. *The portion* $\ker \mu$ *is the group of all nonempty and bounded sets* X *such that functions belonging to* X *are locally equicontinuous on* $R_+ \times [0, b]$ *and the thickness of the bundle formed by functions from* X *tends to zero at infinity. Those properties, will allow us to portray solutions of the integral equation.*

Now to present some significant ideas embraced in the paper, let us consider Ω to be a nonempty subset of the space $BC(R_+ \times [0, b])$ and let R be an operator define on Ω with values in $BC(R_+ \times [0, b])$. Then the solution of the following operator equation

$$u(x, y) = (R\, u)(x, y), \quad (x, y) \in R_+ \times [0, b]. \tag{2.2}$$

can be categorized into two parts.

Definition 2.3 ([8]). *The solutions of equation (2.2) are locally attractive if there exists a closed ball* $B(u_0, \eta)$ *in the space* $BC(R_+ \times [0, b])$ *such that for arbitrary solutions* $u = u(x, y)$ *and* $v = v(x, y)$ *of equation (2.2) belonging to* $B(u_0, \eta) \cap \Omega$*, we have that,*

$$\lim_{x\to\infty} (u(x, y) - v(x, y)) = 0, \quad \textit{for each} \;\; y \in [0, b]. \tag{2.3}$$

When the limit in (2.3) is uniform with respect to the set $B(u_0, \eta) \cap \Omega$*, solutions of equation (2.2) are said to be uniformly locally attractive (or equivalently, that solutions of (2.2) are asymptotically stable).*

Definition 2.4 ([8]). *The solutions* $u = u(x, y)$ *of equation (2.2) is said to be globally attractive if (2.3) holds for each solution* $v = v(x, y)$ *of (2.2). Solutions of equation (2.2) are said to be globally asymptotically stable (or uniformly globally attractive)if condition (2.3) is satisfied uniformly with respect to the set* Ω*.*

Clearly global attractivity imply local attractivity but the converse is not necessarily true and has been justified later in this work.
Now we present two results which will be used in the rest of the paper.

Lemma 2.5 ([16]). *Let* $\kappa : R_+ \to R_+$ *be a concave function with* $\kappa(0) = 0$*. Then* κ *is subadditive(this means that* $\kappa(z_1 + z_2) \le \kappa(z_1) + \kappa(z_2)$ *for any* $z_1, z_2 \in R_+$*.)*

Lemma 2.6 ([16]). *Let* $\kappa : R_+ \to R_+$ *be the function defined by* $\kappa(z) = z^\alpha$*.*

(1) If $\alpha \ge 1$ *and* $z_1, z_2 \in I$ *with* $z_2 > z_1$*, then* $z_2^\alpha - z_1^\alpha \le \alpha(z_2 - z_1)$*.*

(2) If $0 < \alpha < 1$ *and* $z_1, z_2 \in I$ *with* $z_2 > z_1$*, then* $z_2^\alpha - z_1^\alpha \le (z_2 - z_1)^\alpha$*.*

3. Main Results

(H1) The function $g : R_+ \times [0,b] \to R$ is continuous and bounded on $J' = R_+ \times [0,b]$.

(H2) The function $h : J' \times R \to R$ is continuous and there exists a continuous function, $m : J' \to R_+$ such that

$$|h(x,y,u) - h(x,y,v)| \leq m(x,y)\ |u-v|,$$

for any $(x,y) \in J'$ and for all $u,v \in R$.

(H3) The function $f(x,y,s,t,u(s,t)) = f : J' \times J' \times R \to R$ is continuous. In addition, there exists a continuous function, $n : J' \to R_+$, and a continuous and nondecreasing function, $\phi : R_+ \to R_+$, with $\phi(0) = 0$ such that

$$|f(x,y,s,t,u) - f(x,y,s,t,v)| \leq n(x,y)\ \phi(|u-v|),$$

for all $(x,y),(s,t) \in J'$ such that $s \leq x$; $t \leq y$ and for all $u,v \in R$.
Let us define the function $f_1 : J' \to R_+$,

$$f_1(x,y) = \max\{|f(x,y,s,t,0)| :\ 0 \leq s \leq x, 0 \leq t \leq y\}.$$

The function f_1 is continuous on $R_+ \times [0,b]$.
Moreover,it has been assumed that the following conditions are satisfied:

(H4) The functions $p,q,r,s : J' \to R_+$ defined by $p(x,y) = m(x,y)\ n(x,y)\ x^{\alpha\beta}y^{\gamma\delta}$,
$q(x,y) = m(x,y)\ f_1(x,y)\ x^{\alpha\beta}y^{\gamma\delta}$,
$r(x,y) = n(x,y)\ |h(x,y,0)|\ x^{\alpha\beta}y^{\gamma\delta}$,
$s(x,y) = f_1(x,y)\ |h(x,y,0)|\ x^{\alpha\beta}y^{\gamma\delta}$,
are bounded on J' and the functions $p(x,y), r(x,y)$ are such that
$\lim_{x\to\infty} p(x,y) = \lim_{x\to\infty} r(x,y) = 0$, for each fixed $y \in [0,b]$.
In $(H4)$, we may define the following finite constants:
$P = \sup\{p(x,y) : (x,y) \in J'\}$,
$Q = \sup\{q(x,y) : (x,y) \in J'\}$,
$R = \sup\{r(x,y) : (x,y) \in J'\}$,
$S = \sup\{s(x,y) : (x,y) \in J'\}$.

(H5) There exists a positive solution λ_0 of the inequality

$$\|g\| + \frac{1}{\Gamma(1+\beta)\Gamma(1+\delta)}[P\lambda\phi(\lambda) + Q\lambda + R\phi(\lambda) + S] \leq \lambda.$$

Also,

$$Q < \Gamma(1+\beta)\Gamma(1+\delta).$$

Suppose the operators W, U and V defined on the space $BC(R_+ \times [0,b])$ is such that:
$(Wu)(x,y) = h(x,y,u(x,y))$,
$(Uu)(x,y) = \frac{\alpha\gamma}{\Gamma(\beta)\Gamma(\delta)} \int\limits_0^x \int\limits_0^y \frac{s^{\alpha-1}t^{\gamma-1}f(x,y,s,t,u(s,t))}{(x^\alpha - s^\alpha)^{1-\beta}(y^\gamma - t^\gamma)^{1-\delta}} dt\ ds$,
$(Vu)(x,y) = g(x,y) + (Wu)(x,y)(Uu)(x,y)$.

