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Hermite-Hadamard inequality for geometrically quasiconvex functions on co-ordinates

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Abstract

In this paper we introduce the concept of geometrically quasiconvex functions on the co-ordinates and establish some Hermite-Hadamard type integral inequalities for functions defined on rectangles in the plane. Some inequalities for product of two geometrically quasiconvex functions on the co-ordinates are considered.

Keywords: Hermite-Hadamard inequality; convex functions on co-ordinates; geometrically quasiconvex functions.
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1. Introduction and preliminaries

Let $I \subseteq \mathbb{R}$ be a real interval. A function $f : I \to \mathbb{R}$, is said to be convex if, for every $x, y \in I$ and $t \in [0, 1]$,

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

Let $f : I \to \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$ then, we have the following inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

This remarkable result is well known in the literature as Hermite-Hadamard inequality. Both inequalities hold in the reversed direction if f is concave. We note that Hermite-Hadamard inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's

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inequality. There have been several works in the literature which are devoted to investigating refinements and generalizations of the Hermite-Hadamard inequality for convex functions, see for example [1, 2, 3, 5, 7, 8, 9, 11, 12, 14, 15, 16] and references therein. In [4], S.S. Dragomir defined convex functions on the co-ordinates (or co-ordinated convex functions) on the set $\Delta := [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$ as follows:

Definition 1.1. A function $f : \Delta \to \mathbb{R}$ is said to be convex on the co-ordinates on Δ if for every $y \in [c,d]$ and $x \in [a,b]$, the partial mappings,

$$f_y : [a,b] \to \mathbb{R}, \qquad f_y(u) = f(u,y),$$

and

$$f_x : [c,d] \to \mathbb{R}, \qquad f_x(v) = f(x,v),$$

are convex. This means that for every $(x,y),(z,w) \in \Delta$ and $t,s \in [0,1]$,

$$\begin{aligned} & f(tx+(1-t)z, sy+(1-s)w) \\ \leq\ & tsf(x,y)+s(1-t)f(z,y)+t(1-s)f(x,w)+(1-t)(1-s)f(z,w). \end{aligned}$$

Clearly, every convex function is co-ordinated convex. Furthermore, there exist co-ordinated convex functions which are not convex. The following Hermit-Hadamard type inequality for co-ordinated convex functions was also proved in [4].

Theorem 1.2. Suppose that $f : \Delta \to \mathbb{R}$ is convex on co-ordinates on Δ. Then,

$$\begin{aligned} & f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ \leq\ & \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)dy\right] \\ \leq\ & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx \\ \leq\ & \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)dx + \frac{1}{b-a}\int_a^b f(x,d)dx \right. \\ & \left. + \frac{1}{d-c}\int_c^d f(a,y)dy + \frac{1}{d-c}\int_c^d f(a,y)dy\right] \\ \leq\ & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \end{aligned}$$

The above inequalities are sharp.

Since then several important generalizations introduced on this category, see [11, 13, 18, 19, 20, 21] and references therein. Recall that a function $f : I \subseteq \mathbb{R} \to \mathbb{R}$, is said to be quasiconvex if for every $x, y \in I$ and $\lambda \in [0,1]$,

$$f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\}.$$

In [15], M.E. Özdemir et al. introduced the notion of co-ordinated quasiconvex functions which generalize the notion of co-ordinated convex functions as follows:

Definition 1.3. A function $f : \Delta = [a, b] \times [c, d] \to \mathbb{R}$ is said to be quasiconvex on the co-ordinates on Δ if for every $y \in [c, d]$ and $x \in [a, b]$, the partial mapping,

$$f_y : [a, b] \to \mathbb{R}, \qquad f_y(u) = f(u, y),$$

and

$$f_x : [c, d] \to \mathbb{R}, \qquad f_x(v) = f(x, v),$$

are quasiconvex. This means that for every $(x, y), (z, w) \in \Delta$ and $s, t \in [0, 1]$,

$$f(tx + (1-t)z, sy + (1-s)w) \leq \max\{f(x, y), f(x, w), f(z, y), f(z, w)\}.$$

Since then several important generalizations on this category introduced in [11, 14]. On the other hand the notion of geometrically quasiconvex functions is introduced by İ. İşcan in [7] and F. Qi and B.A. Xi in [18] as follows:

Definition 1.4. A function $f : I \subseteq \mathbb{R}_0 := [0, \infty) \to \mathbb{R}_0$, is said to be geometrically quasiconvex on I if for every $x, y \in I$ and $t \in [0, 1]$,

$$f(x^t y^{1-t}) \leq \max\{f(x), f(y)\}.$$

Note that if f decreasing and geometrically quasiconvex then, it is quasiconvex. If f increasing and quasiconvex then, it is geometrically quasiconvex. We recall some results introduced in [18].

Lemma 1.5. Let $f : I \subseteq \mathbb{R}_+ := (0, \infty) \to \mathbb{R}$, be a differentiable function on I° and $a, b \in I^\circ$ with $a < b$. If $f' \in L([a, b])$ then,

$$\begin{aligned} &\frac{(\ln b)f(b) - (\ln a)f(a)}{\ln b - \ln a} - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}dx \\ = &\int_0^1 a^{1-t}b^t \ln(a^{1-t}b^t) f'(a^{1-t}b^t)dt. \end{aligned} \tag{1.1}$$

Theorem 1.6. Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable function on I° and $f' \in L([a, b])$ for $a, b \in I^\circ$ with $a < b$. If $|f'|$ is geometrically quasiconvex on $[a, b]$ then,

$$\begin{aligned} &\left|\frac{(\ln b)f(b) - (\ln a)f(a)}{\ln b - \ln a} - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}dx\right| \\ \leq \; &N(a, b)\sup\{|f'(a)|, |f'(b)|\}, \end{aligned} \tag{1.2}$$

where $N(a, b) := \int_0^1 a^{1-t}b^t |\ln(a^{1-t}b^t)|dt$.

