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On the system of double equations with three unknowns $d + ay + bx + cx^2 = z^2, y + z = x^2$

Mayilrangam Gopalan[a], Aarthy Thangam[b], Ozen Ozer[c,*]

[a] *Department of Mathematics, Shrimati Indira Gandhi College, Trichy-620 002, Tamil Nadu, India*

[b]*Department of Mathematics, Shrimati Indira Gandhi College, Trichy-620 002, Tamil Nadu, India*

[c]*Department of Mathematics, Faculty of Science and Arts, Kirklareli University, 39100, Kirklareli, Turkey*

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Abstract

The system of double equations with three unknowns given by $d + ay + bx + cx^2 = z^2, y + z = x^2$ is analysed for its infinitely many non-zero distinct integer solutions. Different sets of integer solutions have been presented. A few interesting relations among the solutions are given.

Keywords: System of double equations, Pair of equations with three unknowns, Integer solutions, Pell Equations, Special Numbers.
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1. Introduction

Systems of indeterminate quadratic equations of the form $ax + c = u^2, bx + d = v^2$ where a, b, c, d are non-zero distinct constants, have been investigated for solutions by several authors

*Corresponding author
Email addresses: mayilgopalan@gmail.com (Mayilrangam Gopalan), aarthythangam@gmail.com (Aarthy Thangam), ozenozer39@gmail.com (Ozen Ozer)

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[1, 2] and with a few possible exceptions, most of the them were primarily concerned with rational solutions. Even those existing works wherein integral solutions have been attempted, deal essentially with specific cases only and do not exhibit methods of finding integral solutions is a general form. In [3], a general form of the integral solutions to the system of equations $ax + c = u^2, bx + d = v^2$ where a, b, c, d are non-zero distinct constants is presented when the product $a * b$ is a square free integer whereas the product $c * d$ may or may not a square integer. For other forms of system of double diophantine equations, one may refer [4-14].

In this paper, we consider the system of double diophantine equation with three unknowns represented by $d + ay + bx + cx^2 = z^2, y + z = x^2$ for determining its many non-zero distinct integers solutions. A few interesting properties among the solutions are presented.

2. Method of Analysis

The system of double equations to be solved is

$$d + ay + bx + cx^2 = z^2 \tag{2.1}$$

$$y + z = x^2 \tag{2.2}$$

Eliminating y between (2.1) and (2.2), the resulting equation is

$$(a + c)x^2 + bx + (d - az - z^2) = 0 \tag{2.3}$$

Treating (2.3) as a quadratic in x and solving for x, we have

$$x = \frac{1}{2(a + c)}[-b \pm \sqrt{b^2 - 4(a + c)(d - az - z^2)}] \tag{2.4}$$

Let

$$Y^2 = b^2 - 4(a + c)(d - az - z^2) \tag{2.5}$$

Then (2.4) becomes

$$Y^2 = DX^2 + N \tag{2.6}$$

where

$$Y = 2Dx + b, X = 2z + a, D = a + c, N = b^2 - D(4d + a^2) \tag{2.7}$$

The initial solution of (2.6) is (X_0, Y_0).

To find the other solutions of (2.6), consider the pellian

$$Y^2 = DX^2 + 1 \tag{2.8}$$

whose general solution is given by

$$\tilde{Y_n} = \frac{1}{2} f_{n,D}, \tilde{X_n} = \frac{1}{2\sqrt{D}} g_{n,D}$$

where

$$f_{n,D} = (\tilde{Y_0} + \sqrt{D}\tilde{X_0})^{n+1} + (\tilde{Y_0} - \sqrt{D}\tilde{X_0})^{n+1}$$

$$g_{n,D} = (\tilde{Y_0} + \sqrt{D}\tilde{X_0})^{n+1} - (\tilde{Y_0} - \sqrt{D}\tilde{X_0})^{n+1}$$

in which $(\tilde{X_0}, \tilde{Y_0})$ is the initial solution of (2.8).

