New estimates of Gauss–Jacobi and trapezium type inequalities for strongly (h_1, h_2)–preinvex mappings via general fractional integrals

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Abstract

In this paper, authors discover two interesting identities regarding Gauss–Jacobi and trapezium type integral inequalities. By using the first lemma as an auxiliary result, some new bounds with respect to Gauss–Jacobi type integral inequalities for a new class of functions called strongly (h_1, h_2)–preinvex of order $\sigma > 0$ with modulus $\mu > 0$ via general fractional integrals are established. Also, using the second lemma, some new estimates with respect to trapezium type integral inequalities for strongly (h_1, h_2)–preinvex functions of order $\sigma > 0$ with modulus $\mu > 0$ via general fractional integrals are obtained. It is pointed out that some new special cases can be deduced from main results. Some applications to special means for different real numbers and new approximation error estimates for the trapezoidal are provided as well. These results give us the generalizations of some previous known results. The ideas and techniques of this paper may stimulate further research in the fascinating field of inequalities.

Keywords: Hermite–Hadamard inequality, Gauss–Jacobi type quadrature formula, Hölder inequality, power mean inequality, general fractional integrals
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1. Introduction

The theory of inequalities is known to play an important role in almost all areas of pure and applied sciences. Richard Bellman stated succinctly, at the *Second International Conference on Mathematical Inequalities*, Oberwolfach, Germany, July 30-August 5, 1978, . . . there are three reasons for the study of inequalities: practical, theoretical, and aesthetic.

Before we start, the following notations are used throughout this paper. We use I to denote an interval on the real line $\mathbb{R} = (-\infty, +\infty)$ and set of integrable functions on the interval $[q_1, q_2]$, where $q_1 < q_2$ by $L[q_1, q_2]$.

Definition 1.1. *A function $f : I \longrightarrow \mathbb{R}$ is said to be convex, if*

$$f(tq_1 + (1-t)q_2) \leq tf(q_1) + (1-t)f(q_2) \tag{1.1}$$

for all $q_1, q_2 \in I$ and $t \in [0, 1]$.

The following inequality, named Hermite–Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

Theorem 1.2. *Let $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function on I and $q_1, q_2 \in I$ with $q_1 < q_2$. Then the following inequality holds:*

$$f\left(\frac{q_1 + q_2}{2}\right) \leq \frac{1}{q_2 - q_1} \int_{q_1}^{q_2} f(x)dx \leq \frac{f(q_1) + f(q_2)}{2}. \tag{1.2}$$

The inequality (1.2) is also acknowledged as the trapezium inequality.

The trapezium type inequality has remained an area of great interest due to its wide applications in the field of mathematical analysis. For other recent results which generalize, improve and extend the inequality (1.2) through various classes of convex functions interested readers are referred to [1, 2, 4],[7]–[12],[15]–[21],[23, 24, 26, 28, 30, 31].

The Gauss–Jacobi type quadrature formula has the following

$$\int_{q_1}^{q_2} (x - q_1)^p (q_2 - x)^q f(x)dx = \sum_{k=0}^{+\infty} B_{m,k} f(\gamma_k) + R_m^\star |f|, \tag{1.3}$$

for certain $B_{m,k}, \gamma_k$ and rest $R_m^\star |f|$, see [27].

Recently in [14], Liu obtained several integral inequalities for the left-hand side of (1.3). Also in [20], Özdemir et al. established several integral inequalities concerning the left-hand side of (1.3) via some kinds of convexity.

Let us recall some special functions and evoke some basic definitions as follows.

Definition 1.3. *For $k \geq 0$ and $x \in \mathbb{C}$, the k-gamma function is defined by*

$$\Gamma_k(x) = \lim_{n \longrightarrow \infty} \frac{n! k^n (nk)^{\frac{x}{k} - 1}}{(x)_{n,k}}, \tag{1.4}$$

where

$$(x)_{n,k} = x(x + k) \cdot \ldots \cdot (x + (n-1)k).$$

One can note that

$$\Gamma_k(\alpha + k) = \alpha\Gamma_k(\alpha). \tag{1.5}$$

Definition 1.4. *([17]) Let $f \in L[q_1, q_2]$. Then k-fractional integrals of order $\alpha, k > 0$ with $q_1 \geq 0$ are defined as*

$$I_{q_1^+}^{\alpha,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_{q_1}^{x} (x - t)^{\frac{\alpha}{k}-1} f(t)dt, \quad x > q_1$$

and

$$I_{q_2^-}^{\alpha,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_{x}^{q_2} (t - x)^{\frac{\alpha}{k}-1} f(t)dt, \quad q_2 > x. \tag{1.6}$$

For $k = 1$, k-fractional integrals give the classical Riemann–Liouville integrals.

Definition 1.5. *([29]) A set $S \subseteq \mathbb{R}$ is said to be invex set with respect to the mapping $\zeta : S \times S \longrightarrow \mathbb{R}$, if $x + t\zeta(y, x) \in S$ for every $x, y \in S$ and $t \in [0, 1]$.*

The invex set S is also termed an ζ-connected set.

Definition 1.6. *Let $S \subseteq \mathbb{R}$ be an invex set with respect to $\zeta : S \times S \longrightarrow \mathbb{R}$. A function $f : S \longrightarrow [0, +\infty)$ is said to be preinvex with respect to ζ, if for every $x, y \in S$ and $t \in [0, 1]$,*

$$f\big(x + t\zeta(y, x)\big) \leq (1 - t)f(x) + tf(y).$$

For the motivation of them as well as the geometric interpretation of an invex set and a preinvex function with respect to an invex set, respectively, interested readers can see [8, 9, 11, 16, 18, 21, 23, 31].

The notion of strongly convex functions was introduced by Karamardian in [6] and Polyak in [22].

Definition 1.7. *A function $f : S \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be a strongly convex function for modulus $\mu > 0$, if*

$$f((1 - t)x + ty) \leq (1 - t)f(x) + tf(y) - \mu t(1 - t)(y - x)^2$$

for all $x, y \in S$ and $t \in [0, 1]$.

In [6], Karamardian noticed that every strongly monotone has a gradient map if and only if all differentiable function is strongly convex. Higher order strongly convex functions introduced by Lin et al. in [13], to abridge the research of linear programming with equilibrium constraints.

Definition 1.8. *A function $f : S \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be a strongly convex function for modulus $\mu > 0$ with order $\sigma > 0$, if*

$$f((1 - t)x + ty) \leq (1 - t)f(x) + tf(y) - \mu t(1 - t)(y - x)^\sigma$$

for all $x, y \in S$ and $t \in [0, 1]$.

Recently, Rashid et al. in [23], defined the following class of strongly preinvex functions of higher order.

Definition 1.9. *A function $f : S \longrightarrow \mathbb{R}$ is said to be strongly h-preinvex of order $\sigma > 0$ with modulus $\mu > 0$ with respect to $\zeta : S \times S \longrightarrow \mathbb{R}$, if*

$$f(x + t\zeta(y, x)) \leq h(1 - t)f(x) + h(t)f(y) - \mu t(1 - t)\zeta^\sigma(y, x) \tag{1.7}$$

for all $x, y \in S$ and $t \in [0, 1]$.