Lemma 3.1. *The operator V transforms the ball B_{λ_0} in the space $BC(R_+ \times [0,b])$ into itself satisfying the above assumptions$(H1-H5)$ where λ_0 is a number appearing in assumption $(H5)$. Moreover, all solutions of equation (1.1) belonging to the space $BC(R_+ \times [0,b])$ are fixed points of the operator V.*

Proof . It is obvious from the assumptions that, the function Wu is continuous on $R_+ \times [0,b]$ for any $u \in BC(R_+ \times [0,b])$. We will show that the same holds also for the operator U. In order to show that, let us fix $a,b > 0$ and $\epsilon > 0$. Also suppose that $x_1, x_2 \in [0,a]$ and $y_1, y_2 \in [0,b]$ are such that $|x_2 - x_1| \le \epsilon$ and $|y_2 - y_1| \le \epsilon$. Without loss of generality, it is assume that $x_1 < x_2$ and $y_1 < y_2$. Then, by virtue of the above mentioned assumptions, we have

$$
\begin{aligned}
&|(Uu)(x_2,y_2) - (Uu)(x_1,y_1)| \\
&= \frac{\alpha\gamma}{\Gamma(\beta)\Gamma(\delta)} \left[\int_0^{x_1}\int_0^{y_1} \left[\frac{s^{\alpha-1}t^{\gamma-1}f(x_2,y_2,s,t,u(s,t))}{(x_2^\alpha - s^\alpha)^{1-\beta}(y_2^\gamma - t^\gamma)^{1-\delta}} - \frac{s^{\alpha-1}t^{\gamma-1}f(x_1,y_1,s,t,u(s,t))}{(x_1^\alpha - s^\alpha)^{1-\beta}(y_1^\gamma - t^\gamma)^{1-\delta}} \right] dt\ ds \right. \\
&+ \int_{x_1}^{x_2}\int_{y_1}^{y_2} \frac{s^{\alpha-1}t^{\gamma-1}f(x_2,y_2,s,t,u(s,t))}{(x_2^\alpha - s^\alpha)^{1-\beta}(y_2^\gamma - t^\gamma)^{1-\delta}} dt\ ds + \int_0^{x_1}\int_{y_1}^{y_2} \frac{s^{\alpha-1}t^{\gamma-1}f(x_2,y_2,s,t,u(s,t))}{(x_2^\alpha - s^\alpha)^{1-\beta}(y_2^\gamma - t^\gamma)^{1-\delta}} dt\ ds \\
&\left. + \int_{x_1}^{x_2}\int_0^{y_1} \frac{s^{\alpha-1}t^{\gamma-1}f(x_2,y_2,s,t,u(s,t))}{(x_2^\alpha - s^\alpha)^{1-\beta}(y_2^\gamma - t^\gamma)^{1-\delta}} dt\ ds \right] \\
&\le \frac{\alpha\gamma}{\Gamma(\beta)\Gamma(\delta)} \int_0^{x_1}\int_0^{y_1} \frac{s^{\alpha-1}t^{\gamma-1}|f(x_2,y_2,s,t,u(s,t)) - f(x_1,y_1,s,t,u(s,t))|}{(x_2^\alpha - s^\alpha)^{1-\beta}(y_2^\gamma - t^\gamma)^{1-\delta}} dt\ ds \\
&+ \frac{\alpha\gamma}{\Gamma(\beta)\Gamma(\delta)} \int_0^{x_1}\int_0^{y_1} s^{\alpha-1}t^{\gamma-1}|f(x_1,y_1,s,t,u(s,t))| \left[\frac{1}{(x_2^\alpha - s^\alpha)^{1-\beta}(y_2^\gamma - t^\gamma)^{1-\delta}} \right. \\
&\left. - \frac{1}{(x_1^\alpha - s^\alpha)^{1-\beta}(y_1^\gamma - t^\gamma)^{1-\delta}} \right] dt\ ds \\
&+ \frac{\alpha\gamma}{\Gamma(\beta)\Gamma(\delta)} \int_{x_1}^{x_2}\int_{y_1}^{y_2} s^{\alpha-1}t^{\gamma-1} \frac{\{|f(x_2,y_2,s,t,u(s,t)) - f(x_2,y_2,s,t,0)| + |f(x_2,y_2,s,t,0)|\}}{(x_2^\alpha - s^\alpha)^{1-\beta}(y_2^\gamma - t^\gamma)^{1-\delta}} dt\ ds \\
&+ \frac{\alpha\gamma}{\Gamma(\beta)\Gamma(\delta)} \int_0^{x_1}\int_{y_1}^{y_2} s^{\alpha-1}t^{\gamma-1} \frac{\{|f(x_2,y_2,s,t,u(s,t)) - f(x_2,y_2,s,t,0)| + |f(x_2,y_2,s,t,0)|\}}{(x_2^\alpha - s^\alpha)^{1-\beta}(y_2^\gamma - t^\gamma)^{1-\delta}} dt\ ds \\
&+ \frac{\alpha\gamma}{\Gamma(\beta)\Gamma(\delta)} \int_{x_1}^{x_2}\int_0^{y_1} s^{\alpha-1}t^{\gamma-1} \frac{\{|f(x_2,y_2,s,t,u(s,t)) - f(x_2,y_2,s,t,0)| + |f(x_2,y_2,s,t,0)|\}}{(x_2^\alpha - s^\alpha)^{1-\beta}(y_2^\gamma - t^\gamma)^{1-\delta}} dt\ ds \\
&\le \frac{\alpha\gamma}{\Gamma(\beta)\Gamma(\delta)} \int_0^{x_1}\int_0^{y_1} \frac{s^{\alpha-1}t^{\gamma-1}w_1^{a,b}(f,\epsilon;\|u\|)}{(x_2^\alpha - s^\alpha)^{1-\beta}(y_2^\gamma - t^\gamma)^{1-\delta}} dt\ ds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{x_1}\int_0^{y_1} \{s^{\alpha-1}t^{\gamma-1}|f(x_1,y_1,s,t,u(s,t)) - f(x_1,y_1,s,t,0)| + |f(x_1,y_1,s,t,0)|\} \\
&\times \left[\frac{1}{(x_2^\alpha - s^\alpha)^{1-\beta}(y_2^\gamma - t^\gamma)^{1-\delta}} - \frac{1}{(x_2^\alpha - s^\alpha)^{1-\beta}(y_2^\gamma - t^\gamma)^{1-\delta}} \right] dt\ ds \\
&+ \frac{\alpha\gamma}{\Gamma(\beta)\Gamma(\delta)} \int_{x_1}^{x_2}\int_{y_1}^{y_2} \frac{s^{\alpha-1}t^{\gamma-1}\{n(x_2,y_2)\phi(|u(s,t)|) + f_1(x_2,y_2)\}}{(x_2^\alpha - s^\alpha)^{1-\beta}(y_2^\gamma - t^\gamma)^{1-\delta}} dt\ ds \\
&+ \frac{\alpha\gamma}{\Gamma(\beta)\Gamma(\delta)} \int_0^{x_1}\int_{y_1}^{y_2} \frac{s^{\alpha-1}t^{\gamma-1}\{n(x_2,y_2)\phi(|u(s,t)|) + f_1(x_2,y_2)\}}{(x_2^\alpha - s^\alpha)^{1-\beta}(y_2^\gamma - t^\gamma)^{1-\delta}} dt\ ds \\
&+ \frac{\alpha\gamma}{\Gamma(\beta)\Gamma(\delta)} \int_{x_1}^{x_2}\int_0^{y_1} \frac{s^{\alpha-1}t^{\gamma-1}\{n(x_2,y_2)\phi(|u(s,t)|) + f_1(x_2,y_2)\}}{(x_2^\alpha - s^\alpha)^{1-\beta}(y_2^\gamma - t^\gamma)^{1-\delta}} dt\ ds \\
&\le \frac{w_1^{a,b}(f,\epsilon;\|u\|)x_1^{\alpha\beta}y_1^{\gamma\delta}}{\Gamma(\beta+1)\Gamma(\delta+1)} \\
&+ \frac{n(x_1,y_1)\phi(\|u\|) + f_1(x_1,y_1)}{\Gamma(\beta+1)\Gamma(\delta+1)} \left[{x_1}^{\alpha\beta}y_1^{\gamma\delta} - \left\{ (x_2^\alpha - x_1^\alpha)^\beta - x_2^{\alpha\beta} \right\} \times \left\{ (y_2^\gamma - y_1^\gamma)^\delta - y_2^{\gamma\delta} \right\} \right] + \\
&\frac{n(x_2,y_2)\phi(\|u\|) + f_1(x_2,y_2)}{\Gamma(\beta+1)\Gamma(\delta+1)} \quad \times \left[(x_2^\alpha - x_1^\alpha)^\beta (y_2^\gamma - y_1^\gamma)^\delta + (y_2^\gamma - y_1^\gamma)^\delta \left\{ x_2^{\alpha\beta} - (x_2^\alpha - x_1^\alpha)^\beta \right\} \right.
\end{aligned}
$$

