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Perfect 3-colorings Of Heawood Graph

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Abstract

Perfect coloring is a generalization of the notion of completely regular codes, given by Delsarte. A perfect m-coloring of a graph G with m colors is a partition of the vertex set of G into m parts $A_1, \dots, A_m$ such that, for all $i, j \in \{1, \dots, m\}$, every vertex of A_i is adjacent to the same number of vertices, namely, a_{ij} vertices, of A_j. The matrix $A = (a_{ij})i, j \in \{1, 2, , m\}$, is called the parameter matrix. We study the perfect 3-colorings (also known as the equitable partitions into three parts) of the Heawood graph. In particular, we classify all the realizable parameter matrices of perfect 3-colorings for the Haywood graphs.

Keywords: perfect coloring; parameter matrices; cubic graph
2010 MSC: 05C12, 05C50.

1. Introduction

The concept of a perfect m-coloring plays an important role in graph theory, algebraic combinatorics, and coding theory (completely regular codes). There is another term for this concept in the literature as "equitable partition" (see [11]).

The existence of completely regular codes in graphs is a historical problem in mathematics. Completely regular codes are a generalization of perfect codes. In 1973, Delsarte conjectured the non-existence of nontrivial perfect codes in Johnson graphs. Therefore, some effort has been done on enumerating the parameter matrices of some Johnson graphs, including J (6; 3), J (7; 3), J (8; 3), J (8; 4), and J (v; 3) (v odd) (see [4, 5, 9]). Fon-Der-Flass enumerated the parameter matrices (perfect 2-colorings) of n-dimensional hypercube Qn for $n < 24$. He also obtained some constructions and a necessary condition for the existence of perfect 2-colorings of the n-dimensional cube with a given parameter matrix (see [6, 7, 8]). In this article, we enumerate the parameter matrices of all perfect 3-colorings of the Haywood graph.

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Definition 1.1. For a graph G and an integer m, a mapping $T : V(G) \Rightarrow \{1, 2, \ldots, m\}$ is called a perfect m-coloring with matrix $A = (aij)i; j\{1, 2, \ldots, m\}$, if it is surjective, and for all i,j, for every vertex of color i, the number of its neighbors of color j is equal to aij . The matrix A is called the parameter matrix of a perfect coloring. In the case m = 3, we call the first color white that show by W, the second color black that show by B , and the third color red that show by R. In this paper, we generally show a parameter matrix by

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Remark 1.2 In this paper, we consider all perfect 3-colorings, up to renaming the colors; i.e. we identify the perfect 3-coloring with the matrices

$$\begin{bmatrix} a & c & b \\ g & i & h \\ d & f & e \end{bmatrix}, \begin{bmatrix} e & d & f \\ b & a & c \\ h & g & i \end{bmatrix}, \begin{bmatrix} e & f & d \\ h & i & g \\ b & c & a \end{bmatrix}, \begin{bmatrix} i & h & g \\ f & e & d \\ c & b & a \end{bmatrix}, \begin{bmatrix} a & g & h \\ c & a & b \\ f & d & e \end{bmatrix},$$

obtained by switching the colors with the original coloring.

The Heawood graph is 3-regular, an undirected graph with 14 vertices and 21 edges. It has graph diameter 3, graph radius 3, and girth 6. It is a cubic symmetric graph, nonplanar, and Hamiltonian, and the smallest graph is regular with this intersection number. This graph is named Percy John Heawood.

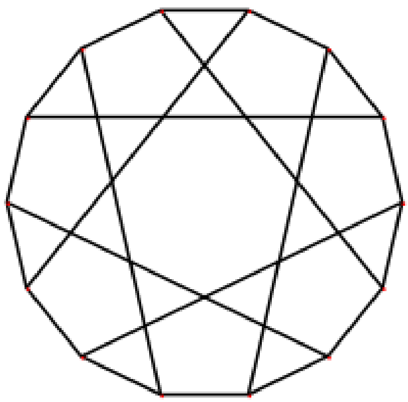

Figure 1: Heawood graph.

