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New common fixed point theorems for contractive self mappings and an application to nonlinear differential equations

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Abstract

In this paper, we prove a new common fixed point in a general topological space with a τ-distance. Then we deduce two common fixed point theorems for two new classes of contractive selfmappings in complete bounded metric spaces. Moreover, an application to a system of differential equations is given.

Keywords: Common fixed point, Shrinking maps, E_θ-Weakly contractive maps, Metric space, Hausdorff topological space
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1. Introduction

The Study of Shrinking or contractive selfmappings on a metric space (X, d) (That is, $d(Tx, Ty) < d(x, y)$, for all $x \neq y \in X$) was initiated by Nemytzki [12]. Moreover, as mentioned in [4], to obtain a fixed point of such mappings, it is necessary either to add the assumption that the space is compact, or else assume that there exists a point $x \in X$ for which $\{T^n x\}$ contains a convergent subsequence. In [14], the author showed that every weakly contractive mapping defined on a complete metric space (X, d) has a unique fixed point, in other words, every selfmapping $T : X \to X$ satisfying $d(Tx, Ty) \leq d(x, y) - \phi(d(x, y))$, for all $x, y \in X$, where $\phi : [0, +\infty) \to [0, +\infty)$ is a continuous nondecreasing function such that $\phi(0) = 0$. Since then, many results have appeared in the literature

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concerning this class of mappings [5, 10, 3, 13].
The authors in [1] introduced the concept of $\tau-$distances in general topological spaces (X, τ) which extend many known spaces in the literature. Further, they proved a version of the well-known Banach's fixed point for this general setting.

In this paper, using the concept of $\tau-$distance, we first establish a new common fixed point theorem for weakly compatible selfmappings which yields the fixed point proved by Jungck and Rhoades [?] in a new setting. As application of this result, we get a new common fixed point theorem for shrinking selfmappings without using neither the compactness of the space nor the fact that there exists a point $x \in X$ for which $\{T^n x\}$ contains a convergent subsequence. In addition, we prove a theorem for a new class of weakly contractive selfmappings, we call it $E_\theta-$weakly contractive selfmappings, where the auxiliary function ϕ satisfies $\phi(1) = 0$ and $\inf\limits_{t>1} \phi(t) > 0$.
Furthermore, based on one of our results, we study the existence and uniqueness of solutions for a system of differential equations.

2. Preliminaries

In this section, we recall some definitions and results needed in the sequel.
Let (X, τ) be a topological space and $p : X \times X \to [0, \infty)$ be a function. For any $\varepsilon > 0$ and any $x \in X$, let $B_p(x, \varepsilon) = \{y \in X : p(x, y) < \varepsilon\}$.

Definition 2.1. *(Definition 2.1 [1]) The function p is said to be $\tau -$ distance if for each $x \in X$ and any neighborhood V of x, there exists $\varepsilon > 0$ such that $B_p(x, \varepsilon) \subset V$.*

Definition 2.2. *A sequence in a Hausdorff topological space X is a p-Cauchy if it satisfies the usual metric condition with respect to p.*

Definition 2.3. *(Definition 3.1 [1])*
Let (X, τ) be a topological space with a τ-distance p.

1. *X is S-complete if for every p-Cauchy sequence (x_n), there exists x in X with $\lim p(x, x_n) = 0$.*
2. *X is p-Cauchy complete if for every p-Cauchy sequence (x_n), there exists x in X with $\lim x_n = x$ with respect to τ.*
3. *X is said to be p-bounded if $\sup\{p(x, y)/x, y \in X\} < \infty$.*

Lemma 2.4. *(Lemma 3.1[1])*
Let (X, τ) be a Hausdorff topological space with a $\tau-$distance p, then

1. *$p(x, y) = 0$ implies $x = y$.*
2. *Let (x_n) be a sequence in X such that $\lim_{n\to\infty} p(x, x_n) = 0$ and $\lim_{n\to\infty} p(y, x_n) = 0$, then $x = y$.*

Definition 2.5. *([?]) Two selfmappings f and g of a set X are said to be weakly compatible if they commute at there coincidence points; i.e., if $fu = gu$ for some $u \in X$, then $f \circ gu = g \circ fu$.*

Definition 2.6. *([5])*
Θ is the class of all functions $\theta : [0, +\infty) \longrightarrow [0, +\infty)$ satisfying:
i) θ is a monotone increasing function,
ii) $\theta(t) = 0$ if and only if $t = 0$.

