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A necessary condition for multiple objective fractional programming

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Abstract

In this paper, we establish a proof for a necessary condition for multiple objective fractional programming. In order to derive the set of necessary conditions, we employ an equivalent parametric problem. Also, we present the related semi parametric model.

Keywords: multiple objective fractional programming; generalized n-set convex function; efficient solution.
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1. Introduction

Suppose that $\mathcal{A}^n$ is the n-fold product of the $\sigma-$algebra $\mathcal{A}$ of the subsets of a given set X. F_i, G_i, $i \in \underline{p} \equiv \{1, 2, \ldots, p\}$, H_j, $j \in \underline{m} \equiv \{1, 2, \ldots, m\}$, are real valued functions defined on $\mathcal{A}^n$. Also for each $i \in \underline{p}$, $G_i(S) > 0$ for all $S \in \mathcal{A}^n$ such that $H_j(S) \leqq 0$, for all $j \in \underline{m}$.

Consider the following multi-objective fractional subset programming problem:

$$\text{(P)} \qquad \text{Minimize} : \left(\frac{F_1(S)}{G_1(S)}, \frac{F_2(S)}{G_2(S)}, \ldots, \frac{F_p(S)}{G_p(S)}\right)$$

$$\text{Subject to} : H_j(S) \leqq 0, j \in \underline{m},\ S \in \mathcal{A}^n,$$

The point-function counterparts of (P) are known in the area of mathematical programming as multi-objective fractional programming problems. These problems have been the focus of interest in the past few years, which has resulted in numerous publications such as [14]. For more information about general multi-objective problems with point-functions see [3, 15, 16].

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In the area of subset programming, multi-objective problems have been investigated in [4, 7], and multi-objective fractional problems in [5, 6]. Much attention has been paid to analysis of optimization problems with set functions, for example see Chou [1], Corley [2], Kim [7], Lai and Lin[8], Lin [9, 10, 11, 12], Liu [13], Morris [17], Preda [18, 19], Preda and Minasian [20, 21, 22] and Zalmai [23, 24, 25].

In this paper, we first present a proof for necessary conditions on multiple objective fractional programming. To this end, we employ an equivalent parametric problem. Then we give its semi parametric model.

2. Preliminaries

In this section, we gather, for convenience of reference, a number of basic definitions and results that will be used often throughout the sequel.

Let $(X, \mathcal{A}, \mu)$ be a finite atomless measure space with $L^1(X, \mathcal{A}, \mu)$ separable, and let d be the pseudo-metric on $\mathcal{A}''$ defined by

$$d(R, S) = \left[\sum_{i=1}^{n} \mu^2(R_i \Delta S_i)\right]^{1/2},$$

for all $R = (R_1, R_2, \dots, R_n),\ S = (S_1, S_2, \dots, S_n) \in \mathcal{A}^n$, where Δ denotes the symmetric difference; thus $(\mathcal{A}^n, d)$ is a pseudo-metric space. For $h \in L^1(X, \mathcal{A}, \mu)$ and $T \in \mathcal{A}$ with characteristic function $\chi_T \in L^\infty(X, \mathcal{A}, \mu)$, the integral $\int_T h d\mu$ will be denoted by $\langle h, \chi_T \rangle$.

The notion of differentiability was originally introduced by Morris [17] for set functions and extended by Corley [2] for n-set functions.

Definition 2.1. A function $F : \mathcal{A} \to R$ is said to be differentiable at S^* if there exists $DF(S^*) \in L^1(X, \mathcal{A}, \mu)$, called the derivative of F at S^*, such that for each $S \in \mathcal{A}$,