Theorem 1.7. Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable function on I° and $f' \in L([a, b])$ for $a, b \in I^\circ$ with $a < b$. If $|f'|^q$ is geometrically quasiconvex on $[a, b]$ for $q > 1$ then,

$$\begin{aligned} &\left|\frac{(\ln b)f(b) - (\ln a)f(a)}{\ln b - \ln a} - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}dx\right| \\ &\leq [M(a, b)]^{\frac{1}{q}} \left[\frac{q-1}{q} N(a^{q/q-1}, b^{q/q-1})\right]^{1-1/q} \times \left[\sup\{|f'(a)|^q, |f'(b)|^q\}\right]^{\frac{1}{q}}, \end{aligned} \tag{1.3}$$

where $M(a, b) := \int_0^1 |\ln(a^{1-t}b^t)|dt$.

Theorem 1.8. Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable function on I° and $f' \in L([a,b])$ for $a, b \in I^\circ$ with $a < b$. If $|f'|^q$ is geometrically quasiconvex on $[a,b]$ for $q > 1$ and $q > r > 0$ then,

$$\begin{aligned} &\left|\frac{(\ln b)f(b) - (\ln a)f(a)}{\ln b - \ln a} - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}dx\right| \\ \leq\ & \left(\frac{q-1}{q-r}\right)^{1-1/q}\left(\frac{1}{r}\right)^{1/q}[N(a^r, b^r)]^{\frac{1}{q}} \\ &\times \left[N(a^{(q-r)/q-1}, b^{(q-r)/q-1})\right]^{1-1/q} \times \left[\sup\{|f'(a)|^q, |f'(b)|^q\}\right]^{\frac{1}{q}}. \end{aligned} \tag{1.4}$$

Theorem 1.9. Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_0$ be a differentiable function on I° and $f \in L([a,b])$ for $a, b \in I^\circ$ with $a < b$. If f is geometrically quasiconvex on $[a,b]$ then,

$$f((ab)^{1/2}) \leq \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}dx \leq \sup\{f(a), f(b)\}. \tag{1.5}$$

In [16], M. E. Özdemir introduced the notion of geometrically convex functions on the co-ordinates as follows:

Definition 1.10. Let $\Delta_+ := [a,b] \times [c,d]$ be a rectangle in ${\mathbb{R}_+}^2$ with $a < b$ and $c < d$. A function $f : \Delta_+ \to \mathbb{R}$ is said to be geometrically convex on the co-ordinates if for every $y \in [c,d]$ and $x \in [a,b]$ the partial mappings,

$$f_y : [a,b] \to \mathbb{R}, \qquad f_y(u) = f(u,y),$$

and

$$f_x : [c,d] \to \mathbb{R}, \qquad f_x(v) = f(x,v),$$

are geometrically convex function. This means that for every $(x,y), (z,w) \in \Delta_+$ and $t, s \in [0,1]$,

$$\begin{aligned} & f(x^t z^{1-t}, y^s w^{1-s}) \\ \leq\ & tsf(x,y) + s(1-t)f(z,y) + t(1-s)f(x,w) + (1-t)(1-s)f(z,w). \end{aligned}$$

The main purpose of this paper is to establish some new results connected to the Hermite-Hadamard type inequality for geometrically quasiconvex functions on the co-ordinates.

2. The main results

In this section we introduce the notion of "geometrically quasiconvex functions on the co-ordinates" for functions defined on rectangles in $\mathbb{R}^2_+$, which is a generalization of the notion " geometrically convex functions on the co-ordinates" given in definition 1.10. Then, we establish some Hermite-Hadamard type inequalities for this class of functions.

Definition 2.1. Let $\Delta_+ := [a,b] \times [c,d]$ be a subset of ${\mathbb{R}_+}^2$ with $a < b$ and $c < d$. A function $f : \Delta_+ \to \mathbb{R}$ is said to be geometrically quasiconvex on the co-ordinates on $\Delta_+ \subseteq \mathbb{R}^2_+$ if for every $y \in [c,d]$ and $x \in [a,b]$ the partial mappings

$$f_y : [a,b] \to \mathbb{R}, \qquad f_y(u) = f(u,y),$$

and

$$f_x : [c,d] \to \mathbb{R}, \qquad f_x(v) = f(x,v),$$

are geometrically quasiconvex. This means that for every $(x,y), (z,w) \in \Delta_+$ and $s, t \in [0,1]$,

$$f(x^t z^{1-t}, y^s w^{1-s}) \leq \max\{f(x,y), f(x,w), f(z,y), f(z,w)\}.$$

Note that every geometrically convex function on co-ordinates is geometrically quasiconvex on co-ordinates, but the converse is not holds. In the following example we give a geometrically quasiconvex function on co-ordinates which is not geometrically convex function on the co-ordinates.