Also, let define a function $\varphi : [0, +\infty) \longrightarrow [0, +\infty)$ satisfying the following conditions:

$$\int_0^1 \frac{\varphi(t)}{t} dt < +\infty, \tag{1.8}$$

$$\frac{1}{A_1} \leq \frac{\varphi(s)}{\varphi(r)} \leq A_1 \text{ for } \frac{1}{2} \leq \frac{s}{r} \leq 2 \tag{1.9}$$

$$\frac{\varphi(r)}{r^2} \leq A_2 \frac{\varphi(s)}{s^2} \text{ for } s \leq r \tag{1.10}$$

$$\left| \frac{\varphi(r)}{r^2} - \frac{\varphi(s)}{s^2} \right| \leq A_3 |r - s| \frac{\varphi(r)}{r^2} \text{ for } \frac{1}{2} \leq \frac{s}{r} \leq 2 \tag{1.11}$$

where $A_1, A_2, A_3 > 0$ are independent of $r, s > 0$. If $\varphi(r) r^\alpha$ is increasing for some $\alpha \geq 0$ and $\frac{\varphi(r)}{r^\beta}$ is decreasing for some $\beta \geq 0$, then φ satisfies (1.8)–(1.11), see [25]. Therefore, we define the following left–sided and right–sided generalized fractional integral operators, respectively, as follows:

$$_{q_1^+} I_\varphi f(x) = \int_{q_1}^x \frac{\varphi(x - t)}{x - t} f(t) dt, \quad x > q_1, \tag{1.12}$$

$$_{q_2^-} I_\varphi f(x) = \int_x^{q_2} \frac{\varphi(t - x)}{t - x} f(t) dt, \quad x < q_2. \tag{1.13}$$

The most important feature of generalized fractional integrals is that they generalize some types of fractional integrals such as Riemann–Liouville fractional integral, k–Riemann–Liouville fractional integral, Katugampola fractional integrals, conformable fractional integral, Hadamard fractional integrals, etc., see [24].

Motivated by the above literatures, the main objective of this paper is to discover in Section 2 and in Section 3, two interesting identities and to established some new bounds regarding Gauss–Jacobi and Hermite–Hadamard type integral inequalities for a new class of functions called strongly (h_1, h_2)–preinvex of order $\sigma > 0$ with modulus $\mu > 0$ via general fractional integrals. By using in Section 2 the first lemma as an auxiliary result, some new estimates with respect to Gauss–Jacobi type integral inequalities for strongly (h_1, h_2)–preinvex mappings of order $\sigma > 0$ with modulus $\mu > 0$ via general fractional integrals will be given. Also, using in Section 3 the second lemma, some new estimates with respect to Hermite–Hadamard type integral inequalities for strongly (h_1, h_2)–preinvex of order $\sigma > 0$ with modulus $\mu > 0$ via general fractional integrals will be obtained. It is pointed out that some new special cases will be deduced from main results. In Section 4, some applications to special means for different real numbers and new approximation error estimates for the trapezoidal will be given. In Section 5, a briefly conclusion and future research is given as well. These results will give us the generalizations of some previous known results. The sharpness for bounds regarding Gauss–Jacobi integral inequalities and approximation error estimates for trapezoidal quadrature formulas compare with other published papers will be given in the future project.

2. Gauss–Jacobi type inequalities via general fractional integral

Throughout this study, for brevity, we denote $P = [q_1, q_1 + \zeta(q_2, q_1)] \subseteq \mathbb{R}$ a closed invex subset, P° is the interior of P and

$$\Lambda^*(t) = \int_0^t \frac{\varphi(\zeta(q_2, q_1)x)}{x} dx < +\infty, \quad \zeta(q_2, q_1) > 0.$$

Also, let define the functions $h_1, h_2 : [0,1] \longrightarrow [0, +\infty)$.

Now, we are in position to introduce the following interesting class of functions.

Definition 2.1. *A function* $f : P \longrightarrow \mathbb{R}$ *is said to be strongly* (h_1, h_2)-*preinvex of order* $\sigma > 0$ *with modulus* $\mu > 0$ *with respect to* $\zeta : P \times P \longrightarrow \mathbb{R}$, *if*

$$f(x + t\zeta(y, x)) \leq h_1(t)f(x) + h_2(t)f(y) - \mu t(1 - t)\zeta^\sigma(y, x) \tag{2.1}$$

for all $x, y \in P$ *and* $t \in [0, 1]$.

Remark 2.2. *Taking* $h_1(t) = h(1 - t)$ *and* $h_2(t) = h(t)$ *in Definition 2.1, we obtain Definition 1.9. If we choose* $\zeta(y, x) = y - x$ *and* $h_1(t) = h(1 - t)$, $h_2(t) = h(t)$ *in Definition 2.1, we get Definition 1.8. We observe that this class unifies several other classes of strong preinvexity.*

Indeed, now we will discuss several special cases of Definition 2.1 as follows:

(i) If $h_1(t) = h_2(t) = 1$, then we attain the following class of strongly P-preinvex functions of order $\sigma > 0$ with modulus $\mu > 0$ with respect to ζ

$$f(x + t\zeta(y, x)) \leq f(x) + f(y) - \mu t(1 - t)\zeta^\sigma(y, x).$$

(ii) If $h_1(t) = 1 - t$ and $h_2(t) = t$, then we attain the following class of strongly preinvex functions of order $\sigma > 0$ with modulus $\mu > 0$ with respect to ζ

$$f(x + t\zeta(y, x)) \leq (1 - t)f(x) + tf(y) - \mu t(1 - t)\zeta^\sigma(y, x).$$

(iii) If $h_1(t) = (1 - t)^s$ and $h_2(t) = t^s$, where $s \in [0, 1]$, then we attain the following class of strongly s-preinvex functions of order $\sigma > 0$ with modulus $\mu > 0$ with respect to ζ

$$f(x + t\zeta(y, x)) \leq (1 - t)^s f(x) + t^s f(y) - \mu t(1 - t)\zeta^\sigma(y, x).$$

(iv) If $h_1(t) = (1 - t)^{-s}$ and $h_2(t) = t^{-s}$, where $s \in [0, 1)$, then we attain the following class of strongly s-preinvex functions of Godunova-–Levin type of order $\sigma > 0$ with modulus $\mu > 0$ with respect to ζ

$$f(x + t\zeta(y, x)) \leq \frac{f(x)}{(1 - t)^s} + \frac{f(y)}{t^s} - \mu t(1 - t)\zeta^\sigma(y, x).$$

(v) If $h_1(t) = h_2(t) = t(1 - t)$, then we attain the following class of strongly tgs-preinvex functions of order $\sigma > 0$ with modulus $\mu > 0$ with respect to ζ

$$f(x + t\zeta(y, x)) \leq t(1 - t)(f(x) + f(y)) - \mu t(1 - t)\zeta^\sigma(y, x).$$

(vi) If $h_1(t) = \frac{\sqrt{t}}{2\sqrt{1-t}}$ and $h_2(t) = \frac{\sqrt{1-t}}{2\sqrt{t}}$, then we attain the following class of strongly MT-preinvex functions of order $\sigma > 0$ with modulus $\mu > 0$ with respect to ζ

$$f(x + t\zeta(y, x)) \leq \frac{\sqrt{t}}{2\sqrt{1 - t}}f(x) + \frac{\sqrt{1 - t}}{2\sqrt{t}}f(y) - \mu t(1 - t)\zeta^\sigma(y, x).$$

Remark 2.3. *If we substitute $\zeta(y, x) = y - x$ in Definition 2.1, then we attain the class of strongly (h_1, h_2)-convex functions of order $\sigma > 0$ with modulus $\mu > 0$ with respect to ζ. To the exceptional of our knowledge, this class is a new addition in convexity theory. It is worth mentioning that this class of functions is quite general and unifying one. This mean that, taking $\mu \to 0^+$ in Definition 2.1, we get the well–known class of functions, respectively, convex, P-convex, s-convex, s-convex of Godunova--Levin type, tgs-convex and MT-convex.*

For establishing some new bounds for Gauss–Jacobi type via general fractional integral, we need the following lemma.