$$F(S) = F(S^*) + \langle DF(S^*), \chi_S - \chi_{S^*} \rangle + V_F(S, S^*),$$

where $V_F(S, S^*)$ is $o\left(d(S, S^*)\right)$, that is

$$\lim_{d(S,S^*) \to 0} V_F(S, S^*)/d(S, S^*) = 0.$$

Definition 2.2. A function $G : \mathcal{A}^n \to R$ is said to have a partial derivative at $S^* = (S_1^*, S_2^*, \dots, S_n^*) \in \mathcal{A}^n$ with respect to its i^{th} argument if the function $F(S_i) = G(S_1^*, \dots, S_{i-1}^*, S_i^*, S_{i+1}^*, \dots, S_n^*)$ has derivative $DF(S_i^*)$, $i \in \underline{n}$; in that case, the i^{th} partial derivative of G at S^* is defined to be $D_iG(S^*) = DF(S_i^*)$, $i \in \underline{n}$. A function $G : \mathcal{A}^n \to R$ is said to be differentiable at S^* if all the partial derivatives $D_iG(S^*)$, $i \in \underline{n}$ exist and satisfy the following equation:

$$G(S) = G(S^*) + \sum_{i=1}^{n} \left\langle DG_i(S^*), \chi_{S_i} - \chi_{S_i^*} \right\rangle + W_G(S, S^*),$$

where $W_G(S, S^*)$ is $o\left(d(S, S^*)\right)$, for all $S \in \mathcal{A}^n$.

Notation 2.3. For $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ and $b = (b_1, \dots, b_n) \in \mathbb{R}^n$, we denote by $a \geqq b$, when $a_i \geq b_i$ for $1 \leq i \leq n$.

3. The main results

In order to derive a set of necessary conditions for (P), we employ a Dinkelbach-type [3] indirect approach via the following auxiliary problem:

$$(\mathrm{P}\lambda) \qquad \text{Minimize}_{S\in F}\left(F_1(S)-\lambda_1 G_1(S),\ldots,F_p(S)-\lambda_p G_p(S)\right),$$

where λ_i, $i \in \underline{p}$, are parameters. This problem is equivalent to (P) in the sense that for particular choices of λ_i, $i \in \underline{p}$, the two problems have the same set of efficient solutions. The equivalence is stated more precisely in the following lemma.

Lemma 3.1. *$S^* \in F$ is an efficient solution of* (P) *if and only if it is an efficient solution of* $(\mathrm{P}\lambda^*)$ *with $\lambda_i^* = F_i(S^*)/G_i(S^*)$, $i \in \underline{p}$.*

Proof . Straightforward. □

Theorem 3.2. *Let F_i, G_i, for $i \in \underline{p}$, and H_j for $j \in \underline{q}$ in S^* be differentiable and for all $i \in \underline{p}$, there exists $\widehat{S}^i$ such that*

$$\begin{cases} H_j(S^*) + \displaystyle\sum_{k=1}^{n}\langle D_k H_j(S^*), \chi_{\widehat{S}^i_k} - \chi_{S_k{}^*}\rangle < 0,\ j \in \underline{q}, \\ \displaystyle\sum_{k=1}^{n}\langle D_k F_l(S^*) - \lambda_l^* D_k G_l(S^*), \chi_{\widehat{S}^l_k} - \chi_{S_k{}^*}\rangle < 0,\ l \in \underline{p}\setminus\{i\}. \end{cases} \qquad (I)$$

If S^* is an efficient solution of (P) *and $\lambda_i^* = F_i(S^*)/G_i(S^*)$, for $i \in \underline{p}$, then there exist $u^* \in U = \left\{u \in \mathbb{R}^n : u > 0, \sum_{i=1}^{p} u_i = 1\right\}$ and $v^* \in \mathbb{R}^q_+$ such that*

$$\sum_{k=1}^{n}\left\langle \sum_{i=1}^{p} u_i^*[D_k F_i(S^*) - \lambda_i^* D_k G_i(S^*)] + \sum_{j=1}^{q} v_j^* D_k H_j(S^*), \chi_{S_k} - \chi_{S_k^*}\right\rangle \geqq 0,$$

and for all $S \in \mathcal{A}^n$, $v_j^* H_j(S^*) = 0$, $j \in \underline{q}$.

