Int. J. Nonlinear Anal. Appl.
Volume 12, Special Issue, Winter and Spring 2021, 71-82
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/IJNAA.2021.4845

Coefficient Bounds of m-Fold Symmetric Bi-Univalent Functions for Certain Subclasses

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(Communicated by Dr.Ehsan Kozegar)

Abstract

In this article, the authors introduce two new subclasses of a class m-fold symmetric biunivalent functions in open unit disk. Coefficient bounds for the Taylor-Maclaurin coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ are are obtain . Furthermore, we solve Fekete-Szegö functional problems for functions in $\mathcal{F}_{\sum,m}(\gamma,\mu,\vartheta)$ and $\mathcal{M}_{\sum,m}(\kappa,\eta,\vartheta)$. Also, several certain special improver results for the associated classes are presented .

Keywords: Analytic functions , Bi-Univalent functions , Fekete-Szegö coefficient , Taylor-Maclaurin series, Univalent functions .

1. Introduction

Indicate by $\mathcal{A}$ the class of normalized functions satisfying the condition $f(0) = f'(0) - 1 = 0$ and given by next Taylor expansion :

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad , \quad z \in \mathbb{D} \tag{1.1}$$

which are analytic in the open unit disk $D = \{z \in \mathbb{C} and |z| < 1\}$, where $\mathbb{C}$ is complex plane . Further, let $\mathcal{H}$ indicate the class of all functions in $\mathcal{A}$ which are univalent in open unit disk . The Koebe one

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Received: October 2020 *accepted:* Janaury 2021

– Quarter Theorem [5] ensures that the image of $\mathbb{D}$ under every univalent function $f \in \mathcal{H}$ contains a disk of radius $\frac{1}{4}$. therefore , every univalent functions f has an inverse f^{-1} define $z = f^{-1}(f(z))$,$(z \in \mathbb{D})$and

$$\omega = f(f^{-1}(\omega)), \left(|\omega| < p_0(f); p_0(f) \geqslant \frac{1}{4}\right) \tag{1.2}$$

where

$$f^{-1}(\omega) = \omega - c_2\omega^2 + (2c_2^2 - c_3)\omega^3 - (5c_2^3 - 5c_2c_3 + c_4)\omega^4 + ... \tag{1.3}$$

If both two functions$f$ and f^{-1} are univalent in $\mathbb{D}$, $f \in A$is known to be bi univalent functions. Indicate for the class of bi-univalent functions in $\mathbb{D}$ by $\sum$,which are normalized by (1.1) .Let f and g be two analytic functions in $\mathbb{D}$. A function f is subordinate to g if there exist $\hbar$ be a Schwarz analytic function in $\mathbb{D}$ with $\hbar(0) = 0$ and $|\hbar(z)| < 1, (z \in \mathbb{D})$satisfying the following condition :

$$f(z) = g(\hbar(z)) \quad , \quad z \in \mathbb{D}$$

This subordination is indicate by $f < g$ or $f(z) < g(z)$, $z \in \mathbb{D}$.
If the function g is univalent in , then$f < g$ if and only if $f(0) = g(0)$ and $f(D) \subset g(D)$ (see [20]) .

Lewin [9] obtained a coefficient bound given by $|a_2| \leqslant 1.51$ for each $f \in \sum$and investigated the class $\sum$ of bi-univalent functions .Thereafter, stimulated by the working of Lewin[9] , Clunie and Brannan [3] guessing that $|a_2| \leq \sqrt{2}$ for each $f \in \sum$.

Actully , in recent years Srivastava et al.[16] have actually enlivev the study of bi-univalent and analytic functions, by Bulut [4] it was followed by such work, Adegani and et al.[1], Guney et al. [6], Srivastava and Wanas [12] and other (see, for example [2, 8,13,14,15,17,18,19]) . We notice that the class $\sum$ is note empty . For example , the functions $z, \frac{z}{1-z}, -log(1-z)$and $\frac{1}{2}log\frac{1+z}{1-z}$ are members of $\sum$. However, the Koebe functions is note a member of . Until now , the coefficient estimate problem for each the following Taylor-Maclaurin coefficients $|a_n|$, $(n \in N = \{1, 2, 3, 4, \ldots\ldots\}, n \geqslant 3)$,for functions $f \in \sum$ is as yet an open problem (see ,for specifics,[16]) .