Lemma 2.4. *Assume that $f : P \longrightarrow \mathbb{R}$ be a continuous mapping on P° with respect to $\eta : P \times P \longrightarrow \mathbb{R}$. Then for any fixed $p, q > 0$, we have*

$$\int_{q_1}^{q_1+\eta(q_2,q_1)} \left[\Lambda^* \left(\frac{x - q_1}{\zeta(q_2, q_1)} \right) \right]^p \left[\Lambda^* \left(\frac{q_1 + \zeta(q_2, q_1) - x}{\zeta(q_2, q_1)} \right) \right]^q f(x) dx$$

$$= \zeta(q_2, q_1) \int_0^1 \left[\Lambda^*(t) \right]^p \left[\Lambda^*(1 - t) \right]^q f(q_1 + t\zeta(q_2, q_1)) dt. \tag{2.2}$$

We denote

$$T_{f,\Lambda^*}^{p,q}(q_1, q_2) = \zeta(q_2, q_1) \int_0^1 \left[\Lambda^*(t) \right]^p \left[\Lambda^*(1 - t) \right]^q f(q_1 + t\zeta(q_2, q_1)) dt. \tag{2.3}$$

Proof . Using (2.3) and changing the variable of integration $x = q_1 + t\zeta(q_2, q_1)$, we have

$$T_{f,\Lambda^*}^{p,q}(q_1, q_2) = \zeta(q_2, q_1)$$

$$\times \int_{q_1}^{q_1+\zeta(q_2,q_1)} \left[\Lambda^* \left(\frac{x - q_1}{\zeta(q_2, q_1)} \right) \right]^p \left[\Lambda^* \left(1 - \frac{x - q_1}{\zeta(q_2, q_1)} \right) \right]^q f(x) \frac{dx}{\zeta(q_2, q_1)}$$

$$= \int_{q_1}^{q_1+\zeta(q_2,q_1)} \left[\Lambda^* \left(\frac{x - q_1}{\zeta(q_2, q_1)} \right) \right]^p \left[\Lambda^* \left(\frac{q_1 + \zeta(q_2, q_1) - x}{\zeta(q_2, q_1)} \right) \right]^q f(x) dx.$$

The proof of Lemma 2.4 is completed. $\square$

Corollary 2.5. *Taking $\zeta(q_2, q_1) = q_2 - q_1$ and $\varphi(x) = x$, in Lemma 2.4, we get the following identity:*

$$\int_{q_1}^{q_2} (x - q_1)^p (q_2 - x)^q f(x) dx = (q_2 - q_1)^{p+q+1} \int_0^1 t^p (1 - t)^q f(q_1 + t(q_2 - q_1)) dt. \tag{2.4}$$

With the help of Lemma 2.4, we have the following results.

Theorem 2.6. *Assume that $f : P \longrightarrow \mathbb{R}$ be a continuous mapping on P° with respect to $\zeta : P \times P \longrightarrow \mathbb{R}$. If $|f|^{\frac{k}{k-1}}$ is a strongly (h_1, h_2)-preinvex mapping of order $\sigma > 0$ with modulus $\mu > 0$ on P, then for $k > 1$ and any fixed $p, q > 0$, we have*

$$\left| T_{f,\Lambda^*}^{p,q}(q_1, q_2) \right| \leq \zeta(q_2, q_1) \sqrt[k]{A_{\Lambda^*}^{p,q}(k)} \tag{2.5}$$

$$\times \left[H_1 |f(q_1)|^{\frac{k}{k-1}} + H_2 |f(q_2)|^{\frac{k}{k-1}} - \frac{\mu}{6} \zeta^{\sigma}(q_2, q_1) \right]^{\frac{k-1}{k}},$$

where

$$A_{\Lambda^*}^{p,q}(k) = \int_0^1 \left[\Lambda^*(t) \right]^{kp} \left[\Lambda^*(1 - t) \right]^{kq} dt, \quad H_i = \int_0^1 h_i(t) dt, \quad \forall\, i = 1, 2.$$

Proof . Since $|f|^{\frac{k}{k-1}}$ is a strongly (h_1, h_2)-preinvex mapping of order $\sigma > 0$ with modulus $\mu > 0$ on P, combining with Lemma 2.4, Hölder inequality and properties of the modulus, we get

$$\left| T_{f,\Lambda^*}^{p,q}(q_1, q_2) \right| \leq \zeta(q_2, q_1) \int_0^1 \left[\Lambda^*(t) \right]^p \left[\Lambda^*(1-t) \right]^q |f(q_1 + t\zeta(q_2, q_1))| dt$$

$$\leq \zeta(q_2, q_1) \left[\int_0^1 \left[\Lambda^*(t) \right]^{kp} \left[\Lambda^*(1-t) \right]^{kq} dt \right]^{\frac{1}{k}} \left[\int_0^1 |f(q_1 + t\zeta(q_2, q_1))|^{\frac{k}{k-1}} dt \right]^{\frac{k-1}{k}}$$

$$\leq \zeta(q_2, q_1) \sqrt[k]{A_{\Lambda^*}^{p,q}(k)}$$

$$\times \left[\int_0^1 \left(h_1(t)|f(q_1)|^{\frac{k}{k-1}} + h_2(t)|f(q_2)|^{\frac{k}{k-1}} - \mu t(1-t)\zeta^\sigma(q_2, q_1) \right) dt \right]^{\frac{k-1}{k}}$$

$$= \zeta(q_2, q_1) \sqrt[k]{A_{\Lambda^*}^{p,q}(k)} \left[H_1|f(q_1)|^{\frac{k}{k-1}} + H_2|f(q_2)|^{\frac{k}{k-1}} - \frac{\mu}{6}\zeta^\sigma(q_2, q_1) \right]^{\frac{k-1}{k}}.$$

The proof of Theorem 2.6 is completed. $\square$
We point out some special cases of Theorem 2.6.

Corollary 2.7. *Under the assumption of Theorem 2.6 taking $\mu \longrightarrow 0^+$, we get*

$$\left| T_{f,\Lambda^*}^{p,q}(q_1, q_2) \right| \leq \zeta(q_2, q_1) \sqrt[k]{A_{\Lambda^*}^{p,q}(k)} \left[H_1|f(q_1)|^{\frac{k}{k-1}} + H_2|f(q_2)|^{\frac{k}{k-1}} \right]^{\frac{k-1}{k}}. \tag{2.6}$$

If we fixed, respectively, $h_1(t) = 1 - t$ and $h_2(t) = t$, then the following corollaries can be obtain.

Corollary 2.8. *Under the assumption of Theorem 2.6 with $\varphi(t) = t$, we get*

$$\left| T_{f,\Lambda_1^*}^{p,q}(q_1, q_2) \right| \leq \zeta^{p+q+1}(q_2, q_1) \sqrt[k]{\beta(kp+1, kq+1)} \tag{2.7}$$

$$\times \left[\frac{|f(q_1)|^{\frac{k}{k-1}} + |f(q_2)|^{\frac{k}{k-1}}}{2} - \frac{\mu}{6}\zeta^\sigma(q_2, q_1) \right]^{\frac{k-1}{k}},$$

where $\Lambda_1^* = \zeta(q_2, q_1)t$.

Corollary 2.9. *Under the assumption of Theorem 2.6 with $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$, we have*

$$\left| T_{f,\Lambda_2^*}^{p,q}(q_1, q_2) \right| \leq \frac{\zeta^{\alpha(p+q)+1}(q_2, q_1)}{\Gamma^{p+q}(\alpha+1)} \sqrt[k]{\beta(\alpha kp+1, \alpha kq+1)} \tag{2.8}$$

$$\times \left[\frac{|f(q_1)|^{\frac{k}{k-1}} + |f(q_2)|^{\frac{k}{k-1}}}{2} - \frac{\mu}{6}\zeta^\sigma(q_2, q_1) \right]^{\frac{k-1}{k}},$$

where $\Lambda_2^* = \frac{\zeta^\alpha(q_2, q_1)}{\Gamma(\alpha+1)}t^\alpha$.

Corollary 2.10. *Under the assumption of Theorem 2.6 with $\varphi(t) = \frac{t^{\frac{\alpha}{k_1}}}{k_1\Gamma_{k_1}(\alpha)}$, we obtain*

$$\left|T_{f,\Lambda_3^*}^{p,q}(q_1,q_2)\right| \leq \frac{\zeta^{\frac{\alpha}{k_1}(p+q)+1}(q_2,q_1)}{\left[k_1\Gamma_{k_1}(\alpha+k_1)\right]^{p+q}} \sqrt[k]{\beta\left(\frac{\alpha kp}{k_1}+1, \frac{\alpha kq}{k_1}+1\right)} \tag{2.9}$$

$$\times \left[\frac{|f(q_1)|^{\frac{k}{k-1}} + |f(q_2)|^{\frac{k}{k-1}}}{2} - \frac{\mu}{6}\zeta^\sigma(q_2,q_1)\right]^{\frac{k-1}{k}},$$

where $\Lambda_3^* = \frac{\zeta^{\frac{\alpha}{k_1}}(q_2,q_1)}{k_1\Gamma_{k_1}(\alpha+k_1)}t^{\frac{\alpha}{k_1}}$.