Example 2.2. Let $\Delta_+ := [1,4] \times [4,9]$ and consider the function $f : \Delta_+ \to \mathbb{R}$ defined by

$$f(x,y) := x^2 - y^2.$$

It is easy to see that the functions

$$f_y(x) = x^2 - y^2, \ x \in [1,4],$$

and

$$f_x(y) = x^2 - y^2, \ y \in [4,9],$$

are geometrically quasiconvex. Hence, f is geometrically quasiconvex on co-ordinates on Δ_+. This function is not geometrically convex on co-ordinates on Δ_+. Indeed, if we choose two points, $(x,y) = (1,4)$, $(z,w) = (4,9)$ and $s = t = \frac{1}{2}$, then

$$f(x^t z^{1-t}, y^s w^{1-s}) = f(2,6) = -32,$$

and

$$\begin{aligned}
& tsf(x,y) + s(1-t)f(z,y) + t(1-s)f(x,w) + (1-t)(1-s)f(z,w) \\
= \ & \frac{1}{4}\{f(x,y), f(x,w), f(z,w), f(z,y)\} = -40 \\
< \ & f(x^t z^{1-t}, y^s w^{1-s}).
\end{aligned}$$

To reach our goal we introduce the following lemma which plays a crucial role in this paper.

Lemma 2.3. *Let $\Delta_+ := [a,b] \times [c,d]$ be a subset of ${\mathbb{R}_+}^2$ with $a < b$ and $c < d$. Suppose that $f : \Delta_+ \to \mathbb{R}$ is a partial differentiable function on $\operatorname{int}(\Delta_+)$. If $\frac{\partial^2 f}{\partial t \partial s} \in L(\Delta_+)$, then*

$$\begin{aligned}
& \frac{1}{(\ln b - \ln a)(\ln d - \ln c)} \\
& \times \left(C + D + \int_a^b \left[(\ln c)\frac{f(x,c)}{x} - (\ln d)\frac{f(x,d)}{x} \right] dx \right. \\
& \left. + \int_c^d \left[(\ln a)\frac{f(a,y)}{y} - (\ln b)\frac{f(b,y)}{y} \right] dy + \int_a^b \int_c^d \frac{f(x,y)}{yx} dy dx \right) \\
= \ & \int_0^1 \int_0^1 a^{1-t} b^t c^{1-s} d^s \ln(a^{1-t} b^t) \ln(c^{1-s} d^s) \frac{\partial^2 f}{\partial t \partial s}(a^{1-t} b^t, c^{1-s} d^s) dt ds,
\end{aligned} \tag{2.1}$$

where

$$C := (\ln d)[(\ln b)f(b,d) - (\ln a)f(a,d)],$$

and

$$D := (\ln c)[(\ln a)f(a,c) - (\ln b)f(b,c)].$$

Proof . If we denote the right hand side of (2.1) by I and integrating by parts on Δ_+ then, we have

$$\begin{aligned}
&(\ln b-\ln a)(\ln d-\ln c)I\\
=\ &(\ln b-\ln a)(\ln d-\ln c)\int_0^1\int_0^1 a^{1-t}b^tc^{1-s}d^s\\
&\times\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)\frac{\partial^2 f}{\partial t\partial s}(a^{1-t}b^t,c^{1-s}d^s)dtds
\end{aligned}$$

$$\begin{aligned}
=&(\ln b-\ln a)(\ln d-\ln c)\int_0^1 c^{1-s}d^s\ln(c^{1-s}d^s)\\
&\times\Big[\int_0^1 a^{1-t}b^t\ln(a^{1-t}b^t)\frac{\partial^2 f}{\partial t\partial s}(a^{1-t}b^t,c^{1-s}d^s)dt\Big]ds\\
=\ &(\ln b-\ln a)(\ln d-\ln c)\\
&\times\Big(\int_0^1 c^{1-s}d^s\ln(c^{1-s}d^s)\Big[\frac{\ln(a^{1-t}b^t)}{(\ln b)-(\ln a)}\frac{\partial f}{\partial s}(a^{1-t}b^t,c^{1-s}d^s)\Big|_0^1\\
&\quad-\int_0^1\frac{\partial f}{\partial s}(a^{1-t}b^t,c^{1-s}d^s)dt\Big]ds\Big)\\
=\ &(\ln b-\ln a)(\ln d-\ln c)\\
&\times\Big(\int_0^1 c^{1-s}d^s\ln(c^{1-s}d^s)\Big[\frac{\ln b}{\ln b-\ln a}\frac{\partial f}{\partial s}(b,c^{1-s}d^s)\\
&-\frac{\ln a}{\ln b-\ln a}\frac{\partial f}{\partial s}(a,c^{1-s}d^s)-\int_0^1\frac{\partial f}{\partial s}(a^{1-t}b^t,c^{1-s}d^s)dt\Big]ds\Big)\\
=\ &(\ln d-\ln c)(\ln b)\int_0^1 c^{1-s}d^s\ln(c^{1-s}d^s)\frac{\partial f}{\partial s}(b,c^{1-s}d^s)ds\\
&-(\ln d-\ln c)(\ln a)\int_0^1 c^{1-s}d^s\ln(c^{1-s}d^s)\frac{\partial f}{\partial s}(a,c^{1-s}d^s)ds\\
&-(\ln b-\ln a)(\ln d-\ln c)\\
&\times\Big(\int_0^1\Big[\int_0^1 c^{1-s}d^s\ln(c^{1-s}d^s)\frac{\partial f}{\partial s}(a^{1-t}b^t,c^{1-s}d^s)ds\Big]dt\Big).
\end{aligned}\tag{2.2}$$