For all$f \in \mathcal{H}$, the function p define by$p(z) = \sqrt[m]{f(z^m)} \quad (m \in N = \{1, 2, 3. \ldots\ldots\})$is maps and univalent in $\mathbb{D}$ into region with m-fold symmetry .Afunction is called m-fold symmetric (see [10],[11]) if the condition of normalized is hold and written as the form :

$$f(z) = z + \sum_{j=1}^{\infty} a_{mj+1}z^{mj+1} \quad , \quad (m \in N = \{1, 2, 3, \ldots\}, z \in \mathbb{D}). \tag{1.4}$$

The class m-fold symmetric univalent functions indicate by $\mathcal{H}_m$ and which are normalized by above series expansion (1.4) . in particular if $m = 1$, the function in class $\mathcal{H}$ are one-fold symmetric. Similar to the notion of m-fold symmetric, one can think of the notion of m-fold symmetric biunivalent functions in a normal way. For all positive integer m, each function f in the class $\sum$ creates an m-fold symmetric biunivalent function. The normalized form of f is define as in (1.4) and f^{-1} is define as follows:

$$\begin{aligned} g(\omega) =& \omega - c_{m+1}\omega^{m+1} + [(m+1)c_{2m+1}^2 - c_{2m+1}]\omega^{2m+1} - \\ & [\frac{1}{2}(m+1)(3m+1)c_{m+1}^3 - (3m+1)c_{m+1}c_{2m+1}]\omega^{3m+1} + ... \end{aligned} \tag{1.5}$$

where $g = f^{-1}$. The class m-fold symmetric biunivalent functions denoted by$\sum_m$. For $m = 1$, the formula (1.5) synchronized with the formula (1.3) of the class $\sum$. Indicate by ς of the class function of the form :

$$h(z) = 1 + h_1 z + h_2 z^2 + ... \quad (z \in \mathbb{D})$$

such that

$$Re(h(z)) > 0 \quad (z \in \mathbb{D}).$$

Pommerenke [10] in see of his working, a symmetric m-fold function h in the class ς of the form

$$h(z) = 1 + d_m z^m + d_{2m} z^{2m} + d_{3m} z^{3m} ... \tag{1.6}$$

Throughout our present investigation , it is assumed that analytic function ϑ with positive real part in $\mathbb{D}$ such that $\vartheta(0) = 0$ and and $\vartheta(\mathbb{D})$ is symmetric with regard to the real part . Such a function has a series expansion of the form:

$$\vartheta(z) = 1 + A_1 z + A_2 z^2 + A_3 z^3 + ..., \quad (A_1 > 0) \tag{1.7}$$

Let two analytic functions $t(z)$ and $u(\omega)$ in $\mathbb{D}$ with

$$t(0) = u(0) \ and \ max|t(z)|, |u(\omega)| < 1.$$

Assume that

$$t(z) = b_m z^m + b_{2m} z^{2m} + b_{3m} z^{3m} + ... \tag{1.8}$$
$$u(\omega) = d_m \omega^m + d_{2m} \omega^{2m} + d_{3m} \omega^{3m} + ... \tag{1.9}$$

Observe that

$$|b_m| \leq 1, |b_{2m}| \leq 1 - |b_m|^2, |d_m| \leq 1, |d_{2m}| \leq 1 - |d_m|^2. \tag{1.10}$$

By simple computations , we have

$$\vartheta(t(z)) = 1 + A_1 b_m z^m + A_2 b_{2m} z^{2m} + A_2 b_m^2 z^{2m} + ..., \quad (|z| < 1) \tag{1.11}$$
$$\vartheta(u(\omega)) = 1 + A_1 d_m \omega^m + A_2 d_{2m} \omega^{2m} + A_2 d_m^2 \omega^{2m} + ..., \quad (|\omega| < 1) \tag{1.12}$$

In this work, two new classes of m-fold biunivalent functions are introduced for this classes and obtain boundary for the Taylor-Maclauain coefficients $|c_{m+1}|$ and $|c_{2m+1}|$. Also, in this two new classes Fekete-Szegö functional problems for afunctions are presented .

2. The function class $\mathcal{F}_{\sum,m}(\gamma, \mu, \vartheta)$

Definition1. Let $f(z)$ be a function, given in (1.4) ,be in the class$\mathcal{F}_{\sum,m}(\gamma, \mu, \vartheta)$ if satisfied the following conditions $f \in \sum_{m'}$ $\left(\frac{zf'(z)}{f(z)}\right)^{\gamma} \left(\frac{zf'(z) + \mu z^2 f''(z)}{(1-\mu)f(z) + \mu z f'(z)}\right) < \vartheta(z)$, where $z \in \mathbb{D}$, $0 \leq \mu \leq 1$ and $\gamma \geq 0$, and
$\left(\frac{\omega g'(\omega)}{g(\omega)}\right)^{\gamma} \left(\frac{\omega g'(\omega) + \mu \omega^2 g''(\omega)}{(1-\mu)g(\omega) + \mu \omega g'(\omega)}\right) < \vartheta(\omega), \quad (g(\omega) = f^{-1}(\omega))$
where g() be a function define by (1.5) .

Note that the particular cases of above class

1- when$m = 1$ reduce to the classes $\mathcal{F}_{\sum,m}(\gamma,\mu,\vartheta) = \mathcal{F}_{\sum,1}(\gamma,\mu,\vartheta)$.

2- when $m = 1$,$\gamma = 0$ and $\mu = 0$ reduce to the classes $\mathcal{F}_{\sum,m}(\gamma,\mu,\vartheta) = \mathcal{F}_{\sum,1}(0,0,\vartheta)$introduce to the class starlike function [7].