$+(x_2^\alpha - x_1^\alpha)^\beta \left\{ y_2^{\gamma\delta} - (y_2^\gamma - y_1^\gamma)^\delta \right\} \Big]$
$\leq \frac{w_1^{a,b}(f,\epsilon;\|u\|)x_1^{\alpha\beta}y_1^{\gamma\delta}}{\Gamma(\beta+1)\Gamma(\delta+1)}$
$+ \frac{n(x_1,y_1)\phi(\|u\|)+f_1(x_1,y_1)}{\Gamma(1+\beta)\Gamma(1+\delta)} \left\{ (x_1^\alpha - x_2^\alpha)^\beta y_2^{\gamma\delta} + y_1^{\gamma\delta}(x_2^\alpha - x_1^\alpha)^\beta + x_2^{\alpha\beta}(y_2^\gamma - y_1^\gamma)^\delta \right\}$

$$+ \frac{n(x_2,y_2)\phi(\|u\|)+f_1(x_2,y_2)}{\Gamma(1+\beta)\Gamma(1+\delta)} \left\{ x_2^{\alpha\beta}(y_2^\gamma - y_1^\gamma)^\delta + y_1^{\gamma\delta}(x_2^\alpha - x_1^\alpha)^\beta \right\} \tag{3.1}$$

where
$w_1^{a,b}(f,\epsilon;\|u\|) = \sup\{|f(x_2,y_2,s,t,z) - f(x_1,y_1,s,t,z)| : (s,t),(x_1,y_1),(x_2,y_2) \in [0,a]\times[0,b]; s \leq x_1, x_2$ and $t \leq y_1, y_2;\ |x_2 - x_1| \leq \epsilon;\ |y_2 - y_1| \leq \epsilon;\ |z| \leq \|u\|\}$.

By the uniform continuity of the function $f(x,y,s,t,z)$ on the set $J \times J \times [-\|u\|, \|u\|]$ we deduce that $w_1^{a,b}(f,\epsilon;\|u\|) \to 0$ as $\epsilon \to 0$. Let us denote

$$\bar{n}(a,b) = \max\{n(x,y) : (x,y) \in [0,a]\times[0,b]\},$$

$$\bar{f}_1(a,b) = \max\{f_1(x,y) : (x,y) \in [0,a]\times[0,b]\}.$$

Now, we have two distinguish cases:
Case 1: $0 < \alpha < 1$, $0 < \gamma < 1$
by Lemma 2, $(x_2^\alpha - x_1^\alpha)^\beta \leq (x_2 - x_1)^{\alpha\beta}$,
$(y_2^\gamma - y_1^\gamma)^\beta \leq (y_2 - y_1)^{\gamma\delta}$
and therefore, from inequality (3.1) it follows
$w^{a,b}(Uu,\epsilon) \leq \frac{a^{\alpha\beta}b^{\gamma\delta}w_1^{a,b}(f,\epsilon;\|u\|)}{\Gamma(1+\beta)\Gamma(1+\delta)}$
$+ \frac{n(x_1,y_1)\phi(\|u\|)+f_1(x_1,y_1)}{\Gamma(1+\beta)\Gamma(1+\delta)}\{\epsilon^{\alpha\beta}(y_1^{\gamma\delta} + y_2^{\gamma\delta}) + \epsilon^{\gamma\delta}x_2^{\alpha\beta}\}$
$+\frac{n(x_2,y_2)\phi(\|u\|)+f_1(x_2,y_2)}{\Gamma(1+\beta)\Gamma(1+\delta)}\{\epsilon^{\gamma\delta}x_2^{\alpha\beta} + \epsilon^{\alpha\beta}y_1^{\gamma\delta}\}$

$$w^{a,b}(Uu,\epsilon) \leq \frac{a^{\alpha\beta}b^{\gamma\delta}w_1^{a,b}(f,\epsilon;\|u\|)}{\Gamma(1+\beta)\Gamma(1+\delta)} + \frac{\bar{n}(a,b)\phi(\|u\|)+\bar{f}_1(a,b)}{\Gamma(1+\beta)\Gamma(1+\delta)}\{3\epsilon^{\alpha\beta}b^{\gamma\delta} + 2\epsilon^{\gamma\delta}a^{\alpha\beta}\}. \tag{3.2}$$

