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Fractal transforms for fuzzy valued images

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Abstract

The aim of this paper is to construct a complete metric space of fuzzy valued image functions and to define a fractal transform operator T. Contraction of T guarantees the existence of its fixed point. A fuzzy point is considered for this purpose as a crisp point and approached through classical method for proving the completeness of the space.

Keywords: Fractal Image compression, Iterated function systems, Fuzzy sets, Fuzzy iterated function systems, Fuzzy valued images, Fuzzy Fractal Image Compression
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1. Introduction

The standard methods of image compression has several varieties such as JPEG,.... Fractal image compression is an exquisite method known for its self-similarity property.The basic principle of fractal image compression (FIC) was introduced by M. Barnsley in 1988 [5, 3]and refined by his doctoral student A.Jacquin [9, 10, 11]. It is also known as fractal image encoding because compressed image is represented by contractive transforms and mathematical functions which are necessary for reconstruction of the original image. Fractal compression is well known for its high theoretical compression rates and fast decompression rates. FIC relies on the similarities within the image.
The ultimate concept of the fractal image encoding is to approximate the given image function by the attractor of the corresponding contractive operator called as fractal transform operator which is composition of iterated function systems (IFS) and grey level maps.Contraction nature of the IFS maps ensures that, the distance between any two points on transformed image will be less than the distance of same points on the original image. Barsnley has derived a special form of the Contractive Mapping Transform (CMT) applied to IFS's called the Collage Theorem [2]. The usual approach

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of fractal image compression is based on the collage theorem, which provides distance between the image to be encoded and the fixed point of a transform, in terms of the distance between the transformed image and the image itself. This distance is known as collage error and it should be as small as possible.

Concisely, FIC seeks to approximate function of an image as a union of its modified copies. The net result is the target is approximated by the attractive fixed point of a contractive fractal transform operator. It performs the shrinking and grey-level modifying operations on image functions.

The approximation of the image is generated by the iteration procedure. To start the iteration, suitable initial value will be a blank screen.The fractal transform is studied and developed in theoretical as well as in algorithm, a lot. The success of an fractal transform operator relies on choosing the suitable complete metric spaces. For detailed study on fractal transform one can refer [14, 20, 8, 4]. In [12], a complete metric space of measure – valued images were constructed and method of fractal transform operator was formulated on it. In [15], they defined fractal transform operator for random measure valued image functions. In this sequence, a complete metric space of fuzzy valued images is constructed in this work.

Fuzzy set theory is a generalization of classical set theory. It was first studied by Lotfi.A. Zadeh[21]. In a fuzzy set, elements have degrees of membership. It helps to study the vagueness in detail. The system used in science and engineering is designed according to two-valued classical set theory and it is inadequate to give all detailed information and proficient suggestion. This situation opens the way to develop fuzzy set theory concepts in science and engineering fields.

Pu Pao -Ming and Liu Ying - Ming [17] defined a fuzzy point as a special case of an ordinary point. They considered fuzzy point as a crisp singleton and introduced the new notion of "$Q-neighborhood$"structure for developing fuzzy topological space concepts potentially. These structures coincide with the crisp topological structures also. From this definition of a fuzzy point, they write a fuzzy set as the union of fuzzy points belonging to the fuzzy set. This idea inspired the authors to apply crisp approach to fuzzy sets and to define fuzzy fractal transform operator on it. Earlier the first study in fractal transform using fuzzy approach is by Cabrelli, etc...[7].Using crisp approach to fuzzy sets, Al-Saidi [1] developed fuzzy metric space. Further Al-Saidi and others [2] introduced fuzzy fractal space and defined IFS on this new space. In our earlier work [19], we have defined fuzzy fractal transform operator on fuzzy fractal space introduced in [2]. Lowen and Peeters [13] presented various pseudo metric functions for calculating distances between fuzzy sets using tolerance level. In this work, we introduced a metric $d^\tau$of fuzzy points which depends on both spatial and membership values of fuzzy points. Using this distance function, we have formed a complete metric space of fuzzy sets and defined a fractal transform operator for fuzzy valued images.

In section 1, basic definitions needed to understand our work was given. Second section was devoted for the construction of complete metric space of fuzzy points and in the section 3, complete metric space of fuzzy sets is formed. Important factor of fractal transform operator, i.e. IFS is defined in section 4. In section 5, we defined fractal transform operator on the constructed complete metric space.

2. The main results

2.1. Basic Definitions

Some important and basic results are presented here to refresh the reader. These are discussed lucidly in [18]. Let (X, d) be a complete metric space, $S(x_0, r)$ and $\bar{S}(x_0, r)$ the open and closed ball of radius r and center x_0 respectively.

Proposition 2.1. *If $F \subseteq X$ and x_0 is a limit point of F, then every open ball $S(x_0, r)$, contains an infinite number of points of F.*

(i) $x \in \bar{F}$.
(ii) $S(x, \epsilon) \cap F \neq \phi$ *for every open ball centered at* x.
(iii) *There exists an infinite sequence* $\{x_n\}$ *of points (not necessarily distinct) of* F *such that* $x_n \longrightarrow x$.

Let A be a non-empty bounded subset of X. Then the diameter of A is defined as,

$$\text{diam}(A) = d(A) = \sup\{d(x, y) : x, y \in A\}.$$

It is interesting to note that $d(A) = d(\bar{A})$. Subsequently we give a vital result. It will be helpful on proving the completeness of the space of fuzzy points.

Theorem 2.2 (Cantor). *A metric space* (X, d) *is complete* $\Longleftrightarrow$ *for every nested sequence* $\{F_n\}_{n\geq 1}$ *of non empty closed subsets of* X *satisfying,*

(a) $F_1 \supseteq F_2 \supseteq \cdots \supseteq F_n \supseteq \cdots$,

(b) $d(F_n) \to 0$ *as* $n \to \infty$, *the intersection* $\bigcap_{n=1}^{\infty} F_n$ *contains one and only point.*

In the following part of the section, we shall discuss the basic results from fuzzy set theory.