3) when $m = 1$,$\gamma = 0$ and $\mu = 0$ reduce to the classes $\mathcal{F}_{\sum,m}(\gamma,\mu,\vartheta) = \mathcal{F}_{\sum,1}(1,0,\vartheta)$ introduce to the class convex function [7].

The next Theorem prove to a class $\mathcal{F}_{\sum,m}(\gamma,\mu,\vartheta)$.
Theorem1. Let $f(z)$ be a function, define by (1.4), in the class$\mathcal{F}_{\sum,m}(\gamma,\mu,\vartheta)$Then:

$$|c_{m+1}| \leqslant \frac{A_1\sqrt{2A_1}}{\sqrt{|[2m^2(1-\mu^2 m-\mu)+2m^2(\mu\gamma m+\mu m+2\gamma)+m(2\mu+\gamma^2-\gamma)]A_1^2-2m^2(1+\mu m+\gamma)^2A_2|+(1+\mu m+\gamma)^2A_1}} \tag{2.1}$$

and

$$|c_{m+1}| \leqslant \begin{cases} \dfrac{(m+1)A_1}{[2m^2(1-\mu^2 m-\mu)+2m^2(\mu\gamma m+\mu m+2\gamma)+m(2\mu+\gamma^2-\gamma)]}; if|A_2| \leqslant A_1 \\[2ex] \dfrac{\langle (m+1)[|2m^2(1-\mu^2 m-\mu)+2m^2(\mu\gamma m+ \mu m+2\gamma)+m(2\mu+\gamma^2-\gamma)A_1^2-2m^2(1+\mu m+\gamma)^2A_2|]A_1+(m+1)(1+\mu m+\gamma)^2|A_2|A_1\rangle}{[2m^2(1-\mu^2 m-\mu)+2m^2(\mu\gamma m+\mu m+2\gamma)+m(2\mu+\gamma^2-\gamma)]|2m^2(1-\mu^2 m-\mu)+2m^2(\mu\gamma m+\mu m+2\gamma)+m(2\mu+\gamma^2-\gamma)A_1^2-2m^2(1+\mu m+\gamma)^2A_2|+(1+\mu m+\gamma)^2A_1} ; if|A_2| > A_1 \end{cases} \tag{2.2}$$

Proof . Let $\mathcal{F}_{\sum,m}(\gamma,\mu,\vartheta)$. Then there is two analytic functions $t : \mathbb{D}ß\mathbb{D}$ and $u : \mathbb{D}ß\mathbb{D}$ with $t(0) = u(0) = 0$, satisfying the next conditions:

$$\left(\frac{zf'(z)}{f(z)}\right)^{\gamma}\left(\frac{zf'(z)+\mu z^2 f''(z)}{(1-\mu)f(z)+\mu zf'(z)}\right) = \vartheta(t(z)), \tag{2.3}$$

and

$$\left(\frac{\omega g'(\omega)}{g(\omega)}\right)^{\gamma}\left(\frac{\omega g'(\omega)+\mu\omega^2 g''(\omega)}{(1-\mu)g(\omega)+\mu\omega g'(\omega)}\right) = \vartheta(u(\omega)), \quad (g(\omega) = f^{-1}(\omega)) \tag{2.4}$$

We get

$$\left(\frac{zf'(z)}{f(z)}\right)^{\gamma}\left(\frac{zf'(z)+\mu z^2 f''(z)}{(1-\mu)f(z)+\mu zf'(z)}\right) = 1+m(1+\mu m+\gamma)c_{m+1}z^m+2m(1+2\mu m+\gamma)c_{2m+1}z^{2m} \\ -\left[m(\mu m+1)^2-m^2\gamma(\mu m+1)-\frac{1}{2}m(\gamma^2-3\gamma)\right]c_{m+1}^2 z^{2m}+\ldots \tag{2.5}$$

and

$$\left(\frac{\omega g'(\omega)}{g(\omega)}\right)^{\gamma}\left(\frac{\omega g'(\omega)+\mu\omega^2 g''(\omega)}{(1-\mu)g(\omega)+\mu\omega g'(\omega)}\right) = 1-m(1+\mu m+\gamma)c_{m+1}\omega^m-2m(1+2\mu m+\gamma)c_{2m+1}\omega^{2m} \\ +\left[m(2m+1)^2-\mu m(2m^2-\mu m^2+2)+\gamma m^2(1+\mu m)+2m(1+m)+\frac{1}{2}m(\gamma^2-3\gamma)\right]c_{m+1}^2\omega^{2m}+\ldots \tag{2.6}$$

from (1.11),(1.12),(2,5) and (2.6), we find that

$$m(1+\mu m+\gamma)c_{m+1} = A_1 b_m, \tag{2.7}$$

$$2m(1+2\mu m+\gamma)c_{2m+1} - \left[m(\mu m+1)^2 - m^2\gamma(\mu m+1) - \frac{1}{2}m(\gamma^2-3\gamma)\right]c_{m+1}^2$$
$$= A_1 b_{2m} + A_2 b_m^2 \tag{2.8}$$
$$-m(1+\mu m+\gamma)c_{m+1} = A_1 d_m \tag{2.9}$$

and

$$-2m(1+2\mu m+\gamma)c_{2m+1}$$
$$+\left[m(2m+1)^2 + \gamma m^2(1+\mu m) + \mu m(2m^2-\mu m^2+2) + 2m(1+m) + \frac{1}{2}m(\gamma^2-3\gamma)\right]c_{m+1}^2$$
$$= A_1 d_{2m} + A_2 d_m^2 \tag{2.10}$$