Similarly integrating by parts in the right hand side of (2.2) deduce that

$$\begin{aligned}
&(\ln b-\ln a)(\ln d-\ln c)I\\
=\ &(\ln b)\Big(\ln(c^{1-s}d^s)f(b,c^{1-s}d^s)\Big|_0^1-(\ln d-\ln c)\int_0^1 f(b,c^{1-s}d^s)ds\Big)\\
&-(\ln a)\Big(\ln(c^{1-s}d^s)f(a,c^{1-s}d^s)\Big|_0^1-(\ln d-\ln c)\int_0^1 f(a,c^{1-s}d^s)ds\Big)\\
&-(\ln b-\ln a)\int_0^1\Big(\ln(c^{1-s}d^s)f(a^{1-t}b^t,c^{1-s}d^s)\Big|_0^1\Big)dt\\
&+(\ln b-\ln a)(\ln d-\ln c)\int_0^1\int_0^1 f(a^{1-t}b^t,c^{1-s}d^s)dtds\\
=\ &(\ln b)\Big([(\ln d)f(b,d)-(\ln c)f(b,c)]
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad - (\ln d - \ln c)\int_0^1 f(b, c^{1-s}d^s)ds\Big)\\
&\quad - (\ln a)\Big([(\ln d)f(a,d) - (\ln c)f(a,c)]\\
&\qquad\qquad - (\ln d - \ln c)\int_0^1 f(a, c^{1-s}d^s)ds\Big)\\
&- (\ln b - \ln a)\Big((\ln d)\int_0^1 f(a^{1-t}b^t, d)dt - (\ln c)\int_0^1 f(a^{1-t}b^t, c)dt\Big)\\
&+ (\ln b - \ln a)(\ln d - \ln c)\int_0^1\int_0^1 f(a^{1-t}b^t, c^{1-s}d^s)dtds.
\end{aligned}
$$

If we using the change of variables $x = a^{1-t}b^t$ and $y = c^{1-s}d^s$ for $t, s \in [0,1]$, we obtain

$$
\begin{aligned}
&(\ln b - \ln a)(\ln d - \ln c)I\\
= \;& (\ln b)\Big([(\ln d)f(b,d) - (\ln c)f(b,c)] - \int_c^d \frac{f(b,y)}{y}dy\Big)\\
&- (\ln a)\Big([(\ln d)f(a,d) - (\ln c)f(a,c)] - \int_c^d \frac{f(a,y)}{y}dy\Big)\\
&- (\ln d)\int_a^b \frac{f(x,d)}{x} + (\ln c)\int_a^b \frac{f(x,c)}{x}dx + \int_a^b\int_c^d \frac{f(x,y)}{yx}dydx.
\end{aligned}
\tag{2.3}
$$

Dividing both sides of (2.3) by $(\ln b - \ln a)(\ln d - \ln c)$ implies that the equation (2.1) holds and proof is completed. ☐

Theorem 2.4. *Let $\Delta_+ := [a,b]\times[c,d]$ be a subset of $\mathbb{R}^2_+$ with $a < b$ and $c < d$. Suppose that $f : \Delta_+ \to \mathbb{R}$ is a partial differentiable function on $\mathrm{int}(\Delta_+)$ and $\frac{\partial^2 f}{\partial t\partial s} \in L(\Delta_+)$. If $\left|\frac{\partial^2 f}{\partial t\partial s}\right|$ is a geometrically quasiconvex function on the co-ordinates on Δ_+ then the following inequality holds:*

$$
\begin{aligned}
&\left|\frac{C+D}{(\ln b - \ln a)(\ln d - \ln c)} + \frac{\int_a^b\int_c^d \frac{f(x,y)}{yx}dydx}{(\ln b - \ln a)(\ln d - \ln c)} - B\right|\\
\le \;& N(a,b)\; N(c,d)\\
&\times \max\left\{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|, \left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|, \left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|, \left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|\right\},
\end{aligned}
\tag{2.4}
$$

where, C, D are defined in Lemma 2.3.

$$
\begin{aligned}
B := \;& \frac{1}{(\ln b - \ln a)(\ln d - \ln c)} \times \Big(\int_a^b \Big[(\ln d)\frac{f(x,d)}{x} - (\ln c)\frac{f(x,c)}{x}\Big]dx\\
&+ \int_c^d \Big[(\ln b)\frac{f(b,y)}{y} - (\ln a)\frac{f(a,y)}{y}\Big]dy\Big).
\end{aligned}
$$

Proof . From Lemma 2.3, it follows that

$$
\begin{aligned}
&\left|\frac{C+D}{(\ln b - \ln a)(\ln d - \ln c)} + \frac{\int_a^b\int_c^d \frac{f(x,y)}{yx}dydx}{(\ln b - \ln a)(\ln d - \ln c)} - B\right|\\
\le \;& \int_0^1\int_0^1 a^{1-t}b^tc^{1-s}d^s|\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)|\\
&\times \left|\frac{\partial^2 f}{\partial t\partial s}(a^{1-t}b^t, c^{1-s}d^s)\right|dtds.
\end{aligned}
$$