Corollary 2.11. *Under the assumption of Theorem 2.6 with $\varphi(t) = t(q_1 + \zeta(q_2,q_1) - t)^{\alpha-1}$ and $f(x)$ is symmetric to $x = q_1 + \frac{\zeta(q_2,q_1)}{2}$, we get*

$$\left|T_{f,\Lambda_4^*}^{p,q}(q_1,q_2)\right| \leq \frac{\zeta^{\frac{k-1}{k}(p+q)+1}(q_2,q_1)}{\alpha^{p+q}} \sqrt[k]{C^{p,q}(\alpha,k)} \tag{2.10}$$

$$\times \left[\frac{|f(q_1)|^{\frac{k}{k-1}} + |f(q_2)|^{\frac{k}{k-1}}}{2} - \frac{\mu}{6}\zeta^\sigma(q_2,q_1)\right]^{\frac{k-1}{k}},$$

where

$$C^{p,q}(\alpha,k) = \int_{q_1}^{q_1+\zeta(q_2,q_1)} \left[(q_1 + \zeta(q_2,q_1))^\alpha - t^\alpha\right]^{kp} \tag{2.11}$$

$$\times \left[(q_1 + \zeta(q_2,q_1))^\alpha - (2q_1 + \zeta(q_2,q_1) - t)^\alpha\right]^{kq} dt$$

and $\Lambda_4^* = \frac{(q_1+\zeta(q_2,q_1))^\alpha - (q_1+(1-t)\zeta(q_2,q_1))^\alpha}{\alpha}$.

Theorem 2.12. *Assume that $f : P \longrightarrow \mathbb{R}$ be a continuous mapping on P° with respect to $\zeta : P \times P \longrightarrow \mathbb{R}$. If $|f|^l$ is a strongly (h_1,h_2)-preinvex mapping of order $\sigma > 0$ with modulus $\mu > 0$ on P, then for $l \geq 1$ and any fixed $p,q > 0$, we have*

$$\left|T_{f,\Lambda^*}^{p,q}(q_1,q_2)\right| \leq \zeta(q_2,q_1)\left[A_{\Lambda^*}^{p,q}(1)\right]^{\frac{l-1}{l}} \tag{2.12}$$

$$\times \sqrt[l]{B_{\Lambda^*,h_1}^{p,q}|f(q_1)|^l + B_{\Lambda^*,h_2}^{p,q}|f(q_2)|^l - \mu W_{\Lambda^*}^{p,q}\zeta^\sigma(q_2,q_1)},$$

where

$$B_{\Lambda^*,h_i}^{p,q} = \int_0^1 \left[\Lambda^*(t)\right]^p\left[\Lambda^*(1-t)\right]^q h_i(t)dt, \quad \forall i = 1,2,$$

$$W_{\Lambda^*}^{p,q} = \int_0^1 \left[\Lambda^*(t)\right]^p\left[\Lambda^*(1-t)\right]^q t(1-t)dt$$

and $A_{\Lambda^*}^{p,q}(1)$ is defined in Theorem 2.6 for value $k = 1$.

Proof . Since $|f|^l$ is a strongly (h_1, h_2)-preinvex mapping of order $\sigma > 0$ with modulus $\mu > 0$ on P, combining with Lemma 2.4, the well–known power mean inequality and properties of the modulus, we get

$$\left| T_{f,\Lambda^*}^{p,q}(q_1, q_2) \right| \le \zeta(q_2, q_1) \int_0^1 \left[\Lambda^*(t) \right]^p \left[\Lambda^*(1-t) \right]^q |f(q_1 + t\zeta(q_2, q_1))| dt$$

$$\le \zeta(q_2, q_1) \left[\int_0^1 \left[\Lambda^*(t) \right]^p \left[\Lambda^*(1-t) \right]^q dt \right]^{\frac{l-1}{l}}$$

$$\times \left[\int_0^1 \left[\Lambda^*(t) \right]^p \left[\Lambda^*(1-t) \right]^q |f(q_1 + t\zeta(q_2, q_1))|^l dt \right]^{\frac{1}{l}}$$

$$\le \zeta(q_2, q_1) \left[A_{\Lambda^*}^{p,q}(1) \right]^{\frac{l-1}{l}}$$

$$\times \left[\int_0^1 \left[\Lambda^*(t) \right]^p \left[\Lambda^*(1-t) \right]^q \left(h_1(t)|f(q_1)|^l + h_2(t)|f(q_2)|^l - \mu t(1-t)\zeta^{\sigma}(q_2, q_1) \right) dt \right]^{\frac{1}{l}}$$

$$= \zeta(q_2, q_1) \left[A_{\Lambda^*}^{p,q}(1) \right]^{\frac{l-1}{l}} \sqrt[l]{B_{\Lambda^*,h_1}^{p,q}|f(q_1)|^l + B_{\Lambda^*,h_2}^{p,q}|f(q_2)|^l - \mu W_{\Lambda^*}^{p,q}\zeta^{\sigma}(q_2, q_1)}.$$

The proof of Theorem 2.12 is completed. $\square$
We point out some special cases of Theorem 2.12.

Corollary 2.13. *Under the assumption of Theorem 2.12 taking* $\mu \longrightarrow 0^+$, *we get*

$$\left| T_{f,\Lambda^*}^{p,q}(q_1, q_2) \right| \le \zeta(q_2, q_1) \left[A_{\Lambda^*}^{p,q}(1) \right]^{\frac{l-1}{l}} \sqrt[l]{B_{\Lambda^*,h_1}^{p,q}|f(q_1)|^l + B_{\Lambda^*,h_2}^{p,q}|f(q_2)|^l}. \tag{2.13}$$

If we fixed, respectively, $h_1(t) = 1 - t$ and $h_2(t) = t$, then from Corollary 2.13, we have the following results.

Corollary 2.14. *Taking* $\varphi(t) = t$, *we get*

$$\left| T_{f,\Lambda_1^*}^{p,q}(q_1, q_2) \right| \le \zeta^{p+q+1}(q_2, q_1)\beta^{\frac{l-1}{l}}(p+1, q+1) \tag{2.14}$$

$$\times \sqrt[l]{\beta(p+1, q+2)|f(q_1)|^l + \beta(q+1, p+2)|f(q_2)|^l}.$$

Corollary 2.15. *Taking* $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$, *we have*

$$\left| T_{f,\Lambda_2^*}^{p,q}(q_1, q_2) \right| \le \frac{\zeta^{\alpha(p+q)+1}(q_2, q_1)}{\Gamma^{p+q}(\alpha+1)}\beta^{\frac{l-1}{l}}(\alpha p + 1, \alpha q + 1) \tag{2.15}$$

$$\times \sqrt[l]{\beta(\alpha p + 1, \alpha q + 2)|f(q_1)|^l + \beta(\alpha q + 1, \alpha p + 2)|f(q_2)|^l}.$$

Corollary 2.16. *Taking* $\varphi(t) = \frac{t^{\frac{\alpha}{k_1}}}{k_1 \Gamma_{k_1}(\alpha)}$, *we obtain*

$$\left| T_{f,\Lambda_3^*}^{p,q}(q_1, q_2) \right| \le \frac{\zeta^{\frac{\alpha}{k_1}(p+q)+1}(q_2, q_1)}{\left[k_1 \Gamma_{k_1}(\alpha + k_1) \right]^{p+q}}\beta^{\frac{l-1}{l}}\left(\frac{p\alpha}{k_1} + 1, \frac{q\alpha}{k_1} + 1 \right). \tag{2.16}$$

$$\times \sqrt[l]{\beta\left(\frac{p\alpha}{k_1} + 1, \frac{q\alpha}{k_1} + 2 \right)|f(q_1)|^l + \beta\left(\frac{q\alpha}{k_1} + 1, \frac{p\alpha}{k_1} + 2 \right)|f(q_2)|^l}.$$