Proof . Suppose that S^* be a solution for (P). Consider the following inequalities system:

$$\begin{cases} (i)\ \displaystyle\sum_{k=1}^{n}\langle D_k F_i(S^*) - \lambda_i^* D_k G_i(S^*), \chi_{S_k} - \chi_{S_k^*}\rangle < 0, \\ (ii)\ H_j(S^*) + \displaystyle\sum_{k=1}^{n}\langle D_k H_j(S^*), \chi_{S_k} - \chi_{S_k^*}\rangle < 0,\ j \in \underline{q}. \end{cases} \qquad (II)$$

We show that for each $S \in \mathcal{A}^n$, the system (II) has no solution. By a contradiction, suppose that there exists $S \in \mathcal{A}^n$ that is a solution for (P). We may rewrite system (II) as follows:

$$\begin{cases} (i)\ \left\langle \displaystyle\sum_{k=1}^{n}(D_k F_i(S^*) - \lambda_i^* D_k G_i(S^*)), \sum_{k=1}^{n}(\chi_{S_k} - \chi_{S_k^*})\right\rangle < 0, \\ (ii)\ H_j(S^*) + \left\langle \displaystyle\sum_{k=1}^{n}\langle D_k H_j(S^*), \sum_{k=1}^{n}(\chi_{S_k} - \chi_{S_k^*})\right\rangle < 0,\ j \in \underline{q}. \end{cases}$$

Since H_j for $j \in \underline{q}$ is differentiable, then we have:

$$\begin{aligned}
&H_j\left(S^* + \lambda\left[\sum_{k=1}^{n}(\chi_{S_k} - \chi_{S_k^*})\right]\right)\\
&= H_j(S^*) + \lambda\left\langle \sum_{k=1}^{n} D_k H_j(S^*), \sum_{k=1}^{n}(\chi_{S_k} - \chi_{S_k^*})\right\rangle + o(\lambda)\\
&= (1-\lambda)H_j(S^*) + \lambda H_j(S^*) + \lambda\left\langle \sum_{k=1}^{n}(\chi_{S_k} - \chi_{S_k^*})\right\rangle + o(\lambda)\\
&\leqq \lambda\left(H_j(S^*) + \left\langle \sum_{k=1}^{n} D_k H_j(S^*), \sum_{k=1}^{n}(\chi_{S_k} - \chi_{S_k^*})\right\rangle\right) + o(\lambda),
\end{aligned}$$

for all $j \in \underline{q}$ and $0 < \lambda < 1$. Note that the last inequality follows from feasibility of S^* and $0 < \lambda < 1$.

From (ii) and by considering λ small enough, we obtain:

$$\left(H_j(S^*) + \left\langle \sum_{k=1}^{n} D_k H_j(S^*), \sum_{k=1}^{n}(\chi_{S_k} - \chi_{S_k^*})\right\rangle\right) + \frac{o(\lambda)}{\lambda} < 0.$$

Since $\lambda > 0$, then

$$H_j\left(S^* + \lambda\left[\sum_{k=1}^{n}(\chi_{S_k} - \chi_{S_k^*})\right]\right) < 0.$$

So $S^* + \lambda\left[\sum_{k=1}^{n}(\chi_{S_k} - \chi_{S_k^*})\right]$ is feasible for (P).

On the other hand, since F_i, G_i, $i \in \underline{p}$, are differentiable, if we set $\widehat{S} := S^* + \lambda\left[\sum_{k=1}^{n}(\chi_{S_k} - \chi_{S_k^*})\right]$, we have:

$$\begin{aligned}
&\left(F_1(\widehat{S}) - \lambda_1^* G_1(\widehat{S}), \ldots, F_p(\widehat{S}) - \lambda_p^* G_p(\widehat{S})\right)\\
&= \left(F_1(S^*) - \lambda_1^* G_1(S^*), \ldots, F_p(S^*) - \lambda_p^* G_p(S^*)\right)\\
&+ \lambda\left\langle \sum_{k=1}^{n}(D_k F_i(S^*) - \lambda_i^* D_k G_i(S^*)), \sum_{k=1}^{n}(\chi_{S_k} - \chi_{S_k^*})\right\rangle\\
&+ o(\lambda).
\end{aligned}$$