Since $\left|\frac{\partial^2 f}{\partial t \partial s}\right|$ is geometrically quasiconvex on the co-ordinates we have

$$\begin{aligned}
&\Big|\frac{\partial^2 f}{\partial t \partial s}(a^{1-t}b^t, c^{1-s}d^s)\Big| \\
\leq\ & \max\Big\{\Big|\frac{\partial^2 f}{\partial t \partial s}(a,c)\Big|, \Big|\frac{\partial^2 f}{\partial t \partial s}(a,d)\Big|, \Big|\frac{\partial^2 f}{\partial t \partial s}(b,c)\Big|, \Big|\frac{\partial^2 f}{\partial t \partial s}(b,d)\Big|\Big\},
\end{aligned}$$

where $t, s \in [0,1]$. From this inequality and Lemma 1.5, it follows that

$$\begin{aligned}
&\int_0^1\int_0^1 a^{1-t}b^t c^{1-s}d^s |\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)|\Big|\frac{\partial^2 f}{\partial t \partial s}(a^{1-t}b^t, c^{1-s}d^s)\Big| dtds \\
\leq\ & \max\Big\{\Big|\frac{\partial^2 f(a,c)}{\partial t \partial s}\Big|, \Big|\frac{\partial^2 f(a,d)}{\partial t \partial s}\Big|, \Big|\frac{\partial^2 f(b,c)}{\partial t \partial s}\Big|, \Big|\frac{\partial^2 f(b,d)}{\partial t \partial s}\Big|\Big\} \\
&\times \int_0^1\int_0^1 a^{1-t}b^t c^{1-s}d^s |\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)| dtds \\
=\ & N(a,b)\ N(c,d) \\
&\times \max\Big\{\Big|\frac{\partial^2 f}{\partial t \partial s}(a,c)\Big|, \Big|\frac{\partial^2 f}{\partial t \partial s}(a,d)\Big|, \Big|\frac{\partial^2 f}{\partial t \partial s}(b,c)\Big|, \Big|\frac{\partial^2 f}{\partial t \partial s}(b,d)\Big|\Big\},
\end{aligned}$$

which is the required inequality (2.4). Note that

$$\begin{aligned}
&\int_0^1\int_0^1 a^{(1-t)}b^t c^{(1-s)}d^s |\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)| dtds \\
=\ & \Big(\int_0^1 a^{(1-t)}b^t |\ln(a^{1-t}b^t)| dt\Big)\Big(\int_0^1 c^{(1-s)}d^s |\ln(c^{1-s}d^s)| ds\Big) \\
=\ & N(a,b)\ N(c,d).
\end{aligned}$$

The proof of theorem is completed. □

The following corollary is an immediate consequence of theorem 2.4.

Corollary 2.5. Suppose that the conditions of the theorem 2.4 are satisfied. Additionally, if

(1) $\left|\frac{\partial^2 f}{\partial t \partial s}\right|$ is increasing on the co-ordinates on Δ_+, then

$$\begin{aligned}
&\Big|\frac{C+D}{(\ln b-\ln a)(\ln d-\ln c)} + \frac{\int_a^b\int_c^d \frac{f(x,y)}{yx} dydx}{(\ln b-\ln a)(\ln d-\ln c)} - B\Big| \\
\leq\ & N(a,b)\ N(c,d)\ \Big|\frac{\partial^2}{\partial t \partial s} f(b,d)\Big|.
\end{aligned} \tag{2.5}$$

(2) $\left|\frac{\partial^2 f}{\partial t \partial s}\right|$ is decreasing on the co-ordinates on Δ_+, then

$$\begin{aligned}
&\Big|\frac{C+D}{(\ln b-\ln a)(\ln d-\ln c)} + \frac{\int_a^b\int_c^d \frac{f(x,y)}{yx} dydx}{(\ln b-\ln a)(\ln d-\ln c)} - B\Big| \\
\leq\ & N(a,b)\ N(c,d)\ \Big|\frac{\partial^2}{\partial t \partial s} f(a,c)\Big|,
\end{aligned} \tag{2.6}$$

where, C, D and B are defined, respectively in Lemma 2.3 and Theorem 2.4.

Theorem 2.6. *Let $\Delta_+ := [a,b]\times[c,d]$ be a subset of $\mathbb{R}_+^2$ with $a<b$ and $c<d$. Suppose that $f:\Delta_+\to\mathbb{R}$ is a partial differentiable function on $\operatorname{int}(\Delta_+)$ and $\frac{\partial^2 f}{\partial t\partial s}\in L(\Delta_+)$. If $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$ is a geometrically quasiconvex function on the co-ordinates on Δ_+ and $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{C+D}{(\ln b-\ln a)(\ln d-\ln c)}+\frac{\int_a^b\int_c^d\frac{f(x,y)}{yx}dydx}{(\ln b-\ln a)(\ln d-\ln c)}-B\right|\\
\leq\ &\left[N(a^p,b^p)\ N(c^p,d^p)\right]^{\frac{1}{p}}\times\left[\max\left\{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q,\right.\right.\\
&\left.\left.\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right\}\right]^{1/q},
\end{aligned}\tag{2.7}$$

where, C, D and B are defined, respectively in Lemma 2.3 and Theorem 2.4.

