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On a class of nonlinear parabolic equations with natural growth in non-reflexive Musielak spaces

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Abstract

An existence result of renormalized solutions for nonlinear parabolic Cauchy-Dirichlet problems whose model

$$\begin{cases} \dfrac{\partial b(x,u)}{\partial t} - \operatorname{div} \mathcal{A}(x,t,u,\nabla u) - \operatorname{div} \Phi(x,t,u) = f & \text{in } \Omega \times (0,T) \\ b(x,u)(t=0) = b(x,u_0) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \times (0,T). \end{cases}$$

is given in the non reflexive Musielak spaces, where $b(x,\cdot)$ is a strictly increasing C^1-function for every $x \in \Omega$ with $b(x,0) = 0$, the lower order term Φ is a non coercive Carathéodory function satisfying only a natural growth condition described by the appropriate Musielak function φ and f is an integrable data.

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1. Introduction

Modular spaces are the adequate setting to model many physical problems, the more general structures are Musielak spaces which generalize classical Sobolev spaces, exponent variable spaces and Orlicz spaces. Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, $\Omega_T = \Omega \times (0,T)$ where T is a positive real number and φ is a Musielak function. Let $\mathrm{A}(u) := -\operatorname{div} \mathcal{A}(x,t,u,\nabla u)$ be a so-called

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Leray-Lions type operator whose prototype is the p-Laplacian operator and $b : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $b(x, \cdot)$ is a strictly increasing C^1-function for any fixed $x \in \Omega$ with $b(x, 0) = 0$.

Consider the following Cauchy-Dirichlet boundary value parabolic problem

$$\begin{cases} \dfrac{\partial b(x,u)}{\partial t} + \mathrm{A}(u) - \operatorname{div} \Phi(x,t,u) = f & \text{in } \Omega_T \\ b(x,u)(t=0) = b(x,u_0) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{1.1}$$

where $u_0 \in L^1(\Omega)$, $f \in L^1(\Omega_T)$.

The problem (1.1) has been studied in different particular cases, we recall some contributions in this directions. In the classical Sobolev spaces, for $\Phi \equiv 0$, b is a maximal monotone graph on $\mathbb{R}$ and $\mathcal{A}(x,t,s,\xi)$ is independent of s, existence and uniqueness of a renormalized solution have been proved by Blanchard and Murat in [18] and by Blanchard and Porretta in the case where $\mathcal{A}(x,t,s,\xi)$ is independent of t in [20]. In [1], Bennouna et al. have studied problem (1.1) for a measure $\mu = f - \operatorname{div}(F)$, with $f \in L^1(\Omega_T)$, $F \in (L^{p'}(\Omega_T))^N$ and Φ satisfies the condition

$$|\Phi(x,t,s)| \leq c(x,t)|s|^\gamma,$$

with $c(x,t) \in L^\tau(\Omega_T)$ for some $\tau = \dfrac{N+p}{p-1}$ and $\gamma = \dfrac{N+2}{N+p}(p-1)$. A renormalized solution to the elliptic case has been rigourously studied by Dal Maso et al. in [25] for a general measure data f.

In Orlicz spaces, Azroul et al. have proved in [13] existence of renormalized solution, where Φ depends only on u (without dependence on x) and $b(x,u) = b(u)$, the same result has been given by Redwane in [40] where $b(x,u)$ depends on x and u. Then, Moussa and Rhoudaf [38] have studied existence of renormalized solution for problem (1.1) in the case $f \in L^1(\Omega_T)$ under a growth condition on Φ prescribed by an N-function P that increases essentially less rapidly than the Orlicz function M defining the framework spaces,

$$|\Phi(x,t,s)| \leq \overline{P}^{-1}(P(|s|)) \text{ with } P \prec\prec M. \tag{1.2}$$

The previous result has been enhanced in [22] under the likely growth condition in the elliptic case,

$$|\Phi(x,s)| \leq \gamma(x) + \overline{M}^{-1}(M(|s|)), \text{ with } \gamma \in E_{\overline{M}}(\Omega). \tag{1.3}$$

In Musielak spaces, for $b(x,u) = u$, an existence and uniqueness results were given in [2] under the more restrictive assumption

$$|\Phi(x,t,s)| \leq \gamma(x,t)\overline{\varphi}_x^{-1}(\varphi(x, \frac{\alpha_0}{\delta}|s|)) \text{ and } \|\gamma\|_{L^\infty(\Omega_T)} < \frac{\alpha}{\alpha_0+1}, \tag{1.4}$$

with $0 < \alpha_0 < 1$, where δ is the constant in the integral Poincaré type inequality and α is the constant of coercivity of the problem. An existence result of entropy solution, for the elliptic case, has been given in [23].