Case 2: $\alpha \geq 1$, $\gamma \geq 1$
by Lemma 2, $(x_2^\alpha - x_1^\alpha)^\beta \leq \alpha^\beta(x_2 - x_1)^\beta$
$(y_2^\gamma - y_1^\gamma)^\beta \leq \gamma(y_2 - y_1)^\delta$
and therefore, from inequality (3.1) it follows

$w^{a,b}(Uu,\epsilon) \leq \frac{a^{\alpha\beta}b^{\gamma\delta}w_1^{a,b}(f,\epsilon;\|u\|)}{\Gamma(1+\beta)\Gamma(1+\delta)}$
$+ \frac{n(x_1,y_1)\phi(\|u\|)+f_1(x_1,y_1)}{\Gamma(1+\beta)\Gamma(1+\delta)}\{\alpha^\beta\epsilon^\beta(y_1^{\gamma\delta} + y_2^{\gamma\delta}) + \gamma^\delta\epsilon^\delta x_2^{\alpha\beta}\}$
$+\frac{n(x_2,y_2)\phi(\|u\|)+f_1(x_2,y_2)}{\Gamma(1+\beta)\Gamma(1+\delta)}\{\gamma^\delta\epsilon^\delta x_2^{\alpha\beta} + \alpha^\beta\epsilon^\beta y_1^{\gamma\delta}\}$

$$w^{a,b}(Uu,\epsilon) \leq \frac{a^{\alpha\beta}b^{\gamma\delta}w_1^{a,b}(f,\epsilon;\|u\|)}{\Gamma(1+\beta)\Gamma(1+\delta)} + \frac{\bar{n}(a,b)\phi(\|u\|)+\bar{f}_1(a,b)}{\Gamma(1+\beta)\Gamma(1+\delta)}\{3\epsilon^{\alpha\beta}b^{\gamma\delta}\alpha^\beta + 2\epsilon^{\gamma\delta}a^{\alpha\beta}\gamma^\delta\}. \tag{3.3}$$

Linking both cases (3.2), (3.3) with the above established facts we conclude that the function Uu is continuous on the subset $[0,a]\times[0,b]$ for any $a,b > 0$. This gives the continuity of Uu on $R_+ \times [0,b]$. Consequently, the function Vu is continuous on $R_+ \times [0,b]$.
Now, for an arbitrary function $u \in BC(R_+ \times [0,b])$ and using our assumptions, for a fixed $(x,y) \in R_+ \times [0,b]$, we have

$|(Vu)(x,y)| \leq |g(x,y)| + \frac{\alpha\gamma\{|h(x,y,u(x,y)) - h(x,y,0)| + |h(x,y,0)|\}}{\Gamma(\beta)\Gamma(\delta)}$
$\times \int\limits_0^x \int\limits_0^y \frac{s^{\alpha-1}t^{\gamma-1}\{|f(x,y,s,t,u(s,t)) - f(x,y,s,t,0)| + |f(x,y,s,t,0)|\}}{(x^\alpha - s^\alpha)^{1-\beta}(y^\gamma - t^\gamma)^{1-\delta}} dt\ ds$
$\leq \|g\| + \frac{\alpha\gamma\{m(x,y)|u(x,y)| + |h(x,y,0)|\}}{\Gamma(\beta)\Gamma(\delta)} \int\limits_0^x \int\limits_0^y \frac{s^{\alpha-1}t^{\gamma-1}\{n(x,y)\phi(|u(s,t)|) + f_1(x,y)\}}{(x^\alpha - s^\alpha)^{1-\beta}(y^\gamma - t^\gamma)^{1-\delta}} dt\ ds$
$\leq \|g\| + \frac{\alpha\gamma\{m(x,y)\|u\| + |h(x,y,0)|\}\{n(x,y)\phi(\|u\|) + f_1(x,y)\}}{\Gamma(\beta)\Gamma(\delta)}$
$\times \int\limits_0^x \int\limits_0^y \frac{s^{\alpha-1}t^{\gamma-1} dt\ ds}{(x^\alpha - s^\alpha)^{1-\beta}(y^\gamma - t^\gamma)^{1-\delta}}$
$\leq\ |g\| + \frac{1}{\Gamma(1+\beta)\Gamma(1+\delta)}\{m(x,y)n(x,y)x^{\alpha\beta}y^{\gamma\delta}\|u\|\phi(\|u\|) + m(x,y)f_1(x,y)$
$x^{\alpha\beta}y^{\gamma\delta}\|u\| + n(x,y)|h(x,y,0)|x^{\alpha\beta}y^{\gamma\delta}\phi(\|u\|) + f_1(x,y)|h(x,y,0)|x^{\alpha\beta}y^{\gamma\delta}\}$

$$\leq \|g\| + \frac{1}{\Gamma(1+\beta)\Gamma(1+\delta)}\{p(x,y)\|u\|\phi(\|u\|) + q(x,y)\|u\| + r(x,y)\phi(\|u\|) + s(x,y)\}. \tag{3.4}$$

As a result, from the above inequality in view of assumption $(H4)$, the function Vu is bounded on $R_+ \times [0,b]$ and also we conclude that $Vu \in BC(R_+ \times [0,b])$, so from the estimate (3.4), we get $\|Vu\| \leq \|g\| + \frac{1}{\Gamma(1+\beta)\Gamma(1+\delta)}\{P\|u\|\phi(\|u\|) + Q\|u\| + R\phi(\|u\|) + S\}$.
Similarly, from the above estimate and assumption $(H5)$, we deduce that there exists positive number λ_0 such that the operator V transforms the ball B_{λ_0} into itself.
Finally, the operator V transforms the space $BC(R_+ \times [0,b])$ into itself, so second assertion of our lemma is obvious. □