From (2.7) and (2.9), we get

$$d_m = -b_m \tag{2.11}$$

and

$$2m^2(1+\mu m+\gamma)^2 c_{m+1}^2 = A_1^2(b_m^2 + d_m^2)$$

By adding (2.8) and (2.10) and , up on some calculations using (2.7) and (2.11), we obtain

$$\left[[2m^2(1-\mu^2 m-\mu) + 2m^2(\mu\gamma m+\mu m+2\gamma) + m(2\mu+\gamma^2-\gamma)]A_1^2 - 2m^2(1+\mu m+\gamma)^2 A_2\right]c_{m+1}^2$$
$$= A_1^3(b_{2m} + d_{2m}) \tag{2.12}$$

Moreover, the equations (2.11) , (2.12), jointly with (1.10), yield

$$\left|[2m^2(1-\mu^2 m-\mu) + m(\gamma^2+2\mu-\gamma)2m^2(\mu\gamma m+\mu m+2\gamma)]A_1^2 - 2m^2(1+\mu m+\gamma)^2 A_2\right|$$
$$\leq 2A_1^3(1-|b_m|^2) \tag{2.13}$$

Now, from (2.7) and (2.13), we get

$$|c_{m+1}| \leqslant$$
$$\frac{A_1\sqrt{2A_1}}{\sqrt{|[2m^2(1-\mu^2 m-\mu) + m(2\mu+\gamma^2-\gamma) + 2m^2(\mu\gamma m+\mu m+2\gamma)]A_1^2 - 2m^2(1+\mu m+\gamma)^2 A_2| + (1+\mu m+\gamma)^2 A_1}}$$

as certain in (2.1).
Subtracting (2.10) from (2.8), and using (2.11) and (2.7), we get

$$\mu(2m^2-\mu m^2+2) + (2m+1) + \gamma m(\mu m+1) + 2\gamma(m+1) + \frac{1}{2}(\gamma^2-3\gamma)A_1 b_{2m}$$
$$+\left[m^2\gamma(\mu m+1) - m(\mu m+1)^2 - \frac{1}{2}m(\gamma^2-3\gamma)\right]A_1 d_{2m} + 2(m+1)[1+2\mu m+\gamma]^2 A_2 b_m^2$$
$$= 4m^2[1+\gamma+2\mu m][1+\mu m+\gamma]c_{2m+1} \tag{2.14}$$

Therefore, by using equation (1.10) in (2.14) , we obtain

$$\begin{aligned}
&2m^2[1+\mu m+\gamma][1+\mu m+\gamma]|c_{2m+1}| \\
&\leq \mu(2m^2-\mu m^2+2)+(2m+1)+\gamma m(\mu m+1)+2\gamma(m+1)+\frac{1}{2}(\gamma^2-3\gamma)A_1 \\
&-(2m+1)+\mu(2m^2-\mu m^2+2)+\gamma m(\mu m+1)+2(1+m)+\frac{1}{2}(\gamma^2-3\gamma)A_1|b_m|^2 \\
&=+(1+m)(1+2\mu m+\gamma)|A_2||b_m|^2
\end{aligned} \tag{2.15}$$

Since

$$|b_m|^2 \leq \frac{[1+\mu m+\gamma]^2 A_1}{\sqrt{|[2m^2(1-\mu^2 m-\mu)+m(2\mu+\gamma^2-\gamma)+2m^2(\mu\gamma m+\mu m+2\gamma)]A_1^2-2m^2(1+\mu m+\gamma)^2A_2|+(1+\mu m+\gamma)^2A_1}} \tag{2.16}$$

Up on substituting from (2.16) into (2.15), we are led easily to the asertion (2.2) of Theorem1.

In case of one-fold symmetric functions of Theorem1 we get the next results.