The approach of this paper is how to deal with the existence of renormalized solutions for problem (1.1) in Musielak spaces where Φ satisfies only the natural growth condition

$$|\Phi(x,t,s)| \leq \gamma(x,t) + \overline{\varphi}_x^{-1}(\varphi(x,|s|)), \text{ where } \gamma \in E_{\overline{\varphi}}(\Omega_T). \tag{1.5}$$

without assuming any restriction on the Musielak function φ neither on its complementary $\overline{\varphi}$, the described problem lives in non reflexive Musielak spaces. We avoid to use the concept of Musielak function grows essentially more slowly than another, we use a more technical method unlike as in [38, 2].

In dealing with this problem, we have encountered some difficulties, essentially, under the natural growth assumption (1.5), it's difficult to prove existence of solution for the regularized problem and proving its convergence, which are the basic results in the proof of such solutions. The improvement in the main proofs follows thanks to an algebraic trick combined with the convexity of φ and Young's inequality on a well-chosen positive quantities. Also, we use some new results of the Log-Hölder continuity restriction on the modular function φ.

This article is organized as follows, in section 2, we recall some well-known preliminaries, results and properties of Musielak-Orlicz-Sobolev spaces and inhomogeneous Musielak-Orlicz-Sobolev spaces. Section 3 is devoted to basic assumptions, problem setting and the proof of the main result.

2. Preliminaries

In this section we list briefly some definitions and facts about Musielak-Orlicz-Sobolev spaces. For further definitions and properties we refer the reader to [35, 15, 39].

2.1. Musielak-Orlicz function

Let Ω be an open subset of $\mathbb{R}^N$ and let φ be real-valued function defined in $\Omega \times \mathbb{R}_+$ and satisfying the following conditions

(a) $\varphi(x, .)$ is an N-function, i.e., convex, nondecreasing, continuous, $\varphi(x, 0) = 0$, $\varphi(x, t) > 0$ for all $t > 0$ and

$$\lim_{t\to 0}\sup_{x\in\Omega}\frac{\varphi(x,t)}{t} = 0 \quad \text{for almost all } x \in \Omega,$$

$$\lim_{t\to \infty}\inf_{x\in\Omega}\frac{\varphi(x,t)}{t} = \infty \quad \text{for almost all } x \in \Omega.$$

(b) $\varphi(., t)$ is a measurable function.

A function $\varphi(x, t)$, which satisfies the condition (a) and (b), is called a Musielak-Orlicz function. For a Musielak-Orlicz function $\varphi(x, t)$ we put $\varphi_x(t) = \varphi(x, t)$ and we associate its nonnegative reciprocal function with respect to t and φ_x^{-1} that is,

$$\varphi_x^{-1}(\varphi(x,t)) = \varphi(x, \varphi_x^{-1}(t)) = t.$$

For any two Musielak-Orlicz functions φ and γ we introduce the following ordering:

(c) If there exists two positive constants c and T such that for almost all $x \in \Omega$

$$\varphi(x,t) \le \gamma(x, ct) \text{ for } t \ge T,$$

then we write $\varphi \prec \gamma$ and we say that γ dominates φ globally if $T = 0$ and near infinity if $T > 0$.

(d) If for every positive constant c and almost everywhere $x \in \Omega$ we have

$$\lim_{t\to 0}\left(\sup_{x\in\Omega}\frac{\varphi(x,ct)}{\gamma(x,t)}\right) = 0 \quad or \quad \lim_{t\to \infty}\left(\sup_{x\in\Omega}\frac{\varphi(x,ct)}{\gamma(x,t)}\right) = 0,$$

then we write $\varphi \prec\prec \gamma$ at 0 or near ∞ respectively, and we say that φ increases essentially more slowly than γ at 0 or near ∞ respectively.