Definition 2.7. *([5]) Ψ is the class of all functions $\psi : [0, +\infty) \longrightarrow [0, +\infty)$ satisfying:*
i) ψ is nondecreasing,
ii) $\lim \psi^n(t) = 0$, for all $t \in [0, \infty)$.

Definition 2.8. *Φ is the class of all functions $\phi : [1, +\infty) \longrightarrow [0, +\infty)$ satisfying:*
i) $\phi(t) = 0$ if and only if $t = 1$,
ii) $\inf_{t>1} \phi(t) > 0$.

3. Main results

In this section, we begin by proving a new Theorem and a Lemma needed in the next.

Theorem 3.1. *Let (X, τ) be a $p-$bounded Hausdorff topological space with a $\tau-$disatnce p. Let f and g be two weakly compatible selfmappings of X, satisfying the following conditions:*
i) $g(X) \subset f(X)$,
ii) $p(gx, gy) \leq \psi(p(fx, fy))$,
for all $x, y \in X$ and $\psi \in \Psi$.
If the range of f or g is $S-$complete subspace of X, then f and g have a unique common fixed point.

Proof . Let $x_0 \in X$. By the condition (i), it follows that there exists $x_1 \in X$ such that $g(x_0) = f(x_1)$, continuing this process, we can choose $x_n \in X$ such that $f(x_n) = g(x_{n-1})$ for any $n \in \mathbb{N}$. Using (ii), we get for all $n, m \in \mathbb{N}$,

$$\begin{aligned} p(fx_n, fx_{n+m}) &= p(gx_{n-1}, gx_{n+m-1}) \\ &\leq \psi(p(fx_{n-1}, fx_{n+m-1})) \\ &\vdots \\ &\leq \psi^n(p(fx_0, fx_m)) \\ &\leq \psi^n(M), \end{aligned} \tag{3.1}$$

where $M = \sup\{p(x, y)/x, y \in X\}$. Since $\lim_{n\to\infty} \psi^n(M) = 0$, we see that $\{fx_n\}$ is a $p-$Cauchy sequence.
Suppose that $f(X)$ is $S-$complete, which implies that there exists $u \in X$ such that $\lim_{n\to\infty} p(fu, fx_n) = 0$, and therefore $\lim_{n\to\infty} p(gu, gx_n) = \lim_{n\to\infty} p(gu, fx_n) = 0$. Using Lemma 2.4, we get $gu = fu$. Now, the assumption that f and g are weakly compatible implies

$$f \circ gu = g \circ fu = g \circ gu = f \circ fu. \tag{3.2}$$

Suppose that $p(g \circ gu, gu) \neq 0$. From (ii), it follows

$$\begin{aligned} p(g \circ gu, gu) &\leq \psi(p(f \circ gu, fu)) \\ &< p(g \circ gu, gu), \end{aligned} \tag{3.3}$$

this leads to a contradiction. Thus $g \circ gu = gu$. Also $f \circ gu = g \circ fu = g \circ gu = gu$, which implies that gu is a common fixed point of f and g.
Now, if the range of g is S-complete subspace of X, then $\lim_{n\to\infty} p(gv, gx_n) = 0$ for some $v \in X$. From (i), there exists $w \in X$ such that $gv = fw$ and the proof that gw is a common fixed point of f and g is the same as that given when $f(X)$ is S-complete.

For the uniqueness, suppose that there exist $u, v \in X$ such that $f(u) = u = g(u)$ and $f(v) = v = g(v)$ with $u \neq v$. Then by condition (ii) and Lemma 2.4 it follows

$$\begin{aligned} p(u,v) &= p(gu,gv) \\ &\leq \psi(p(fu,fv)) \\ &= \psi(p(u,v)) \\ &< p(u,v), \end{aligned}$$

which is a contradiction, then $u = v$. □

For $f = Id_X$ in Theorem 3.1, we get.