2. PRELIMINARIES AND ANALYSIS

In this section, we present some results concerning necessary conditions for the existence of perfect 3-colorings of connected graph of order 8 with a given parameter matrix A. The simplest necessary condition for the existence of perfect 3-colorings of a cubic connected graph with the matrix $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ is

$$a + b + c = d + e + f = g + h + i = 3.$$

Also, it is clear that we cannot have b = c = 0, d = f = 0, or g = h = 0, since the graph is connected. In addition, b = 0, c = 0, f = 0 if d = 0, g = 0, h = 0, respectively.

The number θ is called an eigenvalue of a graph G, if θ is an eigenvalue of the adjacency matrix of this graph. The number λ is called an eigenvalue of a perfect coloring T into three colors with the matrix A, if λ is an eigenvalue of A. The following theorem demonstrates the connection between the introduced notions.

Theorem 2.1 [10] If T is a perfect coloring of a graph G in m colors, then any eigenvalue of T is an eigenvalue of G.

The next theorem can be useful to find the eigenvalues of a parameter matrix.

Theorem 2.2 Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ be a parameter matrix of a k-regular graph. Then the eigenvalues of A are

$$\lambda_{1,2} = \frac{tr(A) - k}{2} \pm \sqrt{(\frac{tra(A) - k}{2})^2 - \frac{det(A)}{k}}, \lambda_3 = k.$$

Proof: By using the condition a + b + c = d + e + f = g + h + i = k, it is clear that one of the eigenvalues is k. Therefore $det(A) = k\lambda_1\lambda_2$. From $\lambda_2 = tr(A)\lambda_1 k$, we get $det(A) = k\lambda_1(tr(A) - \lambda_1 - k) = -k\lambda_1^2 + k(tr(A)k)\lambda_1$. By solving the equation $\lambda^2 + (k - tr(A)) + \frac{det(A)}{k} = 0$, we obtain

$$\lambda_{1,2} = \frac{tr(A) - k}{2} \pm \sqrt{(\frac{tra(A) - k}{2})^2 - \frac{det(A)}{k}},$$

The eigenvalues of the Haywood graph are stated in the next theorem.

Theorem 2.3 [12] The distinct eigenvalues of the Heawood graph are the numbers 3, $1+\sqrt{2}, -1-\sqrt{2}, -3$.

The next proposition gives a formula for calculating the number of white, black and red vertices, in a perfect 3-coloring.

Proposition 2.4 [3] Let T be a perfect 3-coloring of a graph G with the matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

- If $b, c, f \neq 0$, then

$$|W| = \frac{|V(G)|}{\frac{b}{d} + 1 + \frac{c}{g}}, |B| = \frac{|V(G)|}{\frac{d}{b} + 1 + \frac{f}{h}}, |R| = \frac{|V(G)|}{\frac{h}{f} + 1 + \frac{g}{c}}$$

- If $b = 0$, then

$$|W| = \frac{|V(G)|}{\frac{c}{g} + 1 + \frac{ch}{fg}}, |B| = \frac{|V(G)|}{\frac{f}{h} + 1 + \frac{fg}{ch}}, |R| = \frac{|V(G)|}{\frac{h}{f} + 1 + \frac{g}{c}}$$

- If $f = 0$, then

$$|W| = \frac{|V(G)|}{\frac{b}{d} + 1 + \frac{c}{g}}, |B| = \frac{|V(G)|}{\frac{d}{b} + 1 + \frac{cd}{bg}}, |R| = \frac{|V(G)|}{\frac{g}{c} + 1 + \frac{bg}{cd}}$$

In all of this paper, without restriction of generality, we assume $|W| \leq |B| \leq |R|$.

Heawood graph is a 3-regular connected graph of order 14. It can be seen that there are only 109 matrices that can be a parameter matrix corresponding to a perfect 3-coloring in a Heawood graph. By using Remark 1.2 and easy computation shows that we should consider 22 out of 109 matrices. These matrices are listed below.