Proof . Suppose that $p>1$. From Lemma 2.3 and well-known Hölder inequality for double integrals, we obtain

$$\begin{aligned}
&\left|\frac{C+D}{(\ln b-\ln a)(\ln d-\ln c)}+\frac{\int_a^b\int_c^d\frac{f(x,y)}{yx}dydx}{(\ln b-\ln a)(\ln d-\ln c)}-B\right|\\
\leq\ &\int_0^1\int_0^1 a^{1-t}b^tc^{1-s}d^s|\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)|\left|\frac{\partial^2 f}{\partial t\partial s}(a^{1-t}b^t,c^{1-s}d^s)\right|dtds\\
\leq\ &\left(\int_0^1\int_0^1 a^{p(1-t)}b^{pt}c^{p(1-s)}d^{ps}|\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)|^pdtds\right)^{\frac{1}{p}}\\
&\times\left(\int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial t\partial s}(a^{1-t}b^t,c^{1-s}d^s)\right|^qdtds\right)^{\frac{1}{q}}.
\end{aligned}\tag{2.8}$$

Since $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$ is geometrically quasiconvex on the co-ordinates on Δ_+, we obtain

$$\begin{aligned}
&\left|\frac{\partial^2 f}{\partial t\partial s}(a^{1-t}b^t,c^{1-s}d^s)\right|^q\\
\leq\ &\max\left\{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right\}.
\end{aligned}\tag{2.9}$$

Note that

$$\begin{aligned}
&\int_0^1\int_0^1 a^{p(1-t)}b^{pt}c^{p(1-s)}d^{ps}|\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)|^pdtds\\
=\ &\left(\int_0^1 a^{p(1-t)}b^{pt}|\ln(a^{1-t}b^t)|^pdt\right)\left(\int_0^1 c^{p(1-s)}d^{ps}|\ln(c^{1-s}d^s)|^pds\right)\\
=\ &N(a^p,b^p)\ N(c^p,d^p).
\end{aligned}\tag{2.10}$$

Combination of (2.8), (2.9) and (2.10), gives the desired inequality (2.7). Hence the proof of the theorem is completed. □

The following corollary is an immediate consequence of theorem 2.6.

Corollary 2.7. Suppose that the conditions of the Theorem 2.6 are satisfied. Additionally, if

(1) $\left|\frac{\partial^2 f}{\partial t \partial s}\right|^q$ is increasing on the co-ordinates on Δ_+, then

$$\begin{aligned} &\left|\frac{C+D}{(\ln b-\ln a)(\ln d-\ln c)}+\frac{\int_a^b\int_c^d \frac{f(x,y)}{yx}dydx}{(\ln b-\ln a)(\ln d-\ln c)}-B\right| \\ \leq\ & \left[N(a^p,b^p)\ N(c^p,d^p)\right]^{\frac{1}{p}}\left|\frac{\partial^2}{\partial t\partial s}f(b,d)\right|. \end{aligned} \tag{2.11}$$

(2) $\left|\frac{\partial^2 f}{\partial t \partial s}\right|^q$ is decreasing on the co-ordinates on Δ_+, then

$$\begin{aligned} &\left|\frac{C+D}{(\ln b-\ln a)(\ln d-\ln c)}+\frac{\int_a^b\int_c^d \frac{f(x,y)}{yx}dydx}{(\ln b-\ln a)(\ln d-\ln c)}-B\right| \\ \leq\ & \left[N(a^p,b^p)\ N(c^p,d^p)\right]^{\frac{1}{p}}\left|\frac{\partial^2}{\partial t\partial s}f(a,c)\right|. \end{aligned} \tag{2.12}$$

Theorem 2.8. *Let $\Delta_+ := [a,b]\times[c,d]$ be a subset of $\mathbb{R}^2_+$ with $a<b$ and $c<d$. Suppose that $f:\Delta_+\to\mathbb{R}$ is a partial differentiable function on $\operatorname{int}(\Delta_+)$ and $\frac{\partial^2 f}{\partial t\partial s}\in L(\Delta_+)$. If $\left|\frac{\partial^2 f}{\partial t \partial s}\right|^q$ is a geometrically quasiconvex function on the co-ordinates on Δ_+ for $q>1$, then the following inequality holds:*

$$\begin{aligned} &\left|\frac{C+D}{(\ln b-\ln a)(\ln d-\ln c)}+\frac{\int_a^b\int_c^d \frac{f(x,y)}{yx}dydx}{(\ln b-\ln a)(\ln d-\ln c)}-B\right| \\ \leq\ & [M(a,b)\ M(c,d)]^{1/q} \\ & \times\left[\left(\frac{q-1}{q}\right)^2 N\left(a^{q/(q-1)},b^{q/(q-1)}\right)\ N\left(c^{q/(q-1)},d^{q/(q-1)}\right)\right]^{1-1/q} \\ & \times\left[\max\left\{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right\}\right]^{\frac{1}{q}}, \end{aligned} \tag{2.13}$$

where, C, D and B are defined, respectively in Lemma 2.3 and Theorem 2.4.