Theorem 3.2. *The equation (1.1) has at least one solution $u = u(x,y)$ belonging to the space $BC(R_+ \times [0,b])$ under the assumptions$(H1 - H5)$. In addition, solutions of equation (1.1) are uniformly locally attractive.*

Proof . Suppose let us consider a nonempty set $X \subset B_{\lambda_0}$, where B_{λ_0} is a ball in the space $BC(R_+ \times [0,b])$. Then, in virtue of assumptions $(H2)-(H4)$ for $u, v \in X$ and for an arbitrary fixed $(x,y) \in R_+ \times [0,b]$, we get $|(Vu)(x,y) - (Vv)(x,y)|$
$= \Bigg[\frac{\alpha\gamma h(x,y,u(x,y))}{\Gamma(\beta)\Gamma(\delta)} \int\limits_0^x \int\limits_0^y \frac{s^{\alpha-1}t^{\gamma-1}f(x,y,s,t,u(s,t))}{(x^\alpha - s^\alpha)^{1-\beta}(y^\gamma - t^\gamma)^{1-\delta}} dt\ ds$
$- \frac{\alpha\gamma h(x,y,v(x,y))}{\Gamma(\beta)\Gamma(\delta)} \int\limits_0^x \int\limits_0^y \frac{s^{\alpha-1}t^{\gamma-1}f(x,y,s,t,v(s,t))}{(x^\alpha - s^\alpha)^{1-\beta}(y^\gamma - t^\gamma)^{1-\delta}} dt\ ds\Bigg]$
$\leq \frac{\alpha\gamma|h(x,y,u(x,y)) - h(x,y,v(x,y))|}{\Gamma(\beta)\Gamma(\delta)} \int\limits_0^x \int\limits_0^y \frac{s^{\alpha-1}t^{\gamma-1}|f(x,y,s,t,u(s,t))|}{(x^\alpha - s^\alpha)^{1-\beta}(y^\gamma - t^\gamma)^{1-\delta}} dt\ ds$
$+ \frac{\alpha\gamma|h(x,y,v(x,y))|}{\Gamma(\beta)\Gamma(\delta)} \int\limits_0^x \int\limits_0^y \frac{s^{\alpha-1}t^{\gamma-1}|f(x,y,s,t,u(s,t)) - f(x,y,s,t,v(s,t))|}{(x^\alpha - s^\alpha)^{1-\beta}(y^\gamma - t^\gamma)^{1-\delta}} dt\ ds$
$\leq \frac{\alpha\gamma m(x,y)|u(x,y) - v(x,y)|}{\Gamma(\beta)\Gamma(\delta)} \int\limits_0^x \int\limits_0^y \frac{s^{\alpha-1}t^{\gamma-1}\{n(x,y)\phi(|u(s,t)|) + f_1(x,y)\}}{(x^\alpha - s^\alpha)^{1-\beta}(y^\gamma - t^\gamma)^{1-\delta}} dt\ ds$
$+ \frac{\alpha\gamma n(x,y)\{m(x,y)|v(x,y)| + |h(x,y,0)|\}}{\Gamma(\beta)\Gamma(\delta)} \int\limits_0^x \int\limits_0^y \frac{s^{\alpha-1}t^{\gamma-1}\phi(|u(s,t) - v(s,t)|)}{(x^\alpha - s^\alpha)^{1-\beta}(y^\gamma - t^\gamma)^{1-\delta}} dt\ ds$
$\leq \frac{\alpha\gamma 2m(x,y)n(x,y)p_0\phi(p_0)}{\Gamma(\beta)\Gamma(\delta)} \int\limits_0^x \int\limits_0^y \frac{s^{\alpha-1}t^{\gamma-1} dt\ ds}{(x^\alpha - s^\alpha)^{1-\beta}(y^\gamma - t^\gamma)^{1-\delta}} + \frac{\alpha\gamma m(x,y)f_1(x,y)}{\Gamma(\beta)\Gamma(\delta)}$
$\times \operatorname{diam} X(x,y) \int\limits_0^x \int\limits_0^y \frac{s^{\alpha-1}t^{\gamma-1} dt\ ds}{(x^\alpha - s^\alpha)^{1-\beta}(y^\gamma - t^\gamma)^{1-\delta}} + \frac{\alpha\gamma m(x,y)n(x,y)p_0\phi(2p_0)}{\Gamma(\beta)\Gamma(\delta)}$
$\times \int\limits_0^x \int\limits_0^y \frac{s^{\alpha-1}t^{\gamma-1} dt\ ds}{(x^\alpha - s^\alpha)^{1-\beta}(y^\gamma - t^\gamma)^{1-\delta}} + \frac{\alpha\gamma n(x,y)|h(x,y,0)|\phi(2p_0)}{\Gamma(\beta)\Gamma(\delta)}$

$\times \int\limits_0^x \int\limits_0^y \frac{s^{\alpha-1}t^{\gamma-1}dt\,ds}{(x^\alpha-s^\alpha)^{1-\beta}(y^\gamma-t^\gamma)^{1-\delta}}$
$\leq \frac{2p(x,y)}{\Gamma(1+\beta)\Gamma(1+\delta)}\lambda_0\phi(\lambda_0) + \frac{p(x,y)}{\Gamma(1+\beta)\Gamma(1+\delta)}\lambda_0\phi(2\lambda_0)$

$$+ \frac{r(x,y)}{\Gamma(1+\beta)\Gamma(1+\delta)}\phi(2\lambda_0) + \frac{q(x,y)}{\Gamma(1+\beta)\Gamma(1+\delta)}\text{diam } X(x,y). \tag{3.5}$$

Now, applying our assumptions, above estimate yields
diam $(VX)(x,y) \leq \frac{2p(x,y)}{\Gamma(1+\beta)\Gamma(1+\delta)}\lambda_0\phi(\lambda_0) + \frac{p(x,y)}{\Gamma(1+\beta)\Gamma(1+\delta)}\lambda_0\phi(2\lambda_0)$

$$+ \frac{r(x,y)}{\Gamma(1+\beta)\Gamma(1+\delta)}\phi(2\lambda_0) + \frac{q(x,y)}{\Gamma(1+\beta)\Gamma(1+\delta)}\text{diam } X(x,y). \tag{3.6}$$