Corollary1. Let $f(z)$ be a function, define by (1.4), in the clas $\mathcal{F}_{\sum,1}(\gamma,\mu,\vartheta)$. Then

$$|c_2| \leqslant \frac{A_1\sqrt{2A_1}}{\sqrt{|[2(1-\mu^2-\mu)+2(\mu\gamma+\mu+2\gamma)+(2\mu+\gamma^2-\gamma)]A_1^2-2^2(1+\mu+\gamma)^2A_2|+(1+\mu+\gamma)^2A_1}}$$

and

$$|c_3| \leqslant \begin{cases} \dfrac{2A_1}{[2(1-\mu^2-\mu)+2(\mu\gamma+\mu+2\beta)+(2\mu+\gamma^2-\gamma)]}; if|A_2|\leqslant A_1 \\[2ex] \dfrac{\begin{array}{c}\langle 2[|2(1-\mu^2-\mu)+2(\mu\gamma+\mu+2\gamma)\\ +(2\mu+\gamma^2-\gamma)A_1^2-2(1+\mu+\gamma)^2A_2|]A_1+2(1+\mu+\gamma)^2|A_2|A_1\,\rangle\end{array}}{\begin{array}{c}[2(1-\mu^2-\mu)+2(\mu\gamma+\mu+2\gamma)+(2\mu+\gamma^2-\gamma)]|2(1-\mu^2-\mu)+2(\mu\gamma+\mu+2\gamma)+\\ (2\mu+\gamma^2-\gamma)A_1^2-2(1+\mu+\gamma)^2A_2|+(1+\mu+\gamma)^2A_1\end{array}} \quad ; if|A_2|>A_1 \end{cases}$$

Theorem2. Let $f(z)$ be a function, define by (1.4), in the class $\mathcal{F}_{\sum,m}(\gamma,\mu,\vartheta)$. Then

$$|c_{2m+1}-\lambda c_{m+1}^2| \leq \begin{cases} \dfrac{A_1}{2m(1+2\mu m+\gamma)} \ for\ 0\leq|q(\lambda)|<\dfrac{1}{4m(1+2\mu m+\gamma)} \\ 2A_1|q(\lambda)|\ for\ |q(\lambda)|\geq\dfrac{1}{4m(1+2\mu m+\gamma)} \end{cases} \tag{2.17}$$

where

$$q(\lambda)=\frac{A_1^2(m+1-2\lambda)}{2[2m^2(1-\mu^2m-\mu)+m(2\mu+\gamma^2-\gamma)+2m^2(\mu\gamma m+\mu m+2\gamma)]A_1^2-2m^2(1+\mu m+\gamma)^2A_2}$$

$$c_{m+1}^2=\frac{A_1^3(b_{2m}+d_{2m})}{[2m^2(1-\mu^2m-\mu)+2m^2(\mu m+\mu m\gamma^2+2\gamma)+m(2\mu+\gamma^2-\gamma)]A_1^2-2m^2(1+\mu m+\gamma)^2A_2} \tag{2.18}$$

Subtract (2.10) from (2.8), we get

$$c_{2m+1}=\frac{(m+1)A_1^2(b_m^2+d_m^2)}{4m^2[1+\mu m+\gamma]^2}+\frac{A_1(b_{2m}-d_{2m})}{4m[1+2\mu m+\gamma]} \tag{2.19}$$

From (2.18) and (2.19), it follows that

$$c_{2m+1}-\lambda c_{m+1}^{2}=A_1\left[\left(q(\lambda)+\frac{1}{4m[1+2\mu m+\gamma]}\right)b_{2m}+\left(q(\lambda)-\frac{1}{4m[1+2\mu m+\gamma]}\right)d_{2m}\right] \quad (2.20)$$

where

$$q(\lambda)=\frac{A_1^2(m+1-2\lambda)}{2[2m^2(1-\mu^2 m-\mu)+m(2\mu+\gamma^2-\gamma)+2m^2(\mu\gamma m+\mu m+2\gamma)]A_1^2-2m^2(1+\mu m+\gamma)^2A_2}$$

Because each $A_i\in\mathbb{R}$(real)and $_1>0$, this implies that get the equation (2.17) .

In cases of onefold functions symmytric, Theorem2 reduecs to the next .

Corollary2. Let f(z) be a function, define by (1.4), in the class $\mathcal{F}_{\sum,1}(\gamma,\mu,\vartheta)$. Then

$$|c_3-\lambda c_2|\leq\begin{cases}\dfrac{A_1}{2(1+2\mu+\gamma)} & for\ 0\leq|q(\lambda)|<\dfrac{1}{4(1+2\mu+\gamma)}\\ 2A_1|q(\lambda)| & for\ |q(\lambda)|\geq\dfrac{1}{4(1+2\mu+\gamma)}\end{cases}$$

In Theorem2 in case $\lambda=1$, we get the following corollary

Corollary3. Let f(z) be a function, define by (.1.4), in a class$\mathcal{F}_{\sum,m}(\gamma,\mu,\vartheta)$. Then

$$|c_{2m+1}-c_{m+1}^2|\leq\begin{cases}\dfrac{A_1}{2m(1+2\mu m+\gamma)} & for\ 0\leq|q(\lambda)|<\dfrac{1}{4m(1+2\mu m+\gamma)}\\ 2A_1|q(\lambda)| & for\ |q(\lambda)|\geq\dfrac{1}{4m(1+2\mu m+\gamma)}\end{cases}$$

In case of onefold symmyric, then Corollary3 reduecs to the next Corollary.