Proof . By Lemma 2.3, Hölder's inequality, and the geometrically quasiconvexity of $\left|\frac{\partial^2 f}{\partial t \partial s}\right|^q$ on $\triangle_+$, we have

$$\begin{aligned} &\left|\frac{C+D}{(\ln b-\ln a)(\ln d-\ln c)}+\frac{\int_a^b\int_c^d \frac{f(x,y)}{yx}dydx}{(\ln b-\ln a)(\ln d-\ln c)}-B\right| \\ \leq\ & \int_0^1\int_0^1 a^{1-t}b^tc^{1-s}d^s|\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)|\left|\frac{\partial^2 f}{\partial t\partial s}(a^{1-t}b^t,c^{1-s}d^s)\right|dtds \\ \leq\ & \left[\int_0^1\int_0^1 a^{q(1-t)/(q-1)}b^{qt/(q-1)}c^{q(1-s)/(q-1)}d^{qs/(q-1)}\right. \\ & \left.\times|\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)|dtds\right]^{1-1/q} \\ & \times\left[\int_0^1\int_0^1|\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)|\left|\frac{\partial^2 f}{\partial t\partial s}(a^{1-t}b^t,c^{1-s}d^s)\right|^q dtds\right]^{1/q} \end{aligned}$$

$$
\begin{aligned}
&\leq \left[\int_0^1\int_0^1 a^{q(1-t)/(q-1)}b^{qt/(q-1)}c^{q(1-s)/(q-1)}d^{qs/(q-1)}\right.\\
&\quad\left.\times|\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)|dtds\right]^{1-1/q}\\
&\quad\times\left[\int_0^1\int_0^1|\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)|dtds\right]^{1/q}\\
&\quad\times\left[\max\left\{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right\}\right]^{\frac{1}{q}}.
\end{aligned}
$$

Note that by Lemma 1.5 it follows that,

$$
\begin{aligned}
&\int_0^1\int_0^1 a^{q(1-t)/(q-1)}b^{qt/(q-1)}c^{q(1-s)/(q-1)}d^{qs/(q-1)}\\
&\times|\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)|dtds\\
=&\left(\int_0^1 a^{q(1-t)/(q-1)}b^{qt/(q-1)}|\ln(a^{1-t}b^t)|dt\right)\\
&\times\left(\int_0^1 c^{q(1-s)/(q-1)}d^{qs/(q-1)}|\ln(c^{1-s}d^s)|ds\right)\\
=&\frac{(q-1)^2}{q^2}N\big(a^{q/(q-1)},b^{q/(q-1)}\big)\ N\big(c^{q/(q-1)},d^{q/(q-1)}\big).
\end{aligned}
$$

It is easy to see that

$$
\int_0^1\int_0^1|\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)|dtds=M(a,b)\ M(c,d),
$$

and proof is completed. □

Theorem 2.9. *Let $\Delta_+ := [a,b]\times[c,d]$ be a subset of $\mathbb{R}^2_+$ with $a<b$ and $c<d$. Suppose that $f:\Delta_+\to\mathbb{R}$ is a partial differentiable function on $\operatorname{int}(\Delta_+)$ and $\frac{\partial^2 f}{\partial t\partial s}\in L(\Delta_+)$. If $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$ is a geometrically quasiconvex function on the co-ordinates on Δ_+ and $q>\ell>0$, then*

$$
\begin{aligned}
&\left|\frac{C+D}{(\ln b-\ln a)(\ln d-\ln c)}+\frac{\int_a^b\int_c^d\frac{f(x,y)}{yx}dydx}{(\ln b-\ln a)(\ln d-\ln c)}-B\right|\\
\leq&\left(\frac{q-1}{q-\ell}\right)^{2(1-1/q)}\left(\frac{1}{\ell}\right)^{2/q}\left[N\big(a^\ell,b^\ell\big)\ N\big(c^\ell,d^\ell\big)\right]^{1/q}\\
&\times\left[N\big(a^{(q-\ell)/(q-1)},b^{(q-\ell)/(q-1)}\big)\ N\big(c^{(q-\ell)/(q-1)},d^{(q-\ell)/(q-1)}\big)\right]^{(1-1/q)}\\
&\times\left[\max\left\{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right\}\right]^{\frac{1}{q}},
\end{aligned}
\tag{2.14}
$$

where, C, D and B are defined, respectively in Lemma 2.3 and Theorem 2.4.

Proof . From Lemma 2.3, Hölder's inequality, and the geometrically quasiconvexity of $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$ on the co-ordinates on Δ_+ we get