By applying (i) and small enough λ, we get

$$\left(F_1(\widehat{S}) - \lambda_1^* G_1(\widehat{S}), \ldots, F_p(\widehat{S}) - \lambda_p^* G_p(\widehat{S})\right) < \left(F_1(S^*) - \lambda_1^* G_1(S^*), \ldots, F_p(S^*) - \lambda_p^* G_p(S^*)\right),$$

which is a contradiction by the fact that S^* is a feasible solution. So that system (II) has no solution $S \in \mathcal{A}^n$. Then by Gordan's Lemma, there exist $0 \neq \widehat{u} \in \mathbb{R}_+^p$ and $\widehat{v} \in \mathbb{R}_+^q$ such that

$$\sum_{k=1}^{n}\sum_{i=1}^{p}\widehat{u}_i\langle D_kF_i(S^*)-\lambda_i^*D_kG_i(S^*),\chi_{S_k}-\chi_{S_k^*}\rangle$$
$$+\sum_{j=1}^{q}\widehat{v}_jH_j(S^*)+\sum_{j=1}^{q}\sum_{k=1}^{n}\widehat{v}_j\langle D_kH_j(S^*),\chi_{S_k}-\chi_{S_k^*}\rangle \geqq 0. \quad (III)$$

The above inequality is satisfied for all $S\in\mathcal{A}^n$, so if we let $S:=S^*$, we obtain:

$$\sum_{j=1}^{q}\widehat{v}_jH_j(S^*)\geqq 0.$$

On the other hand, $\widehat{v}_j\geqq 0$, for all $j\in \underline{q}$, and S^* is a feasible solution. Then

$$\sum_{j=1}^{q}\widehat{v}_jH_j(S^*)\leqq 0.$$

It follows that

$$\sum_{j=1}^{q}\widehat{v}_jH_j(S^*)=0.$$

So (III) turns to

$$\sum_{k=1}^{n}\left[\sum_{i=1}^{p}\widehat{u}_i\langle D_kF_i(S^*)-\lambda_i^*D_kG_i(S^*),\chi_{S_k}-\chi_{S_k^*}\rangle\right.$$
$$\left.+\sum_{j=1}^{q}\sum_{k=1}^{n}\widehat{v}_j\langle D_kH_j(S^*),\chi_{S_k}-\chi_{S_k^*}\rangle\right]\geqq 0.$$

Then for all $S\in\mathcal{A}^n$, we have:

$$\sum_{k=1}^{n}\left\langle\sum_{i=1}^{p}\widehat{u}_i[D_kF_i(S^*)-\lambda_i^*D_kG_i(S^*)]\right.$$
$$\left.+\sum_{j=1}^{q}\sum_{k=1}^{n}\widehat{v}_jD_kH_j(S^*),\chi_{S_k}-\chi_{S_k^*}\right\rangle\geqq 0.$$

Set

$$u_i^*:=\frac{\widehat{u}_i}{\sum_{i=1}^{p}\widehat{u}_i},\ i\in\underline{p}\qquad\text{and}\qquad v_i^*:=\frac{\widehat{v}_i}{\sum_{i=1}^{p}\widehat{u}_i},\ j\in\underline{q}.$$