Also by applying assumption $(H4)$ we have

$$\lim_{x\to\infty} \sup \text{diam } (VX)(x,y) \leq k \lim_{x\to\infty} \sup \text{diam } X(x,y) \tag{3.7}$$

for each fixed $y \in [0,b]$,

where $$k = \frac{B}{\Gamma(1+\beta)\Gamma(1+\delta)}.$$

It is clear from the assumption $(H5)$, that $k < 1$.
Now, let us suppose arbitrary positive numbers a, b and $\epsilon > 0$. Also we fix an arbitrary function $u \in X$; $x_1, x_2 \in [0,a]$ and $y_1, y_2 \in [0,b]$ such that $|x_2 - x_1| < \epsilon$ and $|y_2 - y_1| < \epsilon$. Without loss of generality, we may assume that $x_1 < x_2$ and $y_1 < y_2$. Then using our assumptions and obtained estimate (3.2), when $0 < \alpha < 1$ we get
$|(Vu)(x_2,y_2) - (Vu)(x_1,y_1)|$
$\leq |g(x_2,y_2) - g(x_1,y_1)| + |(Wu)(x_2,y_2)(Uu)(x_2,y_2) - (Wu)(x_1,y_1)(Uu)(x_2,y_2)|$
$+ |(Wu)(x_1,y_1)(Uu)(x_2,y_2) - (Wu)(x_1,y_1)(Uu)(x_1,y_1)|$
$\leq w^{a,b}(g,\epsilon) + \frac{\alpha\gamma|h(x_2,y_2,u(x_2,y_2)) - h(x_1,y_1,u(x_1,y_1))|}{\Gamma(\beta)\Gamma(\delta)}$
$\times \int\limits_0^{x_2}\int\limits_0^{y_2} \frac{s^{\alpha-1}t^{\gamma-1}f(x_2,y_2,s,t,u(s,t))}{(x^\alpha-s^\alpha)^{1-\beta}(y^\gamma-t^\gamma)^{1-\delta}}dt\,ds + \frac{|h(x_1,y_1,u(x_1,y_1))|}{\Gamma(1+\beta)\Gamma(1+\delta)}[a^{\alpha\beta}b^{\gamma\delta}w_1^{a,b}(f,\epsilon;\lambda_0)$
$+ \{\bar{n}(a,b)\phi(\lambda_0) + \bar{f}_1(a,b)\}\{3\epsilon^{\alpha\beta}b^{\gamma\delta} + 2\epsilon^{\gamma\delta}a^{\alpha\beta}\}]$
$\leq w^{a,b}(g,\epsilon) + \frac{\alpha\gamma\left\{m(x_2,y_2)|u(x_2,y_2)-u(x_1,y_1)| + w_1^{a,b}(h,\epsilon)\right\}}{\Gamma(\beta)\Gamma(\delta)}$
$\times \int\limits_0^{x_2}\int\limits_0^{y_2} \frac{s^{\alpha-1}t^{\gamma-1}\{n(x_2,y_2)\phi(|u(s,t)|) + f_1(x_2,y_2)\}}{(x^\alpha-s^\alpha)^{1-\beta}(y^\gamma-t^\gamma)^{1-\delta}}dt\,ds + \frac{m(x_1,y_1)|u(x_1,y_1)| + |h(x,y,0)|}{\Gamma(1+\beta)\Gamma(1+\delta)}$
$\times [a^{\alpha\beta}b^{\gamma\delta}w_1^{a,b}(\lambda,\epsilon;\lambda_0) + \{\bar{n}(a,b)\phi(\lambda_0) + \bar{f}_1(a,b)\}\{3\epsilon^{\alpha\beta}b^{\gamma\delta} + 2\epsilon^{\gamma\delta}a^{\alpha\beta}\}]$
$\leq w^{a,b}(g,\epsilon) + \frac{w^{a,b}(u,\epsilon)}{\Gamma(1+\beta)\Gamma(1+\delta)}\{m(x_2,y_2)n(x_2,y_2)x_2^{\alpha\beta}y_2^{\gamma\delta}\phi(\lambda_0)$
$+ m(x_2,y_2)f_1(x_2,y_2)x_2^{\alpha\beta}y_2^{\gamma\delta}\} + \frac{w_1^{a,b}(h,\epsilon)a^{\alpha\beta}b^{\gamma\delta}}{\Gamma(1+\beta)\Gamma(1+\delta)}\{\bar{n}(a,b)\phi(\lambda_0) + \bar{f}_1(a,b)\}$
$+ \frac{\bar{m}(a,b)\lambda_0 + \bar{h}(a,b)}{\Gamma(1+\beta)\Gamma(1+\delta)}[a^{\alpha\beta}b^{\gamma\delta}w_1^{a,b}(f,\epsilon;\lambda_0) + \{\bar{n}(a,b)\phi(\lambda_0) + \bar{f}_1(a,b)\}\{3\epsilon^{\alpha\beta}b^{\gamma\delta}$
$+ 2\epsilon^{\gamma\delta}a^{\alpha\beta}\}]$
$\leq w^{a,b}(g,\epsilon) + \frac{P\phi(p_0)+Q}{\Gamma(1+\beta)\Gamma(1+\delta)}w^{a,b}(u,\epsilon) + \frac{w_1^{a,b}(h,\epsilon)a^{\alpha\beta}b^{\gamma\delta}}{\Gamma(1+\beta)\Gamma(1+\delta)}$
$\times \{\bar{n}(a,b)\phi(\lambda_0) + \bar{f}_1(a,b)\} + \frac{\bar{m}(a,b)\lambda_0 + \bar{f}(a,b)}{\Gamma(1+\beta)\Gamma(1+\delta)}[a^{\alpha\beta}b^{\gamma\delta}w_1^{a,b}(f,\epsilon;\lambda_0)$

$$+ \{\bar{n}(a,b)\phi(\lambda_0) + \bar{f}_1(a,b)\}\{3\epsilon^{\alpha\beta}b^{\gamma\delta} + 2\epsilon^{\gamma\delta}a^{\alpha\beta}\}], \tag{3.8}$$