Applying the lemma of Brahmagupta between the solutions (X_0, Y_0) and $(\tilde{X_n}, \tilde{Y_n})$,we have

$$X_{n+1} = \frac{X_0}{2} f_{n,D} + \frac{Y_0}{2\sqrt{D}} g_{n,D}$$

$$Y_{n+1} = \frac{Y_0}{2} f_{n,D} + \frac{X_0}{2} \sqrt{D} g_{n,D}$$

In view of (2.7) and (2.2),the general values for x and y satisfying (2.1) and (2.2) are given by

$$x_{n+1} = \frac{1}{4D}(Y_0 f_{n,D} + X_0 \sqrt{D} g_{n,D} - 2b)$$

$$y_{n+1} = x_{n+1}^2 - \frac{1}{4}(X_0 f_{n,D} + \frac{Y_0}{\sqrt{D}} g_{n,D} - 2a)$$

$$z_{n+1} = \frac{1}{4}(X_0 f_{n,D} + \frac{Y_0}{\sqrt{D}} g_{n,D} - 2a)$$

Observations:

- ❖ $D(x_{n+3} + x_{n+1}) = \tilde{Y_0}(2Dx_{n+2} + b) - b$
- ❖ $2D(x_{n+5} + 2x_{n+1}) + 3b = 2(\tilde{Y_0^2} + D\tilde{X_0^2})(b + 2Dx_{n+3})$
- ❖ $x_{n+3}^2 + x_{n+1}^2 - y_{n+3} - y_{n+1} - 2\tilde{Y_0}(x_{n+2}^2 - y_{n+2}) = a(\tilde{Y_0} - 1)$

Special Cases:

Case: (i)

Let $a = 0, b = kD^{\beta+1}, d = \beta^2, D = 2$

Following the procedure similar to the above, the corresponding values of x, y and z satisfying (2.1) and (2.2)are given by

$$x_{n+1} = \frac{1}{8}(-k2^{\beta+1}f_{n,2} + 2\beta\sqrt{2}g_{n,2} - k2^{\beta+2})$$

$$y_{n+1} = x_{n+1}^2 - \frac{1}{4}(2\beta f_{n,2} - \frac{k2^{\beta+1}}{\sqrt{2}}g_{n,2})$$

$$z_{n+1} = \frac{1}{4}(2\beta f_{n,2} - \frac{k2^{\beta+1}}{\sqrt{2}}g_{n,2})$$

Some interesting relations among the solutions are presented below:

- ❖ $\frac{1}{4\beta^2 - k^2 2^{2\beta+1}}(k2^{\beta+3}x_{2n+2} + 8\beta z_{2n+2} + 8\beta^2)$ is a Perfect square.
- ❖ $\frac{3}{2\beta^2 - k^2 2^{2\beta}}(k2^{\beta+3}x_{2n+2} + 8\beta z_{2n+2} + 8\beta^2)$ is a Nasty number.
- ❖ $\frac{1}{4\beta^2 - k^2 2^{2\beta+1}}(k2^{\beta+3}(x_{3n+3} + x_{n+1}) + 8\beta(z_{3n+3} + 3z_{n+1}) + 4k^2 2^{2\beta+2})$ is a Cubical integer.
- ❖ $\frac{1}{4\beta^2 - k^2 2^{2\beta+1}}(k2^{\beta+3}(x_{4n+4} + 4x_{2n+2}) + 8\beta(z_{4n+4} + 4z_{2n+2}) + 3k^2 2^{2\beta+2} + 16\beta^2)$ is a Bi-quadratic integer.
- ❖ Define $P = k2^{\beta+3}x_{n+1} + 8\beta z_{n+1} + k^2 2^{2\beta+2}$ and $Q = 8\beta x_{n+1} + k2^{\beta+2}z_{n+1} + \beta k2^{\beta+2}$.

  Note that the pair (P, Q) satisfies the hyperbola $P^2 - 2Q^2 = 4(4\beta^2 - k^2 2^{2\beta+1})^2$
- ❖ Define $R = k2^{\beta+3}x_{2n+2} + 8\beta z_{2n+2} + 8\beta^2$ and $Q = 8\beta x_{n+1} + k2^{\beta+2}z_{n+1} + \beta k2^{\beta+2}$.