Definition 2.3. *[17] A fuzzy point in* X *is a fuzzy set such that it has a non-zero membership value for only one point* $x \in X$ *and for all other point* $y \in X$ *its value is zero. Suppose at* x, *its value is* λ, *then we represent it as* p_x^{λ} *or* (x, λ)

$$i.e.\ \ p_x^{\lambda}(y) = \begin{cases} \lambda, & x = y \\ 0, & x \neq y \end{cases}$$

The point x *is called the support of* p_x^{λ}. *Let us denote by* $F_p(X)$, *the class of all fuzzy points in* X. *Let A be a fuzzy set in* X. *A fuzzy point* p_x^{α} *is said to be contained in A or belongs to A, if and only if* $\alpha \leq A(x)$.*From this, a fuzzy set A can be regarded as a collection of all fuzzy points belonging to it. A fuzzy topology for X is a family* $\delta \subseteq I^X$ *of fuzzy sets satisfying the following conditions.*

(i) $\mathcal{X}_{\emptyset}$ *and* $\mathcal{X}_X \in \delta$.
(ii) *for all* $A, B \in \delta \Rightarrow A \cap B \in \delta$.
(iii) *for all* $(A_j)_{j\in I} \in \delta \Rightarrow \cup_{j\in I} A_j \in \delta$, *where* I *is an index set.*

Let us call the pair (X, δ) as a fuzzy topological space. The elements of δ are known as open fuzzy subsets. If the complement of a fuzzy set belongs to δ,then it is called as closed fuzzy set. We call the fuzzy set U on X as a neighborhood of a fuzzy set A if there exists a open fuzzy set B such that $A \subseteq B \subseteq U$. The cover $\mathcal{B}$ of a fuzzy set A of X is a family B of fuzzy sets of X such as $A \subseteq \cup\{B/B \in \mathcal{B}\}$. If each member of the cover are open fuzzy sets of X then the cover is called open cover of A. A subfamily of B which also cover A is called as sub cover. If each open cover of $\mathcal{X}_X$ has a finite sub cover then the pair (X, δ) is called compact fuzzy space.

3. Complete metric space of fuzzy points

In this section, the complete metric space $F_p(X)$ of fuzzy points has been defined. On defining the distance between fuzzy points, we will consider both the spatial value and the membership value of fuzzy points. In the following definition, the notion of a distance between two fuzzy points is defined.

Definition 3.1. *We define the distance between any two fuzzy points p_x^α and p_y^β in $F_p(X)$ as*

$$d^\tau(p_x^\alpha, p_y^\beta) = d(x, y) + \theta|\alpha - \beta|, \ \theta \in R.$$

In the following theorem, we prove that d^τ is a complete metric on the space $F_p(X)$.

Theorem 3.2. *Let (X, d) be a complete metric space. Let $F_p(X)$ be the collection of all the fuzzy points defined on (X, d). Define the distance between any two members of $F_p(X)$ through d^τ. Then the space $(F_p(X), d^\tau)$ is a complete metric space.*

Proof . Let it be first proved that d^τ is a metric on the space $F_p(X)$. When $p_x^\alpha = p_y^\beta$, we have $p_x^\alpha(z) = p_y^\beta(z)$, for all $z \in X$. Which implies $x = y = z$ and $\alpha = \beta$. Conversely, when $d^\tau(p_x^\alpha, p_y^\beta) = 0$, we have $|x - y| = 0$ and $|\alpha - \beta| = 0$, which gives $x = y$ and $\alpha = \beta$. Then, $p_x^\alpha = p_y^\beta$. So, $d^\tau(p_x^\alpha, p_y^\beta) = 0$, when $p_x^\alpha = p_y^\beta$. And, also $d^\tau(p_x^\alpha, p_y^\beta) \geq 0$, for all p_x^α, p_y^β in $F_p(X)$. It can be easily verified that $d^\tau(p_x^\alpha, p_y^\beta) = d^\tau(p_x^\beta, p_y^\alpha)$. For triangular inequality,

$$\begin{aligned} d^\tau(p_x^\alpha, p_z^\gamma) &= d(x, z) + \theta|\alpha - \gamma| \\ &\leq d(x, y) + d(y, z) + \theta\{|\alpha - \beta| + |\beta - \gamma|\} \\ &= d(x, y) + \theta|\alpha - \beta| + d(y, z) + \theta|\beta - \gamma| \\ &= d^\tau(p_x^\alpha, p_y^\beta) + d^\tau(p_y^\beta, p_z^\gamma). \end{aligned}$$

Hence d^τ is a metric on the space $F_p(X)$. It remains to prove the completeness of the metric d^τ. Let $\{p_{x_n}^{\alpha_n}\}$ be Cauchy sequence of fuzzy points in $F_p(X)$. Then for every $\epsilon > 0$, there exists $n_0(\epsilon)$ such that $d^\tau(p_{x_n}^{\alpha_n}, p_{x_m}^{\alpha_m}) < \epsilon$, for all $m, n \leq n_0(\epsilon)$. Let us construct the set A_0 as, $A_0 = \{p_{x_n}^{\alpha_n} : n \geq n_0(\epsilon)\}$. i.e. A_0 contains all the fuzzy points from the sequence $\{p_{x_n}^{\alpha_n}\}$ for $n \geq n_0(\epsilon)$, i.e., it contains all the fuzzy points such that distance between the points is less than ϵ. So, clearly A_0 is an open set. It is possible to choose ϵ so that there exists $n_1(\epsilon)$ such that $d^\tau(p_{x_n}^{\alpha_n}, p_{x_m}^{\alpha_m}) < \frac{\epsilon}{2}$, for all $m, n \leq n_1(\epsilon)$ and assign those elements satisfying the inequality to a set, namely A_1. Then $A_0 \supseteq A_1$.