Note that $\sum_{i=1}^{p}\widehat{u}_i>0$. Hence for all $S\in\mathcal{A}^n$:

$$\sum_{k=1}^{n}\left\langle\sum_{i=1}^{p}u_i^*[D_kF_i(S^*)-\lambda_i^*D_kG_i(S^*)]\right.$$
$$\left.+\sum_{j=1}^{q}\sum_{k=1}^{n}v_j^*D_kH_j(S^*),\chi_{S_k}-\chi_{S_k^*}\right\rangle\geqq 0,$$

where $0 \neq u^* \in \mathbb{R}^p_+$ and $v^* \in \mathbb{R}^q_+$. Now we claim that $u^* > 0$. By a contradiction suppose that there exists $h \in \underline{p}$ such that $u^*_h = 0$. By our hypothesis (I), for all $i \in \underline{p}$,

$$H_j(S^*) + \sum_{k=1}^{n} \langle D_k H_j(S^*), \chi_{\widehat{S}^i_k} - \chi_{S^*_k} \rangle < 0,\ j \in \underline{q}.$$

Furthermore, since for all $j \in \underline{q}$, $\widehat{v}_j \geqq 0$, then

$$\widehat{v}_j H_j(S^*) + \sum_{k=1}^{n} \widehat{v}_j \langle D_k H_j(S^*), \chi_{\widehat{S}^i_k} - \chi_{S^*_k} \rangle \leqq 0,\ j \in \underline{q}.$$

It follows that

$$\sum_{j=1}^{q} \widehat{v}_j H_j(S^*) + \sum_{j=1}^{q} \sum_{k=1}^{n} \langle D_k H_j(S^*), \chi_{\widehat{S}^i_k} - \chi_{S^*_k} \rangle \leqq 0.$$

On the other hand, by considering (I), for all $l \in \underline{p}$ we get

$$\sum_{k=1}^{n} \langle G_l(S^*) D_k F_l(S^*) - F_l(S^*) D_k G_l(S^*), \chi_{\widehat{S}^l_k} - \chi_{S^*_k} \rangle < 0,$$

and by the hypothesis $u^* \neq 0$ we have

$$\sum_{l=1, l \neq i}^{p} \sum_{k=1}^{n} \hat{u}_l \langle G_l(S^*) D_k F_l(S^*) - F_l(S^*) D_k G_l(S^*), \chi_{\widehat{S}^l_k} - \chi_{S^*_k} \rangle < 0.$$

This inequality holds for all $i \in \underline{p}$ and in particular, for $i = h$. Since we assumed that $\widehat{u}_h = 0$, we obtain

$$\sum_{i=1}^{p} \sum_{k=1}^{n} \widehat{u}_i \langle G_i(S^*) D_k F_i(S^*) - F_i(S^*) D_k G_i(S^*), \chi_{\widehat{S}^i_k} - \chi_{S^*_k} \rangle < 0.$$

Whence we get the following inequality:

$$\begin{aligned} \sum_{j=1}^{q} \widehat{v}_j H_j(S^*) \quad &+ \quad \sum_{k=1}^{n} \sum_{i=1}^{p} \widehat{u}_i \langle D_k F_i(S^*) - \lambda^*_i D_k G_i(S^*), \chi_{\widehat{S}_k} - \chi_{S^*_k} \rangle \\ &+ \quad \sum_{j=1}^{q} \sum_{k=1}^{n} \widehat{v}_j \langle D_k H_j(S^*), \chi_{\widehat{S}_k} - \chi_{S^*_k} \rangle < 0, \end{aligned}$$

which contradicts (III). So $u^* > 0$, whence $u^* \in U$. □

The form and contents of the necessary efficiency conditions given in the above theorem are used by Zalmai [26] to derive a number of semi-parametric sufficient efficiency criteria as well as for constructing various duality models for (P).

This result is also applicable, when appropriately specialized to the following three classes of problems with multiple, fractional and conventional objective functions, which are particular cases of (P):

$$\text{Minimize}_{S \in X} : (F_1(S), F_2(S), \ldots, F_p(S)) \qquad \text{(P1)}$$

$$\text{Minimize}_{S\in X}\frac{F_1(S)}{G_1(S)} \qquad (P2)$$

$$\text{Minimize}_{S\in X}F_1(S) \qquad (P3)$$

where X (assumed to be nonempty) is the feasible set of (P), that is,

$$X = \{S \in \Lambda^n : H_j(S) \le 0, j \in \underline{m}\}.$$

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