Corollary 3.2. *(Theorem 4.1 in [1]).*
Let (X, τ) be a Hausdorff topological space with a $\tau-$distance p. Suppose that X is $p-$bounded and $S-$complete. Let f be a selfmapping of X such that

$$p(fx, fy) \leq \psi(p(x,y)),$$

for all $x, y \in X$. Then f has a unique fixed point.

Now, we give an example to support our result.

Example 3.3. *Let $X = [1, 18]$ and $d(x,y) = |x-y|$ the usual metric. Consider the function $p : X \times X \to [0,\infty)$ defined by*

$$p(x,y) = |x-y|e^{|x-y|}, \;\; \forall x, y \in X.$$

It is easy to see that the function p is a $\tau-$distance on X where τ is the usual topology. Define $f, g : X \to X$ by

$$gx = \begin{cases} 2x^2 - 1 & \text{if } x \in [1, \frac{3}{2}] \\ 1 & \text{else} \end{cases},$$

$$fx = \begin{cases} 4x^4 - 3 & \text{if } x \in [1, \frac{3}{2}] \\ 1 & \text{else} \end{cases}.$$

We take $\psi : [0,\infty) \to [0,\infty)$ such that $\psi(t) = \frac{2}{3}t$.
Thus all the assumptions of Theorem 3.1 are satisfied and 1 is the unique common fixed point of f and g.

Lemma 3.4. *Let (X, d) be a metric space and $p : X \times X \to \mathbb{R}^+$ be a function defined by*

$$p(x,y) = e^{\theta(d(x,y))} - 1, \tag{3.4}$$

such that $\theta \in \Theta$. Then p is a τ_d-distance on X where τ_d is the metric topology.

Proof . Let (X, τ_d) be the topological space with the metric topology τ_d and V an arbitrary neighborhood of an arbitrary $x \in X$, then there exists $\varepsilon > 0$ such that $B_d(x, \varepsilon) \subset V$, where $B_d(x, \varepsilon) = \{y \in X, d(x,y) < \varepsilon\}$ is the open ball.
It easy to see that $B_p(x, e^{\theta(\varepsilon)} - 1) \subset B_d(x, \varepsilon)$, indeed:
Let $y \in B_p(x, e^{\theta(\varepsilon)} - 1)$, then $p(x,y) < e^{\theta(\varepsilon)} - 1$, which implies that $e^{\theta(d(x,y))} < e^{\theta(\varepsilon)}$. Since θ supposed increasing, we get $d(x,y) < \varepsilon$. □

Using Theorem 3.1 and Lemma 3.4, we now prove the following Theorem.

Theorem 3.5. *Let (X,d) be a bounded complete metric space (X,d). Let f and g be two weakly compatible selfmapping of X satisfying the following conditions:*

i) $g(X) \subset f(X)$,

ii) $\inf_{x\neq y}\{\theta[d(fx,fy)] - \theta[d(gx,gy)]\} > 0$,

where $\theta \in \Theta$. Then f and g have a unique common fixed point.

Proof . We put $\alpha = \inf_{x\neq y}\{\theta[d(fx,fy)] - \theta[d(gx,gy)]\}$, this implies that

$$\theta(d(gx,gy)) \leq \theta(d(fx,fy)) - \alpha, \tag{3.5}$$

for all $x \neq y \in X$. Hence

$$e^{\theta(d(gx,gy))} \leq k e^{\theta(d(fx,fy))}, \tag{3.6}$$

such that $k = e^{-\alpha} < 1$.
Let's consider the function $p : X \times X \to [0,\infty)$ defined by

$$p(x,y) = e^{\theta(d(x,y))} - 1,$$

which is a τ_d-distance on X as proved in Lemma 3.4, where τ_d is the metric topology. By taking $\psi(t) = kt$ in Theorem 3.1 for all $t \in [0,\infty)$, we get

$$p(gx,gy) \leq kp(fx,fy). \tag{3.7}$$

Finally, we conclude that f and g have a unique common fixed point. □

The following example illustrates Theorem 3.5.