Continuing in this way , we will get the sets $A_0, A_1, A_2, \ldots, A_n$ such that $A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n$. Now fix n. Whenever $n \geq n_0$, each open ball with center $p_{x_n}^{\alpha_n}$ contains at least one point different from $p_{x_n}^{\alpha_n}$. Since $p_{x_n}^{\alpha_n}$ is Cauchy sequence, let $F_0 = \bar{A}_0, F_1 = \bar{A}_1, \ldots, F_n = \bar{A}_n$. Since $A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n$, we have $F_0 \supseteq F_1 \supseteq F_2 \supseteq \cdots \supseteq F_n$. Then $\lim_{n \longrightarrow \infty} d(F_n) = 0$. Which implies that there exists $p_x^\alpha \in F_p(X)$, such that $\cap_{n=1}^\infty F_n = \{p_x^\alpha\}$. The completeness of $F_p(X)$ follows from the Theorem 2.2. □

Now, let us proceed to define contraction mapping on the space of fuzzy points. Let $w : X \to X$ be a contraction mapping on (X, d) with contraction factor s, i.e., $d(w(x), w(y)) \leq sd(x, y)$, for all $x, y \in X$ and $0 < s < 1$.

Definition 3.3. *Let $(F_p(X), d^\tau)$ be a complete metric space of fuzzy points on the space (X, d). Define $w^* : F_p(X) \to F_p(X)$ as $w^*(p_x^\alpha) = p_{w(x)}^{s'\alpha+\gamma}$; where $0 < s' < 1, \forall p_x^\alpha \in F_p(X)$.*

Theorem 3.4. *$w^* : F_p(X) \to F_p(X)$ is a contraction mapping on $F_p(X)$ provided w is contraction.*

Proof .

$$\begin{aligned} d^\tau(w^*(p_x^\alpha), w^*(p_y^\beta)) &= d^\tau(p_{w(x)}^{s'\alpha+\gamma}, p_{w(y)}^{s'\beta+\gamma}) \\ &= d(w(x), w(y)) + \theta|s'\alpha - s'\beta| \\ &\leq sd(x,y) + \theta s'|\alpha - \beta| \\ &\leq \max\{s, s'\}\{d(x,y) + \theta|s'\alpha - s'\beta|\} \\ &= \max\{s, s'\} d^\tau(p_x^\alpha, p_y^\beta). \end{aligned}$$

Thus, w^* is a contraction mapping on $F_p(X)$. □

4. Complete metric space of fuzzy sets

In this section, complete metric space for fuzzy sets on (X, δ) will be constructed. Let $(F_p(X), d^\tau)$ be a complete metric space of fuzzy points on (X, d). Form a particular collection $\mathcal{F}(X)$ of fuzzy sets form δ which satisfies the following properties.

(i)
$$A \in \mathcal{F} \text{ is normal.} \tag{4.1}$$

(ii)
$$A \in \mathcal{F} \text{ is upper semi-continuous.} \tag{4.2}$$

(iii)
$$\text{Support of } A = \{x \in X : A(x) > 0\} \text{ is compact in X.} \tag{4.3}$$

With this properties, we can get, a fuzzy set $A \in \mathcal{F}$ if and only if $A_t \subset X$, where A_t is level set of A and $t \in [0,1]$. Because of the nature of construction of the members of $\mathcal{F}$, we notice that the following conditions are satisfied by the family of t-level sets of A.

(i)
$$A_0 = X. \tag{4.4}$$

(ii)
$$t \leq l \Rightarrow A_t \supset A_l. \tag{4.5}$$

(iii)
$$(iii) t_1 \leq t_2 \leq t_3 \leq \cdots \leq t_n \cdots \text{ and } \lim_{n \longrightarrow \infty} t_n = t > 0 \text{ implies } A_t = \cap_{n=1}^{\infty} A_{t_n}. \tag{4.6}$$

With these conditions, there exists a unique fuzzy set $A \in \mathcal{F}$ such that $\{x \in X / A(x) \geq t\} = A_t$ for each $t \in [0,1]$.The basic notions that required to attain our goal are described in this section. At first, let us define the distance from the point in $F_p(X)$ to the set in $\mathcal{F}(X)$. Let p_x^α be a point in $(F_p(X))$ and the fuzzy sets $A, B \in \mathcal{F}$. We define,

$$d^\tau(p_x^\alpha, B) = \min\{d^\tau(p_x^\alpha, p_y^\beta) : p_y^\beta \in B\},$$

as the distance from the point p_x^α to the set B. And define

$$d^\tau(A, B) = \max\{d^\tau(p_x^\alpha, B) : p_x^\alpha \in A\},$$

as the distance from the set $A \in \mathcal{F}$ to the set $B \in \mathcal{F}$.

Definition 4.1. *Let $(F_p(X), d^\tau)$ be a complete metric space, $p_x^\alpha, p_y^\beta \in F_p(X)$ and $A, B \in \mathcal{F}$. Define the distance between the fuzzy sets A and B as,*

$$d_H(A,B) = \max\{d^\tau(A,B), d^\tau(B,A)\},$$

i.e., $d_H(A,B) = \max\{d^\tau(A,B), d^\tau(B,A)\}$. Since, the members of $\mathcal{F}$ are satisfying (4.1)-(4.3), we can get, $d_H(A,B) = d^\tau(p_x^\alpha, p_y^\beta)$, for some points $p_x^\alpha \in A$, and $p_y^\beta \in B$.