$$\begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 1 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 3 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 3 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix},$$

$$\begin{bmatrix}0&0&3\\0&0&3\\1&1&1\end{bmatrix},\begin{bmatrix}0&0&3\\0&0&3\\1&2&0\end{bmatrix},\begin{bmatrix}0&0&3\\0&1&2\\1&1&1\end{bmatrix},\begin{bmatrix}0&0&3\\0&1&2\\1&2&0\end{bmatrix},$$

$$\begin{bmatrix}0&0&3\\0&0&3\\1&1&1\end{bmatrix},\begin{bmatrix}0&0&3\\0&0&3\\1&2&0\end{bmatrix},\begin{bmatrix}0&0&3\\0&1&2\\1&1&1\end{bmatrix},\begin{bmatrix}0&0&3\\0&1&2\\1&2&0\end{bmatrix},$$

$$\begin{bmatrix}0&0&3\\0&1&2\\2&1&0\end{bmatrix},\begin{bmatrix}0&0&3\\0&2&1\\1&1&1\end{bmatrix},\begin{bmatrix}0&0&3\\0&2&1\\1&2&0\end{bmatrix},\begin{bmatrix}0&0&3\\0&2&1\\2&1&0\end{bmatrix},$$

$$\begin{bmatrix}0&1&2\\1&1&1\\2&1&0\end{bmatrix},\begin{bmatrix}0&1&2\\1&2&0\\1&0&2\end{bmatrix},\begin{bmatrix}0&1&2\\1&2&0\\2&0&1\end{bmatrix},\begin{bmatrix}0&1&2\\1&1&1\\1&2&0\end{bmatrix},$$

$$\begin{bmatrix}0&1&2\\2&1&0\\1&0&2\end{bmatrix},\begin{bmatrix}0&1&2\\2&1&0\\2&0&1\end{bmatrix},\begin{bmatrix}1&0&2\\0&1&2\\1&1&1\end{bmatrix},\begin{bmatrix}1&0&2\\0&2&1\\1&1&1\end{bmatrix},$$

$$\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix},\begin{bmatrix}1&1&1\\1&2&0\\1&0&2\end{bmatrix}.$$

By using the Theorems 2.1 and 2.2, it can be seen that only the following matrices can be parameter ones.

$$A_1=\begin{bmatrix}0&0&3\\0&2&1\\1&1&1\end{bmatrix},A_2=\begin{bmatrix}0&1&2\\2&1&0\\1&0&2\end{bmatrix}.$$

By using the Proposition 2.4, it can be seen just matrix A1 can be a parameter matrix.

Theorem 2.1 There are no perfect 3-colorings with the matrix A1 for the graph 5.

3. PERFECT 3-COLORINGS OF HEAWOOD GRAPH

By using the Theorems 2.1 and 2.2, it can be seen that only the following matrices can be parameter ones.

$$A_1=\begin{bmatrix}0&0&3\\0&2&1\\1&1&1\end{bmatrix},A_2=\begin{bmatrix}0&1&2\\2&1&0\\1&0&2\end{bmatrix}.$$

By using the Proposition 2.4, it can be seen just matrix A1 can be a parameter matrix.

Theorem 3.1 There are perfect 3-colorings with the matrix A1 for the Heawood graph.

Proof: As it has been shown in the above paragraph only the matrix A1 can be parameter matrix. By using Proposition 2.4, we can see $|W|=2,|B|=|R|=6$. The Heawood graph has perfect 3-colorings with the matrix A1. We label the vertices of Heawood graph clockwise by $a_1,a_2,\dots a_{14}$. Consider to the mappings T as follows:

$$T(a_1)=T(a_6)=1,$$

$$T(a_3)=T(a_4)=T(a_8)=T(a_9)=T(a_{12})=T(a_{13})=2,$$

$$T(a_2) = T(a_5) = T(a_7) = T(a_{10}) = T(a_{11}) = T(a_{14}) = 3.$$

It is clear that T is a perfect 3-colorig with the matrix A1.

Consider the following figure.

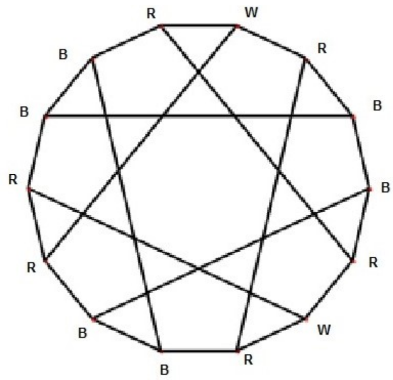

Figure 2: Perfect 3-colorings Heawood graph.

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