$$\begin{aligned}
&\left|\frac{C+D}{(\ln b-\ln a)(\ln d-\ln c)}+\frac{\int_a^b\int_c^d \frac{f(x,y)}{yx}dydx}{(\ln b-\ln a)(\ln d-\ln c)}-B\right| \\
&\le \int_0^1\int_0^1 a^{1-t}b^tc^{1-s}d^s|\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)| \\
&\quad \times\left|\frac{\partial^2 f}{\partial t\partial s}(a^{1-t}b^t,c^{1-s}d^s)\right|dtds \\
&\le \left[\int_0^1\int_0^1 a^{(q-\ell)(1-t)/(q-1)}b^{(q-\ell)t/(q-1)}c^{(q-\ell)(1-s)/(q-1)}\right. \\
&\quad \left.\times d^{(q-\ell)s/(q-1)}\times|\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)|dtds\right]^{1-1/q} \\
&\quad \times\left[\int_0^1\int_0^1 |\ln(a^{\ell(1-t)}b^{\ell t})\ln(c^{\ell(1-s)}d^{\ell s})|\right. \\
&\quad \left.\times\left|\frac{\partial^2 f}{\partial t\partial s}(a^{1-t}b^t,c^{1-s}d^s)\right|^q dtds\right]^{1/q} \\
&\le \left[\int_0^1\int_0^1 a^{(q-\ell)(1-t)/(q-1)}b^{(q-\ell)t/(q-1)}c^{(q-\ell)(1-s)/(q-1)}\right. \\
&\quad \left.\times d^{(q-\ell)s/(q-1)}\times|\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)|dtds\right]^{1-1/q} \\
&\quad \times\left[\int_0^1\int_0^1 |a^{\ell(1-t)}b^{\ell t}c^{\ell(1-s)}d^{\ell s}\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)|dtds\right]^{1/q} \\
&\quad \times\left[\max\left\{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right\}\right]^{\frac{1}{q}} \\
&= \left(\frac{q-1}{q-\ell}\right)^{2(1-1/q)}\Big[N\big(a^{(q-\ell)/(q-1)},b^{(q-\ell)/(q-1)}\big) \\
&\quad \times N\big(c^{(q-\ell)/(q-1)},d^{(q-\ell)/(q-1)}\big)\Big]^{1-1/q} \\
&\quad \times\left(\frac{1}{\ell}\right)^{2/q}\Big[N\big(a^\ell,b^\ell\big)\ N\big(c^\ell,d^\ell\big)\Big]^{1/q} \\
&\quad \times\left[\max\left\{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right\}\right]^{\frac{1}{q}}.
\end{aligned}$$

The proof of theorem is completed. □

Theorem 2.10. *Let $\Delta_+ := [a,b]\times[c,d]$ be a subset of $\mathbb{R}^2_+$ with $a<b$ and $c<d$. Suppose that $f:\Delta_+\to\mathbb{R}$ is a geometrically quasiconvex function on the co-ordinates on Δ_+. If $f\in L(\Delta_+)$, then*

$$\begin{aligned}
f\big((ab)^{1/2},(cd)^{1/2}\big) &\le \frac{1}{(\ln b-\ln a)(\ln d-\ln c)}\int_a^b\int_c^d\frac{f(x,y)}{yx}dydx \\
&\le \max\{f(a,c),f(a,d),f(b,c),f(b,d)\}.
\end{aligned} \tag{2.15}$$

Proof . By geometrically quasiconvexity of f on co-ordinates on Δ_+, for every $t \in [0,1]$ we have

$$\begin{aligned} & f\big((ab)^{1/2}, (cd)^{1/2}\big) \\ \leq\ & \max\{f(a^{1-t}b^t, c^{1-s}d^s), f(a^t b^{1-t}, c^s d^{1-s})\} \\ \leq\ & \max\{f(a,c), f(a,d), f(b,c), f(b,d)\}. \end{aligned} \tag{2.16}$$

Since

$$\begin{aligned} \int_0^1\int_0^1 f(a^{1-t}b^t, c^{1-s}d^s)dtds &= \int_0^1\int_0^1 f(a^t b^{1-t}, c^s d^{1-s})dtds \\ &= \frac{1}{(\ln b - \ln a)(\ln d - \ln c)}\int_a^b\int_c^d \frac{f(x,y)}{yx}dydx, \end{aligned}$$

by integrating in (2.16) we get

$$\begin{aligned} & f\big((ab)^{1/2}, (cd)^{1/2}\big) \\ & \leq \max\left\{\int_0^1\int_0^1 f(a^{1-t}b^t, c^{1-s}d^s)dtds, \int_0^1\int_0^1 f(a^t b^{1-t}, c^s d^{1-s})dtds\right\} \\ = \ & \frac{1}{(\ln b - \ln a)(\ln d - \ln c)}\int_a^b\int_c^d \frac{f(x,y)}{yx}dydx \\ & \leq \max\{f(a,c), f(a,d), f(b,c), f(b,d)\}. \end{aligned}$$

and proof is completed. □

Theorem 2.11. *Let $\Delta_+ := [a,b] \times [c,d]$ be a subset of $\mathbb{R}^2_+$ with $a < b$ and $c < d$. Suppose that $f, g : \Delta_+ \to \mathbb{R}$ are geometrically quasiconvex functions on the co-ordinates on Δ_+. If $fg \in L(\Delta_+)$. Then,*

$$\begin{aligned} & \frac{1}{(\ln b - \ln a)(\ln d - \ln c)}\int_a^b\int_c^d \frac{f(x,y)}{yx}g(x,y)dydx \\ \leq\ & \max\big\{f(u,v)\ g(w,z)\ \big|\ u, w \in \{a,b\},\ v, z \in \{c,d\}\big\}. \end{aligned}$$

Proof . Let $x = a^{1-t}b^t,\ y = a^{1-s}b^s, s, t \in [0,1]$. By using the geometrically quasiconvexity of f, g on Δ_+ we have

$$\begin{aligned} & \frac{1}{(\ln b - \ln a)(\ln d - \ln c)}\int_a^b\int_c^d \frac{f(x,y)}{yx}g(x,y)dydx \\ =\ & \int_0^1\int_0^1 f(a^{1-t}b^t, c^{1-s}d^s)\ g(a^{1-t}b^t, c^{1-s}d^s)dtds \\ \leq\ & \max\{f(a,c), f(a,d), f(b,c), f(b,d)\} \\ & \times \max\{g(a,c), g(a,d), g(b,c), g(b,d)\}, \end{aligned}$$

and proof is completed. □

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