Corollary 2.17. *Taking* $\varphi(t) = t(q_1 + \zeta(q_2, q_1) - t)^{\alpha-1}$ *and* $f(x)$ *is symmetric to* $x = q_1 + \frac{\zeta(q_2,q_1)}{2}$, *we get*

$$\left| T_{f,\Lambda_4^*}^{p,q}(q_1, q_2) \right| \le \zeta(q_2, q_1) \left[\frac{C^{p,q}(\alpha, 1)}{\alpha^{p+q}} \right]^{\frac{l-1}{l}} \sqrt[l]{D^{p,q}|f(q_1)|^l + D^{q,p}|f(q_2)|^l}, \tag{2.17}$$

where

$$D^{p,q} = \frac{1}{\alpha^{p+q}\zeta^2(q_2, q_1)} \int_{q_1}^{q_1+\zeta(q_2,q_1)} (t - q_1) \left[(q_1 + \zeta(q_2, q_1))^\alpha - t^\alpha \right]^p \tag{2.18}$$

$$\times \left[(q_1 + \zeta(q_2, q_1))^\alpha - (2q_1 + \zeta(q_2, q_1) - t)^\alpha \right]^q dt.$$

Remark 2.18. *The above estimates of our Theorems 2.6 and 2.12, respectively, (2.5) and (2.12) are sharp. The corresponding analysis of them is done but we omit here their proofs and the details are left to the interested readers.*

3. Hermite–Hadamard type inequalities via general fractional integral

For establishing some new results regarding Hermite–Hadamard type inequalities via general fractional integral we need to prove the following lemma.

Lemma 3.1. *Let* $f : P \longrightarrow \mathbb{R}$ *be a differentiable mapping on* P°. *If* $f' \in L(P)$, *then the following identity for generalized fractional integrals hold:*

$$\frac{f(q_1) + f(q_1 + \zeta(q_2, q_1))}{2} - \frac{1}{2\Lambda^*(1)} \left[_{q_1^+}I_\varphi f(q_1 + \zeta(q_2, q_1)) + _{(q_1+\zeta(q_2,q_1))^-}I_\varphi f(q_1) \right]$$

$$= \frac{\zeta(q_2, q_1)}{2\Lambda^*(1)} \int_0^1 \left[\Lambda^*(1 - t) - \Lambda^*(t) \right] f'(q_1 + (1 - t)\zeta(q_2, q_1)) dt. \tag{3.1}$$

We denote

$$H_{f,\Lambda^*}(q_1, q_2) = \frac{\zeta(q_2, q_1)}{2\Lambda^*(1)} \int_0^1 \left[\Lambda^*(1 - t) - \Lambda^*(t) \right] f'(q_1 + (1 - t)\zeta(q_2, q_1)) dt. \tag{3.2}$$

Proof . Integrating by parts (3.2), we have

$$H_{f,\Lambda^*}(q_1, q_2) = \frac{\zeta(q_2, q_1)}{2\Lambda^*(1)}$$

$$\times \left\{ \int_0^1 \Lambda^*(1 - t) f'(q_1 + (1 - t)\zeta(q_2, q_1)) dt - \int_0^1 \Lambda^*(t) f'(q_1 + (1 - t)\zeta(q_2, q_1)) dt \right\}$$

$$= \frac{\zeta(q_2, q_1)}{2\Lambda^*(1)} \times \left\{ -\frac{\Lambda^*(1 - t) f(q_1 + (1 - t)\zeta(q_2, q_1))}{\zeta(q_2, q_1)} \Big|_0^1 \right.$$

$$-\frac{1}{\zeta(q_2, q_1)} \int_0^1 \frac{\varphi(\zeta(q_2, q_1)(1 - t))}{1 - t} f(q_1 + (1 - t)\zeta(q_2, q_1)) dt$$

$$\left. +\frac{\Lambda^*(t) f(q_1 + (1 - t)\zeta(q_2, q_1))}{\zeta(q_2, q_1)} \Big|_0^1 - \frac{1}{\zeta(q_2, q_1)} \int_0^1 \frac{\varphi(\zeta(q_2, q_1)t)}{t} f(q_1 + (1 - t)\zeta(q_2, q_1)) dt \right\}$$

$$= \frac{\zeta(q_2, q_1)}{2\Lambda^*(1)} \times \left\{ \frac{\Lambda^*(1)f(q_1 + \zeta(q_2, q_1))}{\zeta(q_2, q_1)} - \frac{1}{\zeta(q_2, q_1)} \times {}_{(q_1+\zeta(q_2,q_1))^-}I_\varphi f(q_1) \right.$$

$$\left. + \frac{\Lambda^*(1)f(q_1)}{\zeta(q_2, q_1)} - \frac{1}{\zeta(q_2, q_1)} \times {}_{q_1^+}I_\varphi f(q_1 + \zeta(q_2, q_1)) \right\}$$

$$= \frac{f(q_1) + f(q_1 + \zeta(q_2, q_1))}{2} - \frac{1}{2\Lambda^*(1)} \left[{}_{q_1^+}I_\varphi f(q_1 + \zeta(q_2, q_1)) + {}_{(q_1+\zeta(q_2,q_1))^-}I_\varphi f(q_1) \right].$$

The proof of Lemma 3.1 is completed. □

Remark 3.2. *Taking $\zeta(q_2, q_1) = q_2 - q_1$ in Lemma 3.1, we get ([24], Lemma 5).*

Theorem 3.3. *Let $f : P \longrightarrow \mathbb{R}$ be a differentiable mapping on P°. If $|f'|^q$ is a strongly (h_1, h_2)-preinvex mapping of order $\sigma > 0$ with modulus $\mu > 0$ on P, then for $q > 1$ and $p^{-1} + q^{-1} = 1$, the following inequality for generalized fractional integrals hold:*

$$\left| H_{f,\Lambda^*}(q_1, q_2) \right| \le \frac{\zeta(q_2, q_1)}{2\Lambda^*(1)} \sqrt[p]{K_{\Lambda^*}(p)} \sqrt[q]{H_1|f'(q_1)|^q + H_2|f'(q_2)|^q - \frac{\mu}{6}\zeta^\sigma(q_2, q_1)}, \tag{3.3}$$

where

$$K_{\Lambda^*}(p) = \int_0^1 \left| \Lambda^*(1 - t) - \Lambda^*(t) \right|^p dt$$

and H_1, H_2 are defined in Theorem 2.6.

Proof . From Lemma 3.1, strongly (h_1, h_2)-preinvexity of order $\sigma > 0$ with modulus $\mu > 0$ of mapping $|f'|^q$, Hölder inequality and properties of the modulus, we have

$$\left| H_{f,\Lambda^*}(q_1, q_2) \right| \le \frac{\zeta(q_2, q_1)}{2\Lambda^*(1)} \int_0^1 \left| \Lambda^*(1 - t) - \Lambda^*(t) \right| \left| f'(q_1 + (1 - t)\zeta(q_2, q_1)) \right| dt$$

$$\le \frac{\zeta(q_2, q_1)}{2\Lambda^*(1)} \left(\int_0^1 \left| \Lambda^*(1 - t) - \Lambda^*(t) \right|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 \left| f'(q_1 + (1 - t)\zeta(q_2, q_1)) \right|^q dt \right)^{\frac{1}{q}}$$

$$\le \frac{\zeta(q_2, q_1)}{2\Lambda^*(1)} \sqrt[p]{K_{\Lambda^*}(p)} \left(\int_0^1 \left(h_1(1 - t)|f'(q_1)|^q + h_2(1 - t)|f'(q_2)|^q - \frac{\mu}{6}\zeta^\sigma(q_2, q_1) \right) dt \right)^{\frac{1}{q}}$$

$$= \frac{\zeta(q_2, q_1)}{2\Lambda^*(1)} \sqrt[p]{K_{\Lambda^*}(p)} \sqrt[q]{H_1|f'(q_1)|^q + H_2|f'(q_2)|^q - \frac{\mu}{6}\zeta^\sigma(q_2, q_1)}.$$

The proof of Theorem 3.3 is completed. □

We point out some special cases of Theorem 3.3.