  Note that the pair (R, Q) satisfies the parabola $(4\beta^2 - k^2 2^{2\beta+1})R - 2Q^2 = 4(4\beta^2 - k^2 2^{2\beta+1})^2$

Case: (ii)

Let $a = r^2 - s^2(r \neq s), b = kD^{\beta+1}, d = r^2 s^2, D = 2$

After performing a few calculations,

$$\begin{cases} x_{n+1} = \frac{1}{4}(-k2^{\beta}(f_{n,2}+2) + \frac{(r^2+s^2)}{\sqrt{2}}g_{n,2}) \\ \\ z_{n+1} = \frac{1}{4}(r^2+s^2)f_{n,2} - \frac{k2^{\beta}}{2\sqrt{2}}g_{n,2} - \frac{(r^2-s^2)}{2} \end{cases} \tag{2.9}$$

As our interest is on finding integer solutions, the following two choices arise:

Choice: 1

The assumptions $r = 2R, s = 2S$ in (2.9) lead to

$$x_{n+1} = -k2^{\beta}(\frac{(\sqrt{2}+1)^{n+1} + (\sqrt{2}-1)^{n+1}}{2})^2 + \frac{(R^2+S^2)}{\sqrt{2}}g_{n,2}$$

$$y_{n+1} = x_{n+1}^2 - R^2((\sqrt{2}+1)^{n+1} - (\sqrt{2}-1)^{n+1})^2 - S^2((\sqrt{2}+1)^{n+1} + (\sqrt{2}-1)^{n+1})^2 + \frac{k2^{\beta}}{2\sqrt{2}}g_{n,2}$$

$$z_{n+1} = R^2((\sqrt{2}+1)^{n+1} - (\sqrt{2}-1)^{n+1})^2 + S^2((\sqrt{2}+1)^{n+1} + (\sqrt{2}-1)^{n+1})^2 - \frac{k2^{\beta}}{2\sqrt{2}}g_{n,2}$$

Choice: 2

The assumptions $r = 2R+1, s = 2S+1$ in (2.9) lead to

$$x_{n+1} = -\frac{k2^{\beta}}{4}(f_{n,2}+2) + \frac{1}{2\sqrt{2}}(2R^2+2S^2+2R+2S+1)g_{n,2}$$

$$y_{n+1} = x_{n+1}^2 - \frac{f_{n,2}}{2}(2R^2+2S^2+2R+2S+1) + \frac{k2^{\beta}}{2\sqrt{2}}g_{n,2} + (2R^2-2S^2+2R-2S)$$

$$z_{n+1} = \frac{f_{n,2}}{2}(2R^2+2S^2+2R+2S+1) - \frac{k2^{\beta}}{2\sqrt{2}}g_{n,2} - (2R^2-2S^2+2R-2S)$$

Case: (iii)

Let $d = n(n+1)a^2, b = kD^{\beta+1}, D = 2$

In this case, the corresponding values of x, y and z satisfying (2.1) and (2.2) are given by

$$x_{n+1} = \frac{1}{8}(-k2^{\beta+1}f_{n,2} + a(2n+1)\sqrt{2}g_{n,2} - k2^{\beta+2})$$

$$y_{n+1} = x_{n+1}^2 - \frac{1}{4}(a(2n+1)f_{n,2} - k2^{\beta}\sqrt{2}g_{n,2} - 2a)$$

$$z_{n+1} = \frac{1}{4}(a(2n+1)f_{n,2} - k2^{\beta}\sqrt{2}g_{n,2} - 2a)$$

Case: (iv)

Let $a^2 + 4d = 0$ and D be a non-square.

After performing some calculations, the corresponding integer values of x, y and z satisfying (2.1) and (2.2) are found to be

$$x_n = \frac{b}{4D}(f_{n,D} - 2)$$

$$y_n = x_n^2 - \frac{1}{4\sqrt{D}}(bg_{n,D} - 2a\sqrt{D})$$

$$z_n = \frac{1}{4\sqrt{D}}(bg_{n,D} - 2a\sqrt{D})$$

Case: (v)

Let $a^2 + 4d = 0$ and $D = \alpha^2$

After some algebra, it is observed that there are 2 sets of integer solutions to (2.1) and (2.2) that are exhibited below:

Set: 1

$x = S^2, y = S^4 - \alpha S^2 - hS + A, z = \alpha S^2 + hS - A$

Set: 2

$x = 2(R-S)^2, y = 4(R-S)^4 - 2\alpha(R^2 - S^2) + A, z = 2\alpha(R^2 - S^2) - A$

3. Conclusion

In this paper, the process of obtaining non-zero distinct integer solutions to the system of double equations of degree two with three unknowns has been illustrated. However, there exists infinitely many systems of diophantine equations with multidegree and multiple variables. The successful completion of exhibiting all integers satisfying the requirements setforth in the problem add further progress to Number Theory.

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