Example 3.6. *Let $X = [0,1]$, with*

$$d(x,y) = \begin{cases} 1 + \max\{x,y\} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}.$$

Define $f,g : X \to X$ by $fx = 1 - x$ and $gx = \frac{1}{2}$, for all $x \in X$ and $\theta : [0,\infty) \to [0,\infty)$, such that $\theta(t) = \ln(1+t)$, for all $t \in [0,\infty)$. It is easy to see that $g(X) \subset f(X)$ and f,g are weakly compatible. On the other hand, we have for all $x < y \in X$

$$\theta(d(fx,fy)) - \theta(d(gx,gy)) = \ln(3-x) \geq \ln 2 > 0.$$

Then f and g satisfy all assumptions of Theorem 3.5 and have the unique fixed point which equal to $\frac{1}{2}$.

Example 3.7. *Let $X = \overline{B}(0,1)$, the unit closed ball of a real Banach space, endowed with the metric*

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}.$$

Define $f,g : X \to X$ by $fx = -x$ and $gx = 0$ for all $x \in X$.

$$\theta(d(fx,fy)) - \theta(d(gx,gy)) = \theta(d(-x,-y)) = 2, \quad \text{for all } x \neq y \in X,$$

with $\theta = 2t$. Therefore f and g satisfy all conditions in Theorem 3.5 such that f and g have the unique fixed point which is equal to 0.

Remark 3.8. *The above examples show that there is no relationship between compactness and the condition* $\inf_{x \neq y}\{\theta[d(fx, fy)] - \theta[d(gx, gy)]\} > 0$. *Indeed:*
In the first example, $X = [0, 1]$ *is compact and* $\inf_{x \neq y}\{\theta[d(fx, fy)] - \theta[d(gx, gy)]\} > 0$. *On the other side, for* $X = [0, 1]$, $fx = \frac{1}{2}x$ *and* $gx = \frac{1}{3}x$, *the space* X *is compact and* $\inf_{x \neq y}\{\theta[d(fx, fy)] - \theta[d(gx, gy)]\} = 0$ *with* $\theta(t) = 2t$. *Moreover, in the second example,* X *is not compact and* $\inf_{x \neq y}\{\theta[d(fx, fy)] - \theta[d(gx, gy)]\} > 0$.

As application of Theorem 3.5, we get a result for a new class of weakly contractive maps defined as follows.

Definition 3.9. *Let* (X, d) *be a metric space and* $f, g : X \longrightarrow X$ *be a two weakly compatible selfmappings of* X *such that* $g(X) \subset f(X)$.
f *and* g *are said to be* E_θ*-weakly contractive if*

i) For all $x \neq y \in X$ *such that* $fx = fy$, *we have* $fx = gx$,

ii) $\theta(d(gx, gy)) \leq \theta(d(fx, fy)) - \phi(\theta(d(fx, fy)) + 1)$, *for all* $x, y \in X$,

where $\theta \in \Theta$ *and* $\phi \in \Phi$.

Theorem 3.10. *Let* (X, d) *be a bounded complete metric space and* f, g *be two* $E_\theta-$*weakly maps on* X. *Then* f *and* g *have a unique common fixed point.*

Proof . Let $x \neq y \in X$, then we have the two following cases:
Case1: If $fx = fy$, then Definition 3.9 implies that $fx = gx = fy = gy$. Since f and g are weakly compatible, we have $g \circ fx = f \circ gx = f \circ fx = g \circ gx$. Again, from Definition 3.9, we get

$$\begin{aligned}\theta(d(g \circ fx, fx)) &= \theta(d(g \circ fx, gx) \\ &\leq \theta(d(f \circ fx, fx)) - \phi(\theta(d(f \circ fx, fx)) + 1) \\ &= \theta(d(g \circ fx, fx)) - \phi(\theta(d(g \circ fx, fx)) + 1).\end{aligned}$$