Corollary 3.4. *Taking $\mu \longrightarrow 0^+$, $h_1(t) = t$, $h_2(t) = 1 - t$ and $\zeta(q_2, q_1) = q_2 - q_1$ in Theorem 3.3, we get ([24], Theorem 7).*

Corollary 3.5. *Under the assumption of Theorem 3.3 taking $\mu \longrightarrow 0^+$, we get*

$$\left| H_{f,\Lambda^*}(q_1, q_2) \right| \le \frac{\zeta(q_2, q_1)}{2\Lambda^*(1)} \sqrt[p]{K_{\Lambda^*}(p)} \sqrt[q]{H_1|f'(q_1)|^q + H_2|f'(q_2)|^q}. \tag{3.4}$$

Corollary 3.6. *Taking $p = q = 2$ in Theorem 3.3, we get*

$$\left|H_{f,\Lambda^*}(q_1, q_2)\right| \leq \frac{\zeta(q_2, q_1)}{2\Lambda^*(1)} \sqrt{K_{\Lambda^*}(2)} \sqrt{H_1|f'(q_1)|^2 + H_2|f'(q_2)|^2 - \frac{\mu}{6}\zeta^\sigma(q_2, q_1)}. \tag{3.5}$$

Corollary 3.7. *Taking $\mu \longrightarrow 0^+$, where $\zeta(q_2, q_1) = q_2 - q_1$, $h_1(t) = 1 - t$, $h_2(t) = t$ and $\varphi(t) = t$ in Theorem 3.3, we have ([3], Theorem 2.3).*

Corollary 3.8. *Taking $\mu \longrightarrow 0^+$, where $\zeta(q_2, q_1) = q_2 - q_1$, $h_1(t) = 1 - t$, $h_2(t) = t$ and $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$ in Theorem 3.3, we get ([19], Theorem 8).*

Corollary 3.9. *Taking $\mu \longrightarrow 0^+$, where $\zeta(q_2, q_1) = q_2 - q_1$, $h_1(t) = 1 - t$, $h_2(t) = t$ and $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Theorem 3.3, we obtain ([5], Theorem 8).*

Corollary 3.10. *Taking $\zeta(q_2, q_1) = q_2 - q_1$, $h_1(t) = 1 - t$, $h_2(t) = t$, where $\varphi(t) = t(q_1 + \zeta(q_2, q_1) - t)^{\alpha-1}$ and $f(x)$ is symmetric to $x = q_1 + \frac{\zeta(q_2, q_1)}{2}$, in Theorem 3.3, we get*

$$\left|H_{f,\Lambda_4^*}(q_1, q_2)\right| \leq \frac{\sqrt[q]{\frac{\zeta(q_2, q_1)}{2}}}{\sqrt[p]{p\alpha + 1}\left[(q_1 + \zeta(q_2, q_1))^\alpha - q_1^\alpha\right]} \tag{3.6}$$

$$\times \sqrt[p]{q_1^{p\alpha+1} + (q_1 + \zeta(q_2, q_1))^{p\alpha+1} - \frac{(2q_1 + \zeta(q_2, q_1))^{p\alpha+1}}{2^{p\alpha}}}$$

$$\times \sqrt[q]{\frac{|f'(q_1)|^q + |f'(q_2)|^q}{2} - \frac{\mu}{6}\zeta^\sigma(q_2, q_1)}.$$

Theorem 3.11. *Let $f : P \longrightarrow \mathbb{R}$ be a differentiable mapping on P°. If $|f'|^q$ is a strongly (h_1, h_2)-preinvex mapping of order $\sigma > 0$ with modulus $\mu > 0$ on P, then for $q \geq 1$, the following inequality for generalized fractional integrals hold:*

$$\left|H_{f,\Lambda^*}(q_1, q_2)\right| \leq \frac{\zeta(q_2, q_1)}{2\Lambda^*(1)}\left[K_{\Lambda^*}(1)\right]^{1-\frac{1}{q}} \tag{3.7}$$

$$\times \sqrt[q]{K_{\Lambda^*,h_1}|f'(q_1)|^q + K_{\Lambda^*,h_2}|f'(q_2)|^q - \mu U_{\Lambda^*}\zeta^\sigma(q_2, q_1)},$$

where

$$K_{\Lambda^*,h_i} = \int_0^1 \left|\Lambda^*(1-t) - \Lambda^*(t)\right|h_i(1-t)dt, \quad \forall\, i = 1, 2,$$

$$U_{\Lambda^*} = \int_0^1 \left|\Lambda^*(1-t) - \Lambda^*(t)\right|t(1-t)dt$$

and $K_{\Lambda^*}(1)$ is defined in Theorem 3.3 for value $p = 1$.

Proof . From Lemma 3.1, strongly (h_1, h_2)-preinvexity of order $\sigma > 0$ with modulus $\mu > 0$ of mapping $|f'|^q$, the well–known power mean inequality and properties of the modulus, we have

$$\left|H_{f,\Lambda^*}(q_1, q_2)\right| \leq \frac{\zeta(q_2, q_1)}{2\Lambda^*(1)}\int_0^1 \left|\Lambda^*(1-t) - \Lambda^*(t)\right|\left|f'(q_1 + (1-t)\zeta(q_2, q_1))\right|dt$$

$$\leq \frac{\zeta(q_2,q_1)}{2\Lambda^*(1)} \left(\int_0^1 \left| \Lambda^*(1-t) - \Lambda^*(t) \right| dt \right)^{1-\frac{1}{q}}$$

$$\times \left(\int_0^1 \left| \Lambda^*(1-t) - \Lambda^*(t) \right| \left| f'(q_1 + (1-t)\zeta(q_2,q_1)) \right|^q dt \right)^{\frac{1}{q}}$$

$$\leq \frac{\zeta(q_2,q_1)}{2\Lambda^*(1)} \left[K_{\Lambda^*}(1) \right]^{1-\frac{1}{q}}$$

$$\times \left(\int_0^1 \left| \Lambda^*(1-t) - \Lambda^*(t) \right| \left(h_1(1-t) \left| f'(q_1) \right|^q + h_2(1-t) \left| f'(q_2) \right|^q - \frac{\mu}{6} \zeta^\sigma(q_2,q_1) \right) dt \right)^{\frac{1}{q}}$$

$$= \frac{\zeta(q_2,q_1)}{2\Lambda^*(1)} \left[K_{\Lambda^*}(1) \right]^{1-\frac{1}{q}} \sqrt[q]{K_{\Lambda^*,h_1} |f'(q_1)|^q + K_{\Lambda^*,h_2} |f'(q_2)|^q - \mu U_{\Lambda^*} \zeta^\sigma(q_2,q_1)}.$$

The proof of Theorem 3.11 is completed. □
We point out some special cases of Theorem 3.11.

Corollary 3.12. *Under the assumption of Theorem 3.11 taking $\mu \longrightarrow 0^+$, we get*

$$\left| H_{f,\Lambda^*}(q_1,q_2) \right| \leq \frac{\zeta(q_2,q_1)}{2\Lambda^*(1)} \left[K_{\Lambda^*}(1) \right]^{1-\frac{1}{q}} \sqrt[q]{K_{\Lambda^*,h_1} |f'(q_1)|^q + K_{\Lambda^*,h_2} |f'(q_2)|^q}. \tag{3.8}$$

Corollary 3.13. *Taking $q = 1$ in Theorem 3.11, we get*

$$\left| H_{f,\Lambda^*}(q_1,q_2) \right| \leq \frac{\zeta(q_2,q_1)}{2\Lambda^*(1)} \left[K_{\Lambda^*,h_1} |f'(q_1)| + K_{\Lambda^*,h_2} |f'(q_2)| - \mu U_{\Lambda^*} \zeta^\sigma(q_2,q_1) \right]. \tag{3.9}$$

If we fixed, respectively, $h_1(t) = t$, $h_2(t) = 1 - t$, then from Corollary 3.12, we have the following results.