Corollary4. Let f(z) be a function, define by (1.4), be in the class $\mathcal{F}_{\sum,1}(\beta,\mu,\vartheta)$. Then

$$|c_3-c_2^2|\leq\frac{A_1}{2(1+2\mu+\gamma)}$$

3. The function class $\mathcal{M}_{\sum,m}(\kappa,\eta,\vartheta)$

Definition2. Let $f(z)$ be a function, define by (1.4) , in a class$\mathcal{M}_{\sum,m}(\kappa,\eta,\vartheta)$ if satisfied next conditions $f\in\sum_{m'}\left(\frac{zf'(z)}{f(z)}\right)^{\kappa}\left((1-\eta)\frac{f(z)}{z}+\eta f'(z)\right)<\vartheta(z)$, where $z\in\mathbb{D}$, $0\leq\eta$ and $\kappa\geq0$, and $\left(\frac{\omega g'(\omega)}{g(\omega)}\right)^{\kappa}\left((1-\eta)\frac{g(\omega)}{\omega}+\eta g'(\omega)\right)<\vartheta(\omega),\quad (g(\omega)=f^{-1}(\omega))$
where$g(\omega)$ be a function define by (1.5) .

Note in above definiation in case $m=1$ the class $\mathcal{M}_{\sum,m}(\kappa,\eta,\vartheta)$ reduce by $\mathcal{M}_{\sum,1}(\kappa,\eta,\vartheta)$.

Theorem3 . Let $f(z)$ be a function , define by (1.4), in the classe $\mathcal{M}_{\sum,m}(\kappa,\eta,\vartheta)$. Thens

$$|c_{m+1}|\leqslant \frac{A_1\sqrt{2A_1}}{\sqrt{|(\kappa m(\eta m+2m+3)+(1+m)(1+2\eta m)+m(\kappa^2-3\kappa))A_1^2-2(1+m(\eta+\kappa)^2A_2|+2(1+m(\eta+\kappa)^2A_1}} \quad (3.1)$$

and

$$|c_{2m+1}| \leqslant \begin{cases} \dfrac{(1+m)A_1}{(1+m)(1+2\eta m)+2m\kappa(m+1)+\frac{1}{2}m(\kappa^2-3\kappa)}; if B_1 \leqslant \psi(B_1) \\ \dfrac{\begin{array}{c}|(\kappa m(\eta m+2m+3)+m(\kappa^2-3\kappa)+(1+m)(1+2\eta m))A_1^2- \\ 2(1+m\kappa+m\eta)^2|A_1+(\kappa^2-3\kappa)+(1+m)(1+2\eta m))A_1^3\end{array}}{\begin{array}{c}(1+2m(\eta+\kappa))a_{2m+1}+\left[\kappa m(1+\eta m)+\frac{1}{2}m(\kappa^2-3\kappa)\right]|(\kappa m(\eta m+2m+3)+ \\ m(\kappa^2-3\kappa)+(1+m)(1+2\eta m))A_1^2-2(1+m(\eta+\kappa))^2A_2|+2(1+m(\eta+\kappa))^2A_1\end{array}} ; if|A_2| > A_1 \end{cases} \tag{3.2}$$

where

$$\psi(A_1) = \frac{2(m\kappa+m\eta+1)^2}{2m\kappa(m+1)+(1+m)(1+2\eta m)+\frac{1}{2}m(\kappa^2-3\kappa)}$$

Proof . Let $\mathcal{M}_{\sum,m}(\kappa,\eta,\vartheta)$. Then there is two analytic functions $t:\mathbb{D}\to\mathbb{D}$ and $u:\mathbb{D}\to\mathbb{D}$ with $t(0)=u(0)=0$, such that satisfying the next conditions:

$$f \in \sum_{m'} \left(\frac{zf'(z)}{f(z)}\right)^{\kappa}\left((1-\eta)\frac{f(z)}{z}+\eta f'(z)\right) < \vartheta(t(z)) \tag{3.3}$$

and

$$\left(\frac{\omega g'(\omega)}{g(\omega)}\right)^{\kappa}\left((1-\eta)\frac{g(\omega)}{\omega}+\eta g'(\omega)\right) < \vartheta(\omega), \quad (g(\omega)=f^{-1}(\omega)) \tag{3.4}$$

Since

$$\left(\frac{zf'(z)}{f(z)}\right)^{\kappa}\left((1-\eta)\frac{f(z)}{z}+\eta f'(z)\right) = 1+(1+m(\eta+\kappa))c_{m+1}z^m+(1+2m(\eta+\kappa))c_{2m+1}z^{2m}$$
$$+\left[\kappa m(\eta m+1)+\frac{1}{2}m(\kappa^2-3\kappa)\right]c_{m+1}^2z^{2m}+...$$

and

$$\left(\frac{\omega g'(\omega)}{g(\omega)}\right)^{\kappa}\left((1-\eta)\frac{g(\omega)}{\omega}+\eta g'(\omega)\right) == 1-(1+m(\eta+\kappa))c_{m+1}\omega^m+(1+2m(\eta+\kappa))c_{2m+1}\omega^{2m}$$
$$+\left[(1+m)(1+2\eta m)+2m\kappa(1+m)+\frac{1}{2}m(\kappa^2-3\kappa)\right]c_{m+1}^2\omega^{2m}+...$$