Then f and g have a unique common fixed point.
Case2: If $fx \neq fy$, it follows from Definition 3.9,

$$0 < \inf_{t>1} \phi(t) \leq \phi(\theta(d(fx, fy)) + 1) \leq \theta(d(fx, fy)) - \theta(d(gx, gy)), \tag{3.8}$$

hence $\inf_{x \neq y}\{\theta(d(fx, fy))) - \theta(d(gx, gy))\} > 0$. According to Theorem 3.5, we conclude that f and g have a unique common fixed point. □

Example 3.11. *Let* $X = \{0, 1, 2, 3\}$ *with the following metric*

$$d(x, y) = \begin{cases} \max\{x, y\} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases} .$$

Define $f, g : X \to X$ *by*

$$f0 = 0,\ f1 = 0,\ f2 = 1,\ f3 = 2$$

and

$$g0 = 0,\ g1 = 0,\ g2 = 0,\ g3 = 1.$$

We take $\theta(t) = 2t$, *for all* $t \in [0, \infty)$ *and*

$$\phi(t) = \begin{cases} 0 & \text{if } t = 1 \\ 1 & \text{if } t > 1 \end{cases} .$$

Then f *and* g *satisfy all conditions in Theorem 3.10 and* 0 *is the unique common fixed point. Note that* ϕ *is not continous at 1.*

4. Application

In this section, we will prove the existence and uniqueness of a common solution for the tow nonlinear integral equations

$$\begin{cases} x'(t) = K(t, \int_0^t K(s, x(s))ds), \\ x(0) = 0 \end{cases} \tag{4.1}$$

and

$$\begin{cases} x'(t) = K(t, x(t)), \\ x(0) = 0 \end{cases} . \tag{4.2}$$

where, $x \in \mathcal{C}[0,T]$, the space of all continuous functions from $[0,T]$ into $\mathbb{R}$, with $T > 0$.
$K : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous mapping.
Let $X = \mathcal{C}[0,T]$ endowed by the metric $d : X \times X \to \mathbb{R}^+$ defined by

$$d(x,y) = \sup_{t\in[0,T]} |x(t) - y(t)|,$$

it is clear that (X,d) is a complete metric space.
The differential equations (4.1) and (4.2) are equivalent to the integral equations

$$x(t) = \int_0^t K(s, \int_0^s K(\xi, x(\xi))d\xi)ds,\ t \in [0,T] \tag{4.3}$$

and

$$x(t) = \int_0^t K(s, x(s))ds,\ t \in [0,T], \tag{4.4}$$

respectively.
Define a mappings $f, g : X \to X$ as follows

$$f(x)(t) = \int_0^t K(s, x(s))ds,\ t \in [0,T] \tag{4.5}$$

and

$$g(x)(t) = \int_0^t K(s, \int_0^s K(\xi, x(\xi))d\xi)ds,\ t \in [0,T] \tag{4.6}$$

for all $x \in X$.
So the problem of the common solution of the differential equations is equivalent to finding the common fixed point of the mappings f and g.
Suppose that the above assumptions hold, then we have the following Theorem:

Theorem 4.1. *If there exists $M > 0$ such that*

$$|K(s,x) - K(s,y)| \leq \frac{1}{T}[|x-y| - M], \tag{4.7}$$

for all $s \in [0,T]$ and $x, y \in X$ such that $x \neq y$. Then the nonlinear integral equations (4.5) *and* (4.6) *have a unique common solution.*

Proof . Clearly, $g(X) \subset f(X)$ and f, g are weakly compatible, so, it remains for us to show that f and g satisfy (ii) in Theorem 3.5.
Let $x \neq y \in X$ and $t \in [0, T]$, then by (4.5), (4.6) and (4.7), we have

$$\begin{aligned} &|g(x)(t) - g(y)(t)| \\ &\quad = \left| \int_0^t K(s, \int_0^s K(\xi, x(\xi))d\xi)ds - \int_0^t K(s, \int_0^s K(\xi, y(\xi))d\xi)ds \right| \\ &\quad \leq \int_0^t \frac{1}{T}\left[\left| \int_0^s K(\xi, x(\xi))d\xi - \int_0^s K(\xi, y(\xi))d\xi \right| - M \right] ds \\ &\quad \leq \int_0^t \frac{1}{T}\left[|f(x)(s) - f(y)(s)| - M \right] ds \\ &\quad \leq d(fx, fy) - M, \end{aligned} \tag{4.8}$$

hence

$$d(gx, gy) \leq d(fx, fy) - M$$

for all $x \neq y \in X$. Then $\inf_{x \neq y}\{d(fx, fy) - d(gx, gy)\} \geq M > 0$, which implies by Theorem 3.5 that there exists a unique common solution of the integral equations (4.5) and (4.6). □