Theorem 4.2. *Let $(F_p(X), d^\tau)$ be a complete metric space of fuzzy points defined on the complete metric space (X,d). Let $\mathcal{F}(X)$ be the collection of fuzzy subsets of (X,d) satisfying the conditions (4.1)-(4.3). Define the distance between the sets A and B in $\mathcal{F}(X)$ as,*

$$d_H(A,B) = \max\{d^\tau(A,B), d^\tau(B,A)\}.$$

Then, the space $(\mathcal{F}, d_H)$ is a complete metric space.

Proof . Let $A, B, C \in \mathcal{F}$. Consider, p_x^α be a point in A, p_y^β be a point in B and p_z^γ be a point in C. In the first part, we prove that d_H is a metric on $\mathcal{F}$. Since $d_H(A,B) = d^\tau(p_x^\alpha, p_y^\beta)$ for some points $p_x^\alpha \in A$, *and* $p_y^\beta \in B$ and d^τ is a complete metric, we have $0 \le d_H(A,B) < \infty$. And,

$$\begin{aligned}
d_H(A,B) = 0 \;&\Rightarrow\; \max\{d^\tau(A,B), d^\tau(B,A)\} = 0\\
&\Rightarrow\; d^\tau(A,B) = 0 \text{ and } d^\tau(B,A) = 0\\
&\Rightarrow\; \max\nolimits_{p_x^\alpha \in A} d^\tau(p_x^\alpha, B) = 0\\
&\Rightarrow\; d^\tau(p_x^\alpha, B) = 0\\
&\Rightarrow\; \min\nolimits_{p_y^\beta \in B} d^\tau(p_x^\alpha, p_y^\beta) = 0\\
&\Leftrightarrow\; d^\tau(p_x^\alpha, p_y^\beta) = 0, \text{ for } p_x^\alpha \in A, p_y^\beta \in B\\
&\Leftrightarrow\; p_x^\alpha = p_y^\beta, \text{ for all } p_x^\alpha \in A, p_y^\beta \in B\\
&\Leftrightarrow\; A = B.
\end{aligned}$$

Also,

$$\begin{aligned}
d_H(A,B) &= \max\{d^\tau(A,B), d^\tau(B,A)\}\\
&= \max\{d^\tau(B,A), d^\tau(A,B)\}\\
&= d_H(B,A).
\end{aligned}$$

To prove the triangular inequality for d_H,we first show that $d^\tau(A,B) \le d^\tau(A,C) + d^\tau(C,B)$. For any $p_x^\alpha \in A$,

$$\begin{aligned}
d^\tau(p_x^\alpha, B) &= \min\left\{d^\tau(p_x^\alpha, p_y^\beta); p_y^\beta \in B\right\}\\
&\le \min\left\{d^\tau(p_x^\alpha, p_z^\gamma) + d^\tau(p_z^\gamma, p_y^\beta) : p_y^\beta \in B\right\} \text{ for } p_z^\gamma \in C\\
&= d^\tau(p_x^\alpha, p_z^\gamma) + \min\left\{d^\tau(p_z^\gamma, p_y^\beta) : p_y^\beta \in B\right\} \text{ for all } p_z^\gamma \in C,
\end{aligned}$$

so,

$$\begin{aligned}
d^\tau(p_x^\alpha, B) &\le \min\{d^\tau(p_x^\alpha, p_z^\gamma) : p_z^\gamma \in C\} + \max\left\{\min(d^\tau(p_z^\gamma, p_y^\beta) : p_y^\beta \in B) : p_z^\gamma \in C\right\}\\
&= d^\tau(p_x^\alpha, C) + d^\tau(C,B).
\end{aligned}$$

Thus,

$$d^\tau(A,B) \leq d^\tau(A,C) + d^\tau(C,B).$$

Similarly, we can prove that $d^\tau(B,A) \leq d^\tau(B,C) + d^\tau(C,A)$. Hence,

$$\begin{aligned} d_H(A,B) &= \max\{d^\tau(A,B), d^\tau(B,A)\} \\ &\leq \max\{d^\tau(B,C), d^\tau(C,B)\} + \max\{d^\tau(A,C), d^\tau(C,A)\} \\ &= d_H(B,C) + d_H(A,C). \end{aligned}$$

So, from the above discussions, it is clear that $(\mathcal{F}, d_H)$ is a metric space. Let $\{A_n\}$ be a Cauchy sequence in $\mathcal{F}$. So, for all $\epsilon > 0$, there exists n_0 such that for all $n, m \geq n_0(\epsilon)$, we have $d_H(A_n, A_m) < \epsilon$. Let $\epsilon = 2^{-k}$ and choose an increasing sequence n_k such that $d_H(A_n, A_{n_k}) < 2^{-k}\ for\ all n \geq n_K$. So, $\{A_{n_k}\}$ is a sub sequence of $\{A_n\}$ whose members are elements of $\mathcal{F}(X)$. By the nature of construction of $\mathcal{F}(X)$, for each $U \in \mathcal{F}$, there exists correspondingly the t-level set $U_t \in X$ for each $t \in [0,1]$. Let us consider the family $\{U_{t_k}\}$ of $t - level$ sets of the sequence $\{A_{n_k}\}$. Each of these is a compact subset of X and satisfies the conditions (4.4)-(4.6). Now, let $U_t = \cap_{k=1}^{\infty} U_{t_k}$. Then, there exists a fuzzy set U in $\mathcal{F}(X)$ such that whose t- level set is U_t. Since $U_t = \cap_{k=1}^{\infty} U_{t_k}$, we get $U \subseteq A_{n_k}, \forall n_k$. Thus, $d_H(A_{n_k}, U) < \epsilon$, which implies that the sequence $\{A_{n_k}\}$ converges to $U \in \mathcal{F}$. Observe that $\{A_{n_k}\}$ is a sub sequence of $\{A_n\}$ and since $\{A_n\}$ is a Cauchy sequence, it follows that $\{A_n\}$ also converges to U. Thus, the space $\mathcal{F}(X)$ is a complete metric space. □

5. Iterated function system on $\mathcal{F}(X)$

In this section, let us define iterated function system (IFS) on the space $\mathcal{F}(X)$ which acts as an important tool on defining fractal transform operator. IFS play on significant roll on constructing self - similar or fractal objects. In the following segment, let us define the contraction mapping on $\mathcal{F}(X)$ out of a contraction mapping on $F_p(X)$ and X.