Similarly, when $\alpha > 1$ and obtained estimate (3.3), we get
$|(Vu)(x_2, y_2) - (Vu)(x_1, y_1)|$
$\leq w^{a,b}(g, \epsilon) + \frac{P\phi(\lambda_0)+Q}{\Gamma(1+\beta)\Gamma(1+\delta)} w^{a,b}(u, \epsilon) + \frac{w_1^{a,b}(h,\epsilon)a^{\alpha\beta}b^{\gamma\delta}}{\Gamma(1+\beta)\Gamma(1+\delta)}$
$\times \{\bar{n}(a, b)\phi(\lambda_0) + \bar{f}_1(a, b)\} + \frac{\bar{m}(a,b)\lambda_0+\bar{f}(a,b)}{\Gamma(1+\beta)\Gamma(1+\delta)} [a^{\alpha\beta}b^{\gamma\delta}w_1^{a,b}(f, \epsilon; \lambda_0)$

$$+ \{\bar{n}(a, b)\phi(\lambda_0) + \bar{f}_1(a, b)\}\{3\epsilon^{\alpha\beta}b^{\gamma\delta}\alpha^{\beta} + 2\epsilon^{\gamma\delta}a^{\alpha\beta}\gamma^{\delta}\}], \tag{3.9}$$

where
$w_1^{a,b}(h, \epsilon) = \sup\{|h(x_2, y_2, u) - h(x_1, y_1, u)| : x_1, x_2 \in [0, a]; y_1, y_2 \in [0, b]; |x_2 - x_1| \leq \epsilon; |y_2 - y_1| \leq \epsilon; u \in [-\lambda_0, \lambda_0]\}$,
$\bar{m}(a, b) = \max\{m(x, y) : (x, y) \in [0, a] \times [0, b]\}$,
$\bar{h}(a, b) = \max\{h(x, y, 0) : (x, y) \in [0, a] \times [0, b]\}$.

Now, by the uniform continuity of the function $f = f(x, y, s, t, u)$ on $J \times J \times [-\lambda_0, \lambda_0]$ and the uniform continuity of the function $h = h(x, y, u)$ on $J \times [-\lambda_0, \lambda_0]$, it has been concluded from both the estimates (3.8) and (3.9) that

$$w_0^{a,b}(VX) \leq k w_0^{a,b}(X).$$

Consequently,

$$w_0(VX) \leq k w_0(X). \tag{3.10}$$

Now linking equations (3.7) and (3.10), we get

$$\mu(VX) \leq k\mu(X), \tag{3.11}$$

Further let us construct a nonempty, bounded, closed and convex set S, for which the sequence $(B^n_{\lambda_0})$ has been constructed, where $B^1_{\lambda_0} = ConvV(B_{\lambda_0})$, $B^2_{\lambda_0} = ConvV(B^1_{\lambda_0})$.... . Obviously all sets of this sequence are nonempty, bounded, convex and closed. A part from this it has been observed that $B^{n+1}_{\lambda_0} \subset B^n_{\lambda_0} \subset B_{\lambda_0}$ for $n = 1, 2, 3, ...$. Thus, applying $k < 1$ and taking into account equation (3.11), it has been concluded that $\lim_{n\to\infty} \mu(B^n_{\lambda_0}) = 0$. Hence, using the axiom (vi) of Definition (2.1), the set $S = \bigcap_{n=1}^{\infty} B^n_{\lambda_0}$ is nonempty, bounded, convex and closed. However, by the remark 1, $S \in \ker\mu$. In particular,

$$\lim_{x\to\infty} \sup \operatorname{diam} S(x, y) = \lim_{x\to\infty} \operatorname{diam} S(x, y) = 0, \tag{3.12}$$

for each fixed $y \in [0, b]$. It has been witnessed that the operator V maps S into itself and also V is continuous on the set S.
Let us fix $\epsilon > 0$ and suppose $u, v \in S$ be arbitrary functions such that $\|u - v\| \leq \epsilon$. Then by linking (3.12) and $VS \subset S$ it has been derived that there exists $a, b > 0$ such that for an arbitrary $x \geq a$, it follows

$$|(Vu)(x, y) - (Vv)(x, y)| \leq \epsilon. \tag{3.13}$$

Further, take $(x, y) \in J$. Then, proceeding as above, we obtain
$|(Vu)(x, y) - (Vv)(x, y)|$
$\leq \frac{\alpha\gamma m(x,y)|u(x,y)-v(x,y)|}{\Gamma(\beta)\Gamma(\delta)} \int\limits_0^x \int\limits_0^y \frac{s^{\alpha-1}t^{\gamma-1}\{n(x,y)\phi(|u(s,t)|)+f_1(x,y)\}}{(x^{\alpha}-s^{\alpha})^{1-\beta}(y^{\gamma}-t^{\gamma})^{1-\delta}} dt\ ds$
$+ \frac{\alpha\gamma\{m(x,y)|v(x,y)|+|h(x,y,0)|\}n(x,y)}{\Gamma(\beta)\Gamma(\delta)} \int\limits_0^x \int\limits_0^y \frac{s^{\alpha-1}t^{\gamma-1}\phi(|u(s,t)-v(s,t)|)}{(x^{\alpha}-s^{\alpha})^{1-\beta}(y^{\gamma}-t^{\gamma})^{1-\delta}} dt\ ds$
$\leq \frac{\alpha\gamma\{m(x,y)n(x,y)\phi(\lambda_0)+m(x,y)f_1(x,y)\}\epsilon}{\Gamma(\beta)\Gamma(\delta)} \int\limits_0^x \int\limits_0^y \frac{s^{\alpha-1}t^{\gamma-1}dt\ ds}{(x^{\alpha}-s^{\alpha})^{1-\beta}(y^{\gamma}-t^{\gamma})^{1-\delta}}$

$$+\frac{\alpha\gamma\{m(x,y)n(x,y)\lambda_0+|h(x,y,0)|n(x,y)\}\phi(\epsilon)}{\Gamma(\beta)\Gamma(\delta)}\int_0^x\int_0^y\frac{s^{\alpha-1}t^{\gamma-1}dt\,ds}{(x^\alpha-s^\alpha)^{1-\beta}(y^\gamma-t^\gamma)^{1-\delta}}$$

$$\leq\frac{p(x,y)\phi(\lambda_0)+q(x,y)}{\Gamma(1+\beta)\Gamma(1+\delta)}\epsilon+\frac{p(x,y)\lambda_0+r(x,y)}{\Gamma(1+\beta)\Gamma(1+\delta)}\phi(\epsilon)\leq\frac{P\phi(\lambda_0)+Q}{\Gamma(1+\beta)\Gamma(1+\delta)}\epsilon+\frac{P\lambda_0+R}{\Gamma(1+\beta)\Gamma(1+\delta)}\phi(\epsilon).$$