Corollary 3.14. *Taking $\varphi(t) = t$, we have*

$$\left| H_{f,\Lambda_1^*}(q_1,q_2) \right| \leq \frac{\zeta(q_2,q_1)}{2^{2+\frac{1}{q}}} \sqrt[q]{|f'(q_1)|^q + |f'(q_2)|^q}. \tag{3.10}$$

Corollary 3.15. *Taking $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$, we obtain*

$$\left| H_{f,\Lambda_2^*}(q_1,q_2) \right| \leq \left(\frac{2^\alpha - 1}{2^{\alpha+1}} \right) \sqrt[q]{\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+2)}} \zeta(q_2,q_1) \sqrt[q]{|f'(q_1)|^q + |f'(q_2)|^q}. \tag{3.11}$$

Corollary 3.16. *Taking $\varphi(t) = \frac{t^{\frac{\alpha}{k_1}}}{k_1 \Gamma_{k_1}(\alpha)}$, we get*

$$\left| H_{f,\Lambda_3^*}(q_1,q_2) \right| \leq \left(\frac{2^{\frac{\alpha}{k_1}} - 1}{2^{\frac{\alpha}{k_1}+1}} \right) \sqrt[q]{\frac{\Gamma_{k_1}(\alpha+k_1)}{\Gamma_{k_1}(\alpha+k_1+1)}} \zeta(q_2,q_1) \sqrt[q]{|f'(q_1)|^q + |f'(q_2)|^q}. \tag{3.12}$$

Corollary 3.17. *Taking* $\varphi(t) = t(q_1 + \zeta(q_2, q_1) - t)^{\alpha-1}$ *and* $f(x)$ *is symmetric to* $x = q_1 + \frac{\zeta(q_2,q_1)}{2}$, *we have*

$$\left| H_{f,\Lambda_4^*}(q_1, q_2) \right| \le \frac{\zeta(q_2, q_1)}{2\overline{\Lambda^*(1)}} \left[\overline{K_{\Lambda^*}(1)} \right]^{1-\frac{1}{q}} \sqrt[q]{\overline{K_{\Lambda^*}}} \sqrt[q]{|f'(q_1)|^q + |f'(q_2)|^q}, \tag{3.13}$$

where

$$\overline{\Lambda^*(1)} = \frac{(q_1 + \zeta(q_2, q_1))^\alpha}{\alpha},$$

$$\overline{K_{\Lambda^*}(1)} = \frac{2}{\alpha} \left[(q_1 + \zeta(q_2, q_1))^{\alpha+1} - 2\left(q_1 + \frac{\zeta(q_2, q_1)}{2} \right)^{\alpha+1} + q_1^{\alpha+1} \right],$$

$$\overline{K_{\Lambda^*}} = \frac{1}{\alpha} \left[F_{11} - F_{12} + F_{21} - F_{22} \right]$$

and

$$F_{11} = \frac{1}{\zeta^2(q_2, q_1)} \left\{ \frac{(q_1 + \zeta(q_2, q_1))}{\alpha + 1} \left[(q_1 + \zeta(q_2, q_1))^{\alpha+1} - \left(q_1 + \frac{\zeta(q_2, q_1)}{2} \right)^{\alpha+1} \right] \right.$$
$$\left. - \frac{1}{\alpha + 2} \left[(q_1 + \zeta(q_2, q_1))^{\alpha+2} - \left(q_1 + \frac{\zeta(q_2, q_1)}{2} \right)^{\alpha+2} \right] \right\},$$

$$F_{12} = \frac{1}{\zeta^2(q_2, q_1)} \left\{ \frac{1}{\alpha + 2} \left[\left(q_1 + \frac{\zeta(q_2, q_1)}{2} \right)^{\alpha+2} - q_1^{\alpha+2} \right] \right.$$
$$\left. - \frac{q_1}{\alpha + 1} \left[\left(q_1 + \frac{\zeta(q_2, q_1)}{2} \right)^{\alpha+1} - q_1^{\alpha+1} \right] \right\},$$

$$F_{21} = \frac{1}{\zeta^2(q_2, q_1)} \left\{ \frac{1}{\alpha + 2} \left[(q_1 + \zeta(q_2, q_1))^{\alpha+2} - \left(q_1 + \frac{\zeta(q_2, q_1)}{2} \right)^{\alpha+2} \right] \right.$$
$$\left. - \frac{q_1}{\alpha + 1} \left[(q_1 + \zeta(q_2, q_1))^{\alpha+1} - \left(q_1 + \frac{\zeta(q_2, q_1)}{2} \right)^{\alpha+1} \right] \right\},$$

$$F_{22} = \frac{1}{\zeta^2(q_2, q_1)} \left\{ \frac{(q_1 + \zeta(q_2, q_1))}{\alpha + 1} \left[\left(q_1 + \frac{\zeta(q_2, q_1)}{2} \right)^{\alpha+1} - q_1^{\alpha+1} \right] \right.$$
$$\left. - \frac{1}{\alpha + 2} \left[(q_1 + \zeta(q_2, q_1))^{\alpha+2} - q_1^{\alpha+2} \right] \right\}.$$

Remark 3.18. *Applying our Theorems 3.3 and 3.11 for appropriate choices of function* $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$, $\frac{t^{\frac{\alpha}{k_1}}}{k_1 \Gamma_{k_1}(\alpha)}$; $\varphi(t) = t(q_1 + \zeta(q_2, q_1) - t)^{\alpha-1}$, *where* $f(x)$ *is symmetric to* $x = q_1 + \frac{\zeta(q_2,q_1)}{2}$, *and* $\varphi(t) = \frac{t}{\alpha} \exp\left[\left(-\frac{1-\alpha}{\alpha} \right) t \right]$ *for* $\alpha \in (0, 1)$, *for suitable choices of functions* $h_1(t)$ *and* $h_2(t)$, *for example:* 1; t^s; t^{-s}; $(1-t)^s$; $(1-t)^{-s}$; $t(1-t)$; $\frac{\sqrt{t}}{2\sqrt{1-t}}$; $\frac{\sqrt{1-t}}{2\sqrt{t}}$, *such that* $|f'|^q$ *to be strongly* (h_1, h_2)*-preinvex mapping of order* $\sigma > 0$ *with modulus* $\mu > 0$, *we can deduce some new general fractional integral inequalities using special means. Also, if we choose* $\zeta(q_2, q_1) = q_2 - q_1$, *we can establish some new fascinating general fractional integral inequalities for strongly* (h_1, h_2)*-convex functions of order* $\sigma > 0$ *with modulus* $\mu > 0$ *using special means. Finally, taking* $\mu \longrightarrow 0^+$, *we can obtain some new general fractional integral inequalities for* (h_1, h_2)*-preinvex mappings. We omit their proofs and the details are left to the interested readers.*

4. Applications

Consider the following special means for different real numbers q_1, q_2 and $q_1 q_2 \neq 0$, as follows:

1. The arithmetic mean:
$$A(q_1, q_2) = \frac{q_1 + q_2}{2},$$

2. The harmonic mean:
$$H(q_1, q_2) = \frac{2}{\frac{1}{q_1} + \frac{1}{q_2}},$$

3. The logarithmic mean:
$$L(q_1, q_2) = \frac{q_2 - q_1}{\ln |q_2| - \ln |q_1|},$$

4. The generalized log–mean:
$$L_n(q_1, q_2) = \left[\frac{q_2^{n+1} - q_1^{n+1}}{(n+1)(q_2 - q_1)} \right]^{\frac{1}{n}}; \quad n \in \mathbb{Z} \setminus \{-1, 0\}.$$

Now, using the theory results in Section 3, we give some applications to special means for different real numbers.