Now,frome (1.11) ,(1.12),(3.3) and (3.4) we obtain

$$(1+m\kappa+m\eta)c_{m+1} = A_1b_m \tag{3.5}$$

$$(1+2m(\eta+\kappa))c_{2m+1}+\left[\frac{1}{2}m(\kappa^2-3\kappa)+\kappa m(1+\eta m)\right]c_{m+1}^2 = A_1b_{2m}+ = A_2b_m^2 \tag{3.6}$$

$$-(1+m\kappa+m\eta)c_{m+1} = A_1d_m \tag{3.7}$$

and

$$-(1+2m(\eta+\kappa))c_{2m+1}+\left[(1+m)(1+2\eta m)+2m\kappa(1+m)+\frac{1}{2}m(\kappa^2-3\kappa)\right]c_{m+1}^2=A_1d_{2m}+A_2d_m^2 \tag{3.8}$$

From (3.5) and (3.7), we get

$$d_m=-d_m \tag{3.9}$$

adding (3.6) and (3.8) and up on some calculations use (3.5) and (3.9),we obtain

$$|(\kappa m(\eta m+2m+3)+(1+m)(1+2\eta m)+m(\kappa^2-3\kappa))A_1^2-2(1+m(\eta+\kappa)^2A_2|a_{m+1}^2 \\ =A_1^3(b_{2m}+d_{2m}) \tag{3.10}$$

Also , from (3.9) and (3.10), jointly with (1.10), implies that

$$|(\kappa m(\eta m+2m+3)+(1+m)(1+2\eta m)+m(\kappa^2-3\kappa))A_1^2-2(1+m(\eta+\kappa)^2A_2|a_{m+1}^2\leq A_1^3(1-|b_m^2|) \tag{3.11}$$

Now , from (3.5) and (3.11), conclude that

$$|c_{m+1}|\leqslant \\ \frac{A_1\sqrt{2A_1}}{\sqrt{|(\kappa m(\eta m+2m+3)+(1+m)(1+2\eta m)+m(\kappa^2-3\kappa))A_1^2-2(1+m(\eta+\kappa)^2A_2|+2(1+m(\eta+\kappa)^2A_1}}$$

as asserted in (3.1).
Next, subtracting (3.8) from (3.6), we find

$$2(1+2m(\eta+\kappa))c_{2m+1}=[(1+m)(1+2\eta m)-m\kappa(1+\eta m)-\kappa m(1+\eta m)+2\kappa m(m+1)]\,c_{m+1}^2 \\ +A_1(b_{2m}-d_{2m}) \tag{3.12}$$

By using (1.10) and (3.5) in (3.12), if follows that

$$2(1+2m(\eta+\kappa))|c_{2m+1}|\leq[2\kappa m(m+1)+(m+1)(1+2\eta m)+\kappa m(1+\eta m)]\,|c_{m+1}^2| \\ +A_1(|b_{2m}|-|d_{2m}|)$$

Which ,implies that in view (3.5).
By applying (3.1) in (3.13),we get (3.2).
Theorem3 is complete .

Remark: in case one-fold symmetric functions, Theorem3 which we recall as next Corollary.

Corollary5 . Let $f(z)$ be a function , define by (1.4), be in the classe $\mathcal{M}_{\sum,m}(\kappa,\eta,\vartheta)$. Then

$$|c_2|\leqslant \\ \frac{A_1\sqrt{2A_1}}{\sqrt{|(\kappa(\eta+5)+5(1+2\eta 5)+(\kappa^2-3\kappa))A_1^2-2(1+\eta+\kappa)^2A_2|+2(1+\eta+\kappa)^2A_1}} \tag{3.13}$$

and

$$|c_3| \leqslant \begin{cases} \dfrac{A_1}{(1+2\eta)+2\kappa+\frac{1}{4}m(\kappa^2-3\kappa)}; if|A_2| \leq \dfrac{(1+\eta+\kappa)^2}{(1+2\eta)+2\kappa+\frac{1}{4}m(\kappa^2-3\kappa)} \\ \dfrac{|(\kappa(\eta+5)+(\kappa^2-3\kappa)+2(1+2\eta))A_1^2-2(1+\eta+\kappa)^2A_2|A_1 2(1+2\eta)+4\kappa+\frac{1}{2}m(\kappa^2-3\kappa)A_1^3}{2(1+2\eta)+4\kappa+\frac{1}{2}(\kappa^2-3\kappa)|(\kappa(\eta+5)+(\kappa^2-3\kappa)+2(1+2\eta))A_1^2-2(1+\eta+\kappa)^2A_2|+2(1+\eta+\kappa)^2A_1}; \\ if|A_2| \leq \dfrac{(1+\eta+\kappa)^2}{(1+2\eta)+2\kappa+\frac{1}{4}m(\kappa^2-3\kappa)} \end{cases}$$