Now, connecting (3.13) and (3.14) with the assumption $(H4)$, it has been concluded that the operator V is a self continuous mapping from the set S into itself.
Lastly, by the classical Schauder fixed point principle, it has been concluded that V has at least one fixed point u in the set S . Let us observe that the function $u = u(x, y)$ is a solution of the quadratic fractional integral equation (1.1) by the Lemma (3.1).
By using Definition (2.3), we deduce that

$$\lim_{x\to\infty}(u(x,y)-v(x,y))=0,$$

for each fixed $y \in [0, b]$. Consequently, equation (1.1) has a solution and all solutions are uniformly locally attractive on $R_+ \times [0, b]$.
□

Remark 3.3. *The behavior of solution can be analysed from the fact that as $S \in ker\mu$ where S is a nonempty bounded convex and closed subset, then by the remark 2 we can conclude that the thickness of the bundle formed by solutions of equation (1.1) from S tends to zero at infinity.*

4. Numerical Example

Example 4.1. *Consider the following quadratic fractional integral equation of fractional order:*

$$u(x,y)=e^{-(x^2+y^2)}+\frac{5}{3}\left(\frac{x+y+x^{\frac{2}{3}}y^{\frac{2}{3}}u(x,y)}{\Gamma(\frac{3}{4})\Gamma(\frac{4}{5})}\right)\int_0^x\int_0^y\frac{s^{1/3}t^{1/4}(e^{-5xy-3st}\sqrt{u(s,t)}+\frac{1}{7x^{5/3}y^{5/3}})}{(x^{4/3}-s^{4/3})^{1/4}(y^{5/4}-t^{5/4})^{1/5}}dt\,ds,\quad(4.1)$$

where $(x, y) \in R_+ \times [0, 1]$.
The above equation is a particular case of equation (1.1).

$$g(x,y)=e^{-(x^2+y^2)},$$
$$h(x,y,u)=x+y+x^{\frac{2}{3}}y^{\frac{2}{3}}u(x,y),$$
$$f(x,y,s,t,u)=e^{-5xy-3st}\sqrt{u(s,t)}+\frac{1}{7x^{5/3}y^{5/3}},$$
$$\alpha=4/3,\beta=3/4,\gamma=5/4,\delta=4/5$$

which satisfied the assumptions of Theorem 3.2. Obviously the function $g(x, y)$ satisfy assumption $(H1)$ with $\|g\| = e^0 = 1$.
Further, observe that the assumption $(H2)$ is satisfied with $m(x, y) = x^{2/3}y^{2/3}$ and $|h(x, y, 0)| = x+y$. Moreover, from assumption $(H3)$, we have

$$|f(x,y,s,t,u)-f(x,y,s,t,v)|\leq e^{-5xy}|\sqrt{u}-\sqrt{v}|.$$

Since

$$|u-v|=|\sqrt{u}-\sqrt{v}||\sqrt{u}+\sqrt{v}|$$

$$i.e., |\sqrt{u} - \sqrt{v}| = \frac{|u-v|}{|\sqrt{u} + \sqrt{v}|}$$
$$i.e., |\sqrt{u} - \sqrt{v}| < |u-v|.$$

Observe that the function $f(x, y, s, t, u)$ *satisfies assumption* $(H3)$ *with*

$$n(x, y) = e^{-5xy},$$
$$\phi(\lambda) = \sqrt{\lambda},$$

and

$$f_1(x, y) = f(x, y, s, t, 0) = \frac{1}{7x^{5/3}y^{5/3}}.$$

Also

$$p(x, y) = x^{5/3}y^{5/3}e^{-5xy},$$
$$q(x, y) = \frac{1}{7},$$
$$r(x, y) = (x + y)xye^{-5xy},$$
$$s(x, y) = \frac{x + y}{7x^{2/3}y^{2/3}}.$$

Let us observe that $p(x, y) \to 0$ *as* $x \to \infty$ *and* $P = (1/3)^{5/3}e^{-5/3} = 0.030267\ldots$. *It is also seen that the function* $q(x, y)$ *is constant on* $R_+ \times [0, 1]$ *and* $Q = 0.142857\ldots$. *Let us see that* $r(x, y) \to 0$ *as* $x \to \infty$ *and* $R = 2(3/10)^{3/2}e^{-3/2} = 0.073328\ldots$. *Also, we check that* $s(x, y) \to 0$ *as* $x \to \infty$ *and* $S = 0.22907\ldots.$.
Finally, the inequality from the assumption $(H5)$ *comes in the form*

$$e^0 + \frac{1}{\Gamma(7/4)\Gamma(9/5)}[P\lambda^{3/2} + Q\lambda + R\lambda^{1/2} + S] \leq \lambda.$$

Let us write the inequality in the form:

$$\Gamma(7/4)\Gamma(9/5) + P\lambda^{3/2} + Q\lambda + R\lambda^{1/2} + S \leq \lambda\ \Gamma(7/4)\Gamma(9/5). \tag{4.2}$$

Let us denote the left-hand side of this inequality by $L(\lambda)$, *i.e.,*

$$L(\lambda) = \Gamma(7/4)\Gamma(9/5) + P\lambda^{3/2} + Q\lambda + R\lambda^{1/2} + S.$$

Here, keeping in mind the above established values of the constants P, Q, R, S *for* $\lambda = 2$, *we obtain*

$$L(2) = \Gamma(7/4)\Gamma(9/5) + 0.08562 + 0.28572 + \ldots.$$

Hence, it has been observed that $\lambda_0 = 2$ *is a solution of the inequality (4.2), since* $\Gamma(7/4) \simeq 0.9197$ *and* $\Gamma(9/5) \simeq 0.9322$.
Moreover,

$$Q \simeq 0.14286.. < \Gamma(7/4)\Gamma(9/5).$$

Thus, on the basis of the Theorem 3.2, equation (4.1) has at least one solution in the space $BC(R_+ \times [0, 1])$ *which belongs to the ball* B_{λ_0}. *Moreover, solutions of equation (4.1) are uniformly locally attractive in the sense of Definition (2.3) which means that for arbitrary solutions* $u(x, y)$ *and* $v(x, y)$ *of equation (4.1) belonging to* B_{λ_0} *for each fixed* $y \in [0, 1]$, *we have that*

$$\lim_{x\to\infty} (u(x, y) - v(x, y)) = 0,$$

uniformly with respect to the ball B_{λ_0}.

5. Conclusion

In this work we have derived the sufficient condition for the existence of solution of a quadratic fractional integral equation in two variables on an unbounded interval. Also we have shown that the the solutions are uniformly locally attractive. Finally, an example has been given to substantiate the result.

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