Proposition 4.1. Let $q_1, q_2 \in \mathbb{R} \setminus \{0\}$, where $q_1 < q_2$ and $\zeta(q_2, q_1) > 0$, Then, for $n \in \mathbb{Z} \setminus \{-1, 0\}$, where $q > 1$ and $p^{-1} + q^{-1} = 1$, the following inequality hold:

$$\left| A\left(q_1^n, (q_1 + \zeta(q_2, q_1))^n\right) - L_n\left(q_1, q_1 + \zeta(q_2, q_1)\right) \right| \leq \frac{|n|\zeta(q_2, q_1)}{2\sqrt[p]{p+1}} \tag{4.1}$$

$$\times \sqrt[q]{A\left(|q_1|^{q(n-1)}, |q_2|^{q(n-1)}\right)}.$$

Proof. Taking $\mu \longrightarrow 0^+$ and applying Theorem 3.3 for $f(x) = x^n$, $h_1(t) = t$, $h_2(t) = 1 - t$ and $\varphi(t) = t$, one can obtain the result immediately. $\square$

Proposition 4.2. Let $q_1, q_2 \in \mathbb{R} \setminus \{0\}$, where $q_1 < q_2$ and $\zeta(q_2, q_1) > 0$, Then, for $q > 1$ and $p^{-1} + q^{-1} = 1$, the following inequality hold:

$$\left| \frac{1}{H\left(q_1, q_1 + \zeta(q_2, q_1)\right)} - \frac{1}{L\left(q_1, q_1 + \zeta(q_2, q_1)\right)} \right| \leq \frac{\zeta(q_2, q_1)}{2\sqrt[p]{p+1}} \frac{1}{\sqrt[q]{H\left(q_1^{2q}, q_2^{2q}\right)}}. \tag{4.2}$$

Proof. Taking $\mu \longrightarrow 0^+$ and applying Theorem 3.3 for $f(x) = \frac{1}{x}$, $h_1(t) = t$, $h_2(t) = 1 - t$ and $\varphi(t) = t$, one can obtain the result immediately. $\square$

Proposition 4.3. Let $q_1, q_2 \in \mathbb{R} \setminus \{0\}$, where $q_1 < q_2$ and $\zeta(q_2, q_1) > 0$, Then, for $n \in \mathbb{Z} \setminus \{-1, 0\}$ and $q \geq 1$, the following inequality hold:

$$\left| A\left(q_1^n, (q_1 + \zeta(q_2, q_1))^n\right) - L_n\left(q_1, q_1 + \zeta(q_2, q_1)\right) \right| \leq \frac{|n|\zeta(q_2, q_1)}{4\sqrt[q]{2}} \tag{4.3}$$

$$\times \sqrt[q]{A\left(|q_1|^{q(n-1)}, |q_2|^{q(n-1)}\right)}.$$

Proof . Taking $\mu \longrightarrow 0^+$ and applying Theorem 3.11 for $f(x) = x^n$, $h_1(t) = t$, $h_2(t) = 1 - t$ and $\varphi(t) = t$, one can obtain the result immediately. $\square$

Proposition 4.4. *Let $q_1, q_2 \in \mathbb{R} \setminus \{0\}$, where $q_1 < q_2$ and $\zeta(q_2, q_1) > 0$, Then for $q \geq 1$, the following inequality hold:*

$$\left| \frac{1}{H(q_1, q_1 + \zeta(q_2, q_1))} - \frac{1}{L(q_1, q_1 + \zeta(q_2, q_1))} \right| \leq \frac{\zeta(q_2, q_1)}{4\sqrt[q]{2}} \frac{1}{\sqrt[q]{H(q_1^{2q}, q_2^{2q})}}. \tag{4.4}$$

Proof . Taking $\mu \longrightarrow 0^+$ and applying Theorem 3.11 for $f(x) = \dfrac{1}{x}$, $h_1(t) = t$, $h_2(t) = 1 - t$ and $\varphi(t) = t$, one can obtain the result immediately. $\square$

Next, we provide some new error estimates for the trapezoidal formula.

Let Q be the partition of the points $q_1 = x_0 < x_1 < \ldots < x_n = q_2$ of the interval $[q_1, q_2]$. Let consider the following quadrature formula:

$$\int_{q_1}^{q_2} f(x) dx = T(f, Q) + E(f, Q),$$

where

$$T(f, Q) = \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2} (x_{i+1} - x_i)$$

is the trapezoidal version and $E(f, Q)$ is denote their associated approximation error.

Proposition 4.5. *Let $f : [q_1, q_2] \longrightarrow \mathbb{R}$ be a differentiable function on (q_1, q_2), where $q_1 < q_2$. If $|f'|^q$ is convex on $[q_1, q_2]$ for $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality hold:*

$$\left| E(f, Q) \right| \leq \frac{1}{2\sqrt[q]{2}\sqrt[p]{p+1}} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2 \sqrt[q]{|f'(x_i)|^q + |f'(x_{i+1})|^q}. \tag{4.5}$$

Proof . Taking $\mu \longrightarrow 0^+$ and applying Theorem 3.3 for $\zeta(q_2, q_1) = q_2 - q_1$, $h_1(t) = t$, $h_2(t) = 1 - t$ and $\varphi(t) = t$ on the subintervals $[x_i, x_{i+1}]$ $(i = 0, \ldots, n - 1)$ of the partition Q, we have

$$\left| \frac{f(x_i) + f(x_{i+1})}{2} - \frac{1}{x_{i+1} - x_i} \int_{x_i}^{x_{i+1}} f(x) dx \right|$$

$$\leq \frac{(x_{i+1} - x_i)}{2\sqrt[p]{p+1}} \left[\frac{|f'(x_i)|^q + |f'(x_{i+1})|^q}{2} \right]^{\frac{1}{q}}. \tag{4.6}$$

Hence from (4.6), we get

$$\left| E(f, Q) \right| = \left| \int_{q_1}^{q_2} f(x) dx - T(f, Q) \right| \leq \left| \sum_{i=0}^{n-1} \left\{ \int_{x_i}^{x_{i+1}} f(x) dx - \frac{f(x_i) + f(x_{i+1})}{2} (x_{i+1} - x_i) \right\} \right|$$

$$\leq \sum_{i=0}^{n-1} \left| \left\{ \int_{x_i}^{x_{i+1}} f(x) dx - \frac{f(x_i) + f(x_{i+1})}{2} (x_{i+1} - x_i) \right\} \right|$$

$$\leq \frac{1}{2\sqrt[q]{2}\sqrt[p]{p+1}} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2 \sqrt[q]{|f'(x_i)|^q + |f'(x_{i+1})|^q}.$$

The proof of Proposition 4.5 is completed. $\square$

Proposition 4.6. *Let $f : [q_1, q_2] \longrightarrow \mathbb{R}$ be a differentiable function on (q_1, q_2), where $q_1 < q_2$. If $|f'|^q$ is convex on $[q_1, q_2]$ for $q \geq 1$, then the following inequality holds:*

$$\left| E(f, Q) \right| \leq \frac{1}{4\sqrt[q]{2}} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2 \sqrt[q]{|f'(x_i)|^q + |f'(x_{i+1})|^q}. \tag{4.7}$$

Proof . The proof is analogous as to that of Proposition 4.5 but use Theorem 3.11 taking $\mu \longrightarrow 0^+$ for $\zeta(q_2, q_1) = q_2 - q_1$, $h_1(t) = t$, $h_2(t) = 1 - t$ and $\varphi(t) = t$. $\square$

Remark 4.7. *From Remark 3.18 such that $|f'|^q$ to be strongly (h_1, h_2)-convex function of order $\sigma > 0$ with modulus $\mu > 0$, we can provide some new error estimates for the trapezoidal formula using ideas and techniques of Propositions 4.5 and 4.6. We omit their proofs and the details are left to the interested reader.*

Remark 4.8. *The error estimates for the trapezoidal quadrature rules in (4.5) and (4.7) are given assuming some differentiability properties of the function, but it depends on the choice of nodes. Some numerical experiments to compare with standard estimates of the trapezoidal rule would be needed.*

5. Conclusion

Since convex functions has large applications in many mathematical areas, this new class of functions called strongly (h_1, h_2)-preinvex of order $\sigma > 0$ with modulus $\mu > 0$ can be applied to obtain several results in convex analysis, special functions, quantum mechanics, related optimization theory, mathematical inequalities and may stimulate further research in different areas of pure and applied sciences. The error estimates for the trapezoidal quadrature rules in (4.5) and (4.7) are given assuming some differentiability properties of the function, but it depends on the choice of nodes. Some numerical experiments to compare with standard estimates of the trapezoidal rule would be needed. Therefore, future research and other projects will explain this problem much better.

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