Theorem4. Let $f(z)$ be a function, define by (1.4), in aclass $\mathcal{M}_{\sum,m}(\kappa,\eta,\vartheta)$. Then

$$|c_{2m+1}-\lambda c_{m+1}^2| \leq \begin{cases} \dfrac{A_1}{(1+2m(\eta+\kappa))+\left[\kappa m(\eta m+1)+\frac{1}{2}(\kappa^2-3\kappa)\right]} \; for\ 0 \leq |q(\lambda)| < \dfrac{1}{2(1+2m(\eta+\kappa))} \\ 2A_1|q(\lambda)| \; for\ |q(\lambda)| \geq \dfrac{1}{2(1+2m(\eta+\kappa))} \end{cases} \tag{3.14}$$

where

$$q(\lambda) = \frac{A_1^2(m+1-2\lambda)}{2[(\kappa m(\eta m+2m+3)+(1+m)(1+2\eta m)+m(\kappa^2-3\kappa))B_1^2-2(1+m(\eta+\kappa))^2B_2]}$$

Proof . Adding (3.6) and (3.8), we get

$$c_{m+1}^2 = \frac{A_1^3(b_{2m}+d_{2m})}{2[(\kappa m(\eta m+2m+3)+(1+m)(1+2\eta m)+m(\kappa^2-3\kappa))A_1^2-2(1+m(\eta+\kappa))^2A_2]} \tag{3.15}$$

Subtract (3.8) from (3.6), we get

$$c_{2m+1} = \frac{(m+1)A_1^2(b_m^2+d_m^2)}{2(1+m(\eta+\kappa))^2} + \frac{A_1(b_{2m}-d_{2m})}{2(1+2m(\eta+\kappa))} \tag{3.16}$$

From (3.15) and (3.16), it follows that

$$c_{2m+1}-\lambda c_{m+1}^2 = B_1\left[\left(q(\lambda)+\frac{1}{2(1+2m(\eta+\kappa))}\right)b_{2m}+\left(q(\lambda)-\frac{1}{2(1+2m(\eta+\kappa))}\right)d_{2m}\right] \tag{3.17}$$

where

$$q(\lambda) = \frac{A_1^2(m+1-2\lambda)}{2[(\kappa m(\eta m+2m+3)+(1+m)(1+2\eta m)+m(\kappa^2-3\kappa))A_1^2-2(1+m(\eta+\kappa))^2A_2]}$$

Since all i are real and $A_1 > 0$, which implies the assertion (3.14) .
In case of the one- fold symmytric function , Theorem4 reduce to the next .

Corollary6. Let $f(z)$ be the function , define by (1.4), in a classe $\mathcal{M}_{\sum,m}(\kappa,\eta,\vartheta)$. Then

$$|c_{2m+1}-c_{m+1}^2| \leq \begin{cases} \dfrac{A_1}{(1+2(\eta+\kappa))+\left[\kappa(1+\eta)+\dfrac{1}{2}(\kappa^2-3\kappa)\right]} \; for \; 0 \leq |q(\lambda)| < \dfrac{1}{2(1+2(\eta+\kappa))} \\ 2A_1|q(\lambda)| \; for \; |q(\lambda)| \geq \dfrac{1}{2(1+2(\eta+\kappa))} \end{cases}$$

In Theorem4 in case $\lambda = 1$, we get the following corollary.

Corollary7. Let $f(z)$ be a function, define by (1.4), in a clas $\mathcal{M}_{\sum,m}(\kappa,\eta,\vartheta)$. Then

$$|c_{2m+1}-c_{m+1}^2| \leq \begin{cases} \dfrac{A_1}{(1+2m(\eta+\kappa))+\left[\kappa m(1+\eta m)+\dfrac{1}{2}m(\kappa^2-3\kappa)\right]} \; for \; 0 \leq |q(\lambda)| < \dfrac{1}{2(1+2m(\eta+\kappa))} \\ 2A_1|q(\lambda)| \; for \; |q(\lambda)| \geq \dfrac{1}{2(1+2m(\eta+\kappa))} \end{cases}$$

In case of one-fold symmyric, then Corollary7 reduces to the next Corollary.

Corollary8. Let $f(z)$ be a function, define by (1.4), in a class $\mathcal{M}_{\sum,m}(\kappa,\eta,\vartheta)$. Then

$$|c_3-c_2^2| \leq \frac{A_1}{(1+\eta(2+\kappa))+\dfrac{1}{2}m(\kappa^2+3\kappa)}$$

4. Conclusions

We conclude from this study, in the case of applying the two new classes m-fold symmetric bi univalent to the geometric functions, it was determine $|c_{m+1}|$ and $|c_{2m+1}|$ for all class m-fold symmetric bi-univalent , it is useful in complex analysis ,also derived the Fekete-Szegö functional problems for functions are obtains and several many improver results for this two new classes are presented inside new open unit disk

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