Definition 5.1. *Let w be a contraction mapping on X with contraction factor s and w^* be a contraction mapping on $F_p(X)$ with contraction factor s'. Let $A \in \mathcal{F}$. Define the mapping, W from $\mathcal{F}(X)$ onto itself as, $W(A) = \{w^*(p_x^\alpha) : p_x^\alpha \in A\} = \left\{p_{w(x)}^{s\alpha+\gamma} : p_x^\alpha \in A\right\}$.*

Theorem 5.2. *W is a contraction mapping on $\mathcal{F}$.*

Proof . Let $A, B \in \mathcal{F}$. Since the space $\mathcal{F}(X)$ is a metric space and $w^* : F_p(X) \to F_p(X)$ is a contraction mapping implies that W is continuous.Therefore W maps $\mathcal{F}$ into itself. Now,

$$\begin{aligned} d_H(W(A), W(B)) &= \max\{d^\tau(W(A), W(B)), d^\tau(W(B), W(A))\} \\ &= \max\left\{\max_{p_{w(x)}^{s'\alpha+\gamma} \in W(A)} d^\tau(p_{w(x)}^{s'\alpha+\gamma}, W(B)), \max_{p_{w(y)}^{s'\beta+\gamma} \in W(B)} d^\tau(p_{w(y)}^{s'\beta+\gamma}, W(A))\right\} \end{aligned}$$

$$\begin{aligned} &= \max\left\{\max_{p_{w(x)}^{s'\alpha+\gamma} \in W(A)} \min_{p_{w(y)}^{s'\beta+\gamma} \in W(B)} d^\tau(p_{w(x)}^{s'\alpha+\gamma}, p_{w(y)}^{s'\beta+\gamma}), \max_{p_{w(y)}^{s'\beta+\gamma} \in W(B)} \min_{p_{w(x)}^{s'\alpha+\gamma} \in W(A)} d^\tau(p_{w(y)}^{s'\beta+\gamma}, p_{w(x)}^{s'\alpha+\gamma})\right\} \\ &\leq \max\left\{\max_{p_x^\alpha \in A} \min_{p_y^\beta \in B} \max\{s, s'\} d^\tau(p_x^\alpha, p_y^\beta), \max_{p_y^\beta \in B} \min_{p_x^\alpha \in A} \max\{s, s'\} d^\tau(p_y^\beta, p_x^\alpha)\right\} \\ &= \max\{s, s'\} \max\{d^\tau(A,B), d^\tau(B,A)\} \\ &= \max\{s, s'\} d_H(A,B). \end{aligned}$$

□

Theorem 5.3. *Let $(F_p(X), d^\tau)$ be a complete metric space. Let $\{W_n, n = 1, 2, 3, ..., N\}$ be contraction mappings on $(\mathcal{F}(X), d_H)$. Let the contraction factor for each W_n be denoted by s_n. For $B, C \in \mathcal{F}$, define $W^* : \mathcal{F}(X) \to \mathcal{F}(X)$ by $W^*(B) = \{p_x^\alpha : p_x^\alpha \in W_n(B),$ for any $n = 1, 2, 3, ..., N\}$, i.e., $W^*(B)$ contains the fuzzy points which are belongs to $W_n(B)$ for $n = 1, 2, ..., N$, for each $B \in \mathcal{F}$. Then W^* is a contraction mapping on $\mathcal{F}(X)$.*

Proof . The case $N = 1$ is trivial. We consider the case when $N = 2$. Let $B, C \in \mathcal{F}$. Then,$W^*(B) = \{p_x^\alpha : p_x^\alpha \in W_1(B)$ or $W_2(B)\}$, i.e., $W^*(B) = \{p_x^\alpha : p_x^\alpha \in W_1(B) \cup W_2(B)\}$. Let $p_x^\alpha \in B$ and $p_y^\beta \in C$. Now,

$$\begin{aligned} &d_H(W^*(B), W^*(C)) = d_H\big(W_1(B) \cup W_2(B), W_1(C) \cup W_2(C)\big) \\ &= \max\{d^\tau\big(W_1(B) \cup W_2(B), W_1(C) \cup W_2(C)\big), d^\tau\big(W_1(C) \cup W_2(C), W_1(B) \cup W_2(B)\big)\}. \end{aligned}$$