5. Some open problems

Very recently, Gordji et al.[6] introduced the concept of orthogonal sets as follows

Definition 5.1. *[6]. Let $X \neq \emptyset$ and let $\perp \subset X \times X$ be a binary relation. If $\perp$ satisfies the following hypothesis:*

$$\exists x_0 : (\forall y, y \perp x_0) \text{ or } (\forall y, x_0 \perp y),$$

then it called an orthogonal set (briefly O-set). we denote this $O - set$ by $(X, \perp)$.

For more details, we refer the reader to see [6]. Then, an extension of Banach's contraction principle is presented. In the same direction of research, in 2017 Baghani et al. [2] defined an extension of F-contraction on orthogonal sets namely $\perp_F$-contraction and proved a fixed point theorem for this contractions. In the setting of orthogonal sets we can see also ([7, 11]). Motivated by this notion, the first open problem can be introduced as follows:
Open problem I: Common fixed point theorems for contractive selfmappings on orthogonal metric spaces and their applications to nonlinear differential equations.
On the other hand, the authors in [9], start the R-metric spaces, via this idea we can state:
Open problem II: Common fixed point theorems for contractive selfmappings on R-metric spaces and their applications to nonlinear differential equations.

References

[1] M. Aamri and D. El Moutawakil, *τ-distance in general topological spaces with application to fixed point theory*, Southwest J. Pure Appl. Math. 2 (2003) 1–5.
[2] H. Baghani, M. Eshaghi Gordji and M. Ramezani, *Orthogonal sets, the axiom of choice and proof of a fixed point theorem*, J. Fixed Point Theory Appl. 18 (2016) 465–477.
[3] L. Ćirić, *Some recent results in metrical fixed point theory*, University of Belgrade, Serbia, 2003.
[4] M. Edelstein, *On fixed and periodic points under contractive mappings*, J. London Math. Soc., 37 (1962) 74-–79.
[5] D. Djorić, *Common fixed point for generalized (ψ, ϕ)-weak contractions,* Appl. Math. Lett. 22 (2009) 1896–1900.

[6] M. E. Gordji, M. Rameani, M. De La Sen and Y. Je Cho, *On orthogonal sets and Banach fixed point theorem*, Fixed Point Theory, 18(2) (2017) 569–578.

[7] M. E. Gordji and H. Habibi, *Fixed point theory in generalized orthogonal metric space,* J. Linear Top. Alg. 6(3) (2017) 251–260.

[8] G. Jungck and B. E. Rhoades, *Fixed point for set valued functions without continuity*, Indian J. Pure Appl. Math. 29(3) (1998) 227—238.

[9] S. Khalehoghli, H. Rahimi and M. Eshaghi Gordji, *Fixed point theorems in R-metric spaces with applications*, AIMS Mathematics, 5(4) (2020) 3125–3137.

[10] W. Kirk and N. Shahzad, *Fixed point theory in distance spaces*, Springer International Publishing Switzerland, 2014.

[11] M. Ramezani and H. Baghani, *Contractive gauge functions in strongly orthogonal metric spaces*, Int. J. Nonlinear Anal. Appl. 8(2) (2017) 23–28.

[12] V. V. Nemytzki, *The fixed point method in analysis*, (Russian), Usp. Mat. Nauk, 1 (1936) 141–174.

[13] S. Radenović, Z. Kadelburg, D. Jandrlić and A. Jandrlić, *Some results on weak contraction maps*, Bull. Iran. Math. Soc. 38 (2012) 625–645.

[14] B. E. Rhoades, *Some theorems on weakly contractive maps*, Nonlinear Anal. 47 (2001) 2683–2693.