Now,

$$\begin{aligned} &d^\tau\Bigg(W_1(B) \cup W_2(B), W_1(C) \cup W_2(C)\Bigg) = \max_{p_x^\alpha \in W_1(B)\cup W_2(B)} d^\tau\Bigg(p_x^\alpha, W_1(C) \cup W_2(C)\Bigg) \\ &= \max_{p_x^\alpha \in W_1(B)\cup W_2(B)} \Bigg\{\min_{p_y^\beta \in W_1(C)\cup W_2(C)} d^\tau\Bigg(p_x^\alpha, p_y^\beta\Bigg)\Bigg\} \\ &= \max_{p_x^\alpha \in W_1(B)\cup W_2(B)} \Bigg\{\min\Bigg[\min\Bigg\{d^\tau(p_x^\alpha, p_y^\beta) : p_y^\beta \in W_1(C)\Bigg\}, \min\Bigg\{d^\tau(p_x^\alpha, p_y^\beta) : p_y^\beta \in W_2(C)\Bigg\}\Bigg]\Bigg\} \\ &= \max_{p_x^\alpha \in W_1(B)\cup W_2(B)} \min\Bigg\{d^\tau(p_x^\alpha, W_1(C)), d^\tau(p_x^\alpha, W_2(C))\Bigg\} \\ &= \max\Bigg\{\min\Bigg\{d^\tau(p_x^\alpha, W_1(C)) : p_x^\alpha \in W_1(B)\Bigg\}, \min\Bigg\{d^\tau(p_x^\alpha, W_2(C)) : p_x^\alpha \in W_2(B)\Bigg\}\Bigg\} \\ &= \max\{d^\tau(W_1(B), W_1(C)), d^\tau(W_2(B), W_2(C))\}. \end{aligned}$$

Similarly,

$$d^\tau(W_1(C) \cup W_2(C), W_1(B) \cup W_2(B)) = \max\Bigg\{d^\tau(W_1(C), W_1(B)), d^\tau(W_2(C), W_2(B))\Bigg\}.$$

$$\begin{aligned} &d_H(W^*(B), W^*(C)) = d_H(W_1(B) \cup W_2(B), W_1(C) \cup W_2(C)) \\ &= \max\{\max\{d^\tau(W_1(B), W_1(C)), d^\tau(W_2(B), W_2(C))\}, \max\{d^\tau(W_1(C), W_1(B)), d^\tau(W_2(C), W_2(B))\}\} \\ &\le \max\{\max\{d^\tau(W_1(B), W_1(C)), d^\tau(W_1(C), W_1(B))\}, \max\{d^\tau(W_2(B), W_2(C)), d^\tau(W_2(C), W_2(B))\}\} \\ &= \max\{d_H(W_1(B), W_1(C)), d_H(W_2(B), W_2(C))\} \\ &\le \max\{s_1\ d_H(B, C), s_2\ d_H(B, C)\} \\ &\le \max\{s_1, s_2\}\ d_H(B, C) \\ &= s' d_H(B, C), \text{where } s' = \max\{s_1, s_2\}. \end{aligned}$$

Hence the result is true for $N = 2$. From this, the general result i.e. $N = N$ will also follows. □

6. Fractal transform operator on the fuzzy space

In this section, we define fractal transform operator for fuzzy valued image functions. We first establish fuzzy valued image function space and prove that it is a complete metric space.

Definition 6.1. *Let f be a set valued function from the space $F_p(X)$ into the space $\mathcal{F}(X)$.Let $y \in X$, $p_x^\alpha \in F_p(X)$ and $A \in \mathcal{F}(X)$. Then, the image of a point p_x^α in $F_p(X)$ is a fuzzy set A in $\mathcal{F}(X)$ and it is defined as,*

$$f(p_x^\alpha)(y) = A(y) = Sup_{p_x^\alpha \in F_p(X)}(\alpha \wedge \beta))$$

where β is the grade of membership value of $f(p_x^\alpha)$ in A. Now we define our space of fuzzy valued image functions.

Definition 6.2. *Let us define a space $\mathfrak{F}$ be the collection of all fuzzy functions from $F_p(X)$ to $\mathcal{F}(X)$ and it is denoted by*

$$\mathfrak{F}(F_p(X)) = \{f | f : F_p(X) \to \mathcal{F}(X)\}.$$

We should remember that $f(p_x^\alpha)$ also satisfies the properties (4.1)-(4.3). Now, we define a distance function on $\mathfrak{F}$ as, for $f, g \in \mathfrak{F}$,

$$d_\infty^\tau(f, g) = max\{d_H(f(p_x^\alpha), g(p_x^\alpha)) : p_x^\alpha \in F_p(X)\}.$$

In the following result, we will prove that $(\mathfrak{F}, d_\infty^\tau)$ is a complete metric space.

Theorem 6.3. *Let $F_p(X)$ be the space of fuzzy points and $\mathcal{F}(X)$ be the collection of fuzzy subsets of (X, d). Then the space, $(\mathfrak{F}, d_\infty^\tau)$ collection of all fuzzy functions from $F_p(X)$ to $\mathcal{F}(X)$ with respect to the metric,*

$$d_\infty^\tau(f, g) = \max\{d_H(f(p_x^\alpha), g(p_x^\alpha)) : p_x^\alpha \in F_p(X)\}$$

is a complete metric space, whenever $(\mathcal{F}(X), d_H)$is a complete metric space.

Proof . Let $f, g, h \in \mathfrak{F}$. It is clear that $d_\infty^\tau(f, g) \geq 0, f, g \in \mathfrak{F}$. Now, $f = g \Leftrightarrow f(p_x^\alpha)(y) = g(p_x^\alpha)(y), \forall y \in X \Leftrightarrow d_H(f(p_x^\alpha), g(p_x^\alpha)) = 0 \Leftrightarrow d_\infty^\tau(f, g) = 0$. It is obvious that $d_\infty^\tau(f, g) = d_\infty^\tau(g, f)$ for all $f, g \in \mathfrak{F}$.

$$\begin{aligned}
d_\infty^\tau(f, g) &= \max\{d_H(f(p_x^\alpha), g(p_x^\alpha)) : p_x^\alpha \in F_p(X)\} \\
&\leq \max\{d_H(f(p_x^\alpha), h(p_x^\alpha)) + d_H(h(p_x^\alpha), g(p_x^\alpha)) : p_x^\alpha \in F_p(X)\} \\
&\leq \max\{d_H(f(p_x^\alpha), h(p_x^\alpha)) : p_x^\alpha \in F_p(X)\} + \max\{d_H(h(p_x^\alpha), g(p_x^\alpha)) : p_x^\alpha \in F_p(X)\} \\
&= d_\infty^\tau(f, h) + d_\infty^\tau(h, g).
\end{aligned}$$

From the above discussion, it is proved that $(\mathfrak{F}, d_\infty^\tau)$ is a metric space whenever d_H is a metric on $\mathcal{F}(X)$. We now proceed to prove that d_∞^τ is a complete metric. Let f_n be a Cauchy sequence in $(\mathfrak{F}, d_\infty^\tau)$. Then for every $\epsilon > 0$, there exists a $n_0(\epsilon)$ such that $d_\infty^\tau(f_n, f_m) < \epsilon$, $\forall n \geq n_0(\epsilon)$. Which implies that, $\max\{d_H(f_n(p_x^\alpha), f_m(p_x^\alpha)) : p_x^\alpha \in F_p(X)\} < \epsilon$. Then, of course $d_H(f_n(p_x^\alpha), f_m(p_x^\alpha)) < \epsilon$. Let $\epsilon = 3^{-k}$. Choose a increasing sequence n_k such that $d_H(f_n(p_x^\alpha), f_{n_k}(p_x^\alpha)) < 3^{-k}$, for $n \geq n_k$. Then $\{f_{n_k}\}$ is a Cauchy sequence in $\mathcal{F}(X)$. Since $\mathcal{F}(X)$ is a complete metric space, $\{f_{n_k}\}$ converges to a limit in $\mathcal{F}(\mathcal{X})$. Let the limit be f. Besides, f satisfies the relation, $d_H(f_n(p_x^\alpha), f(p_x^\alpha)) < 3^{-k}$, for $n \geq n_k$. From this, as $n \to \infty$, we get f_n also converges to f. □

Now let us define a fractal transform operator T on the space $(\mathfrak{F}, d_\infty^\tau)$. We first present the ingredients for T.

(i) A set of N one-to-one, invertible contraction maps $w_i^* : F_p(X) \to F_p(X)$, with the condition that $\cup_{i=1}^N w_i^*(F_p(X)) = F_p(X)$. Take the contraction factor of the mappings w_i^* as s_i respectively.

(ii) A set of gray scale maps $\phi_i : \mathcal{F}(\mathcal{X}) \to \mathcal{F}(\mathcal{X})$ assumed to be contraction maps with contraction factors $c_i, i = 1, 2, ..., N$.

Let $f \in \mathfrak{F}$ be the given fuzzy valued image function.We define the N fractal components u_i of the fractal transform operator $T : \mathfrak{F} \to \mathfrak{F}$ defined by the above are as follows.

$$u_i(p_x^\alpha) = \begin{cases} \phi_i\Big(f(w_i^{-1}(p_x^\alpha))\Big) & ; p_x^\alpha \in w_i(F_p(X)) \\ 0 & ; p_x^\alpha \notin w_i(F_p(X)) \end{cases}$$

and the fractal transform operator T is defined as,

$$(Tf)(p_x^\alpha) = \sum_{i=1}^N u_i(p_x^\alpha) = \sum_{i=1}^N \phi_i\Big(f(w_i^{-1}(p_x^\alpha))\Big).$$

We now give a condition for the contraction of the operator T.

Theorem 6.4. *For $f, g \in \mathfrak{F}$,*

$$d_\infty^\tau(T(f), T(g)) \le \Big(\sum_{i=1}^N c_i s_i\Big) d_\infty^\tau(f, g)$$

Proof .

$$\begin{aligned} d_\infty^\tau(T(f), T(g)) &= d_\infty^\tau\Big(\sum_{i=1}^N \phi_i\Big(f(w_i^{*-1}(p_x^\alpha))\Big), \sum_{i=1}^N \phi_i\Big(g(w_i^{*-1}(p_x^\alpha))\Big)\Big) \\ &= \max_{p_x^\alpha \in F_p(X)} d_H\Big(\sum_{i=1}^N \phi_i\Big(f(w_i^{*-1}(p_x^\alpha))\Big), \sum_{i=1}^N \phi_i\Big(g(w_i^{*-1}(p_x^\alpha))\Big)\Big) \\ &\le \max_{p_x^\alpha \in F_p(X)} \sum_{i=1}^N h\Big(\phi_i\Big(f(w_i^{*-1}(p_x^\alpha))\Big), \phi_i\Big(g(w_i^{*-1}(p_x^\alpha))\Big)\Big) \\ &\le \max_{p_x^\alpha \in F_p(X)} \sum_{i=1}^N c_i d_H\Big(f(w_i^{*-1}(p_x^\alpha)), g(w_i^{*-1}(p_x^\alpha))\Big) \\ &\le \max_{p_x^\alpha \in F_p(X)} \sum_{i=1}^N c_i r_i d_H\Big(f(p_x^\alpha), g(p_x^\alpha)\Big) \\ &\le \sum_{i=1}^N c_i s_i d_\infty^\tau(f, g). \end{aligned}$$

□

Corollary 6.5. *When $\sum_{i=1}^N c_i r_i < 1, T$ is a contraction mapping.*

Theorem 6.6. *Given an $f \in \mathfrak{F}$, suppose that there exists a contraction operator T such that $d_\infty^\tau(f, T(f)) < \epsilon$. If $\bar{f}$ is the fixed point of T and $c = \max_{p_x^\alpha \in F_p(X)} \sum_{i=1}^N c_i s_i$, then $d_\infty^\tau(f, \bar{f}) \le \frac{\epsilon}{1-c}$.*

Proof . Using the triangular property of the metric d_∞^τ, we may easily prove the result. □

7. Examples

Let the base be $X = [0, 1]$ and the sample image be $u(x) = 4x(1 - x)$, $x \in X$. See Fig.1. Then fuzzification 'f'of the sample image by triangular fuzzifying function is shown in Figure.2.

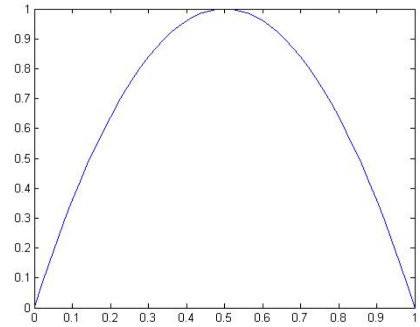

Figure 1: Sample Image

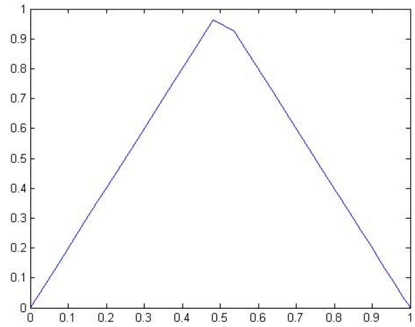

Figure 2: Fuzzy Image

We define the members of the IFS on X as, $w_1(x) = 0.6x$, $w_2(x) = 0.6x + 0.4$ and grey level maps as $\phi_1(x) = 0.5x + 0.5$, $\phi_2(x) = 0.75x$ $x \in X$. On applying fractal transform operator, the original image is shrunken into number of copies. It is clearly indicated in Fig.3.While applying fuzzy fractal transform operator to the fuzzified image, we have got compressed version of input but it is relatively smaller compared to Fig.3. It will be represented in Fig.4.

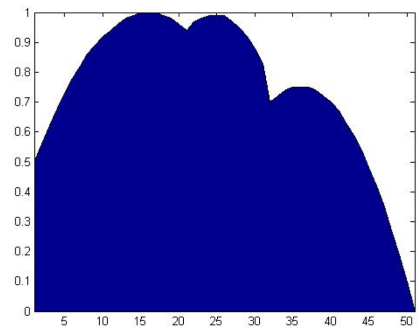

Figure 3:

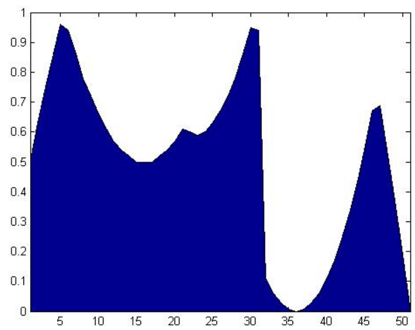

Figure 4:

In the second example, we take sample image as $u(x) = x^2$ in $[0, 1]$(see Fig-5). In this case, we used Gaussian membership function with parameters 0.1and1 to convert the image as fuzzy image (see Fig-6).

With the same IFS and grey level members used in the previous example, the results obtained are shown in Figure-7,8. In this case also, the fuzzy fractal transform operator gives better result compared to the classical fractal transform operator.

8. Conclusion

In this work, we consider fuzzy points as crisp points.Using classical approach we have constructed a complete metric space of compact fuzzy subsets which is similar to Hausdorff space. In that space,

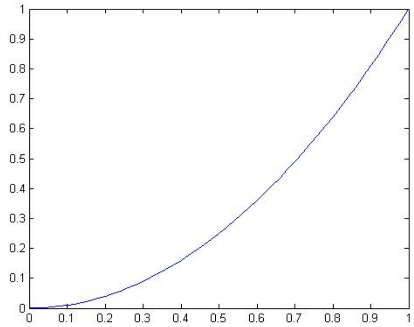

Figure 5: Sample Image $u = x^2$

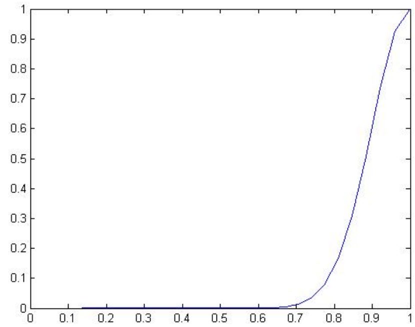

Figure 6: Fuzzy Image of $u = x^2$

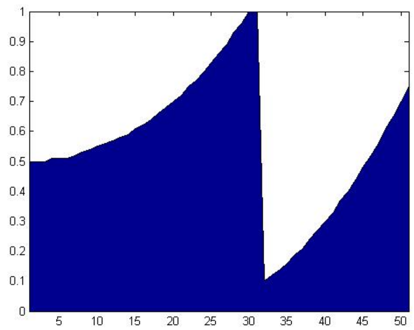

Figure 7:

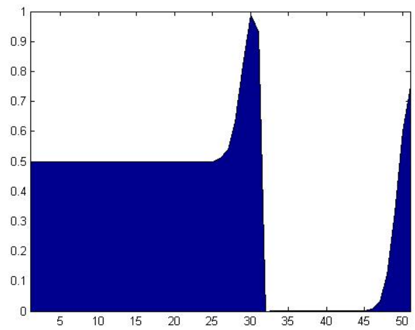

Figure 8:

we defined a fractal transform operator and proved it as a contraction operator. By contraction mapping theorem, there exists a fixed point. The contraction mapping W defined on fuzzy sets compresses both the spatial and membership value of a fuzzy point.We used affine mappings for compressing membership value of a fuzzy point. It will give more flexibility than one defined by Nadia [2]. Further research in this direction can carried out on finding suitable membership function for fuzzification of image. In such cases, it will give more compression ratio. It could be interesting to study the applications of this work, in other areas of image processing.

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