We recall that the Musielak function φ is said to satisfy the Δ_2-condition (or doubling condition) if for some $k > 0$, and a non-negative function c, integrable on Ω, we have

$$\varphi(x, 2t) \leq k\varphi(x, t) + c(x) \text{ for all } x \in \Omega \text{ and all } t \geq 0.$$

Remark 2.1. *[26, 10] If $\gamma \prec\prec \varphi$, then for all $\epsilon > 0$ there exists a constant $k(\epsilon)$ such that:*

$$\gamma(x, t) \leq k(\epsilon)\varphi(x, \epsilon t) \textit{ for all } t \geq 0 \textit{ and a.e } x \in \Omega.$$

Example 2.2. *We give some examples of Musielak-Orlicz functions:*

1. $\varphi(x, t) = \varphi(t)$, *(classical Orlicz spaces),*
2. $\varphi(x, t) = t^{p(x)}$, *such that* $\sup\limits_{x\in\Omega} p(x) < \infty$ *(variable exponent Lebesgue spaces),*
3. $\varphi(x, t) = t^{p(x)} \log(1 + t)$,
4. $\varphi(x, t) = t(\log(1 + t))^{p(x)}$,
5. $\varphi(x, t) = (\exp(t))^{p(x)} - 1$.

2.2. Musielak-Orlicz-Sobolev spaces

For a Musielak function φ and a measurable function $u : \Omega \to \mathbb{R}$ we define the functional

$$\varrho_{\varphi,\Omega}(u) = \int_\Omega \varphi(x, |u(x)|)\, dx.$$

The set $K_\varphi(\Omega) = \Big\{u : \Omega \to \mathbb{R} \quad \textit{measurable} : \varrho_{\varphi,\Omega}(u) < \infty\Big\}$ is called the Musielak class (or the Musielak-Orlicz class or generalized Orlicz class). The Musielak space (or Musielak-Orlicz space or generalized Orlicz space) $L_\varphi(\Omega)$ is the vector space generated by $K_\varphi(\Omega)$, that is the smallest linear space containing the set $K_\varphi(\Omega)$. Equivalently

$$L_\varphi(\Omega) = \Big\{u : \Omega \to \mathbb{R} \quad \textit{measurable} : \varrho_{\varphi,\Omega}(\frac{u}{\lambda}) < \infty \quad \text{for some} \quad \lambda > 0\Big\}.$$

For a Musielak function φ we put

$$\overline{\varphi}(x, s) = \sup_{t\geq 0} \Big\{st - \varphi(x, t)\Big\}.$$

$\overline{\varphi}$ is called the Musielak function complementary to φ (or conjugate of φ) in the sense of Young with respect to s.
we say that a sequence of function $u_n \in L_\varphi(\Omega)$ is modular convergent to $u \in L_\varphi(\Omega)$ if there exists a constant $\lambda > 0$ such that

$$\lim_{n\to\infty} \varrho_{\varphi,\Omega}\Big(\frac{u_n - u}{\lambda}\Big) = 0.$$

In the space $L_\varphi(\Omega)$ we can define two norms, the first is called the Luxemburg norm, that is

$$\|u\|_{\varphi,\Omega} = \inf\Big\{\lambda > 0 : \int_\Omega \varphi(x, \frac{|u(x)|}{\lambda})\, dx \leq 1\Big\}$$

and the second so-called the Orlicz norm, that is

$$|||u|||_{\varphi,\Omega} = \sup_{\|v\|_{\overline{\varphi}} \leq 1} \int_\Omega |u(x)\, v(x)|\, dx,$$

where $\overline{\varphi}$ is the Musielak function complementary to φ. These two norms are equivalent and we have a Musielak class $K_\varphi(\Omega)$ is a convex subset of the Musielak space $L_\varphi(\Omega)$.
The closure in $L_\varphi(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_\varphi(\Omega)$. It is a separable space and $(E_{\overline{\varphi}}(\Omega))^* = L_\varphi(\Omega)$. We have $E_\varphi(\Omega) = K_\varphi(\Omega)$ if and only if $K_\varphi(\Omega) = L_\varphi(\Omega)$ if and only if φ satisfies the Δ_2-condition for large values of t or for all values of t, according to whether Ω has finite measure or not.
We define

$$W^1L_\varphi(\Omega) = \Big\{ u \in L_\varphi(\Omega) : D^\alpha u \in L_\varphi(\Omega), \forall |\alpha| \leq 1 \Big\}$$

$$W^1E_\varphi(\Omega) = \Big\{ u \in E_\varphi(\Omega) : D^\alpha u \in E_\varphi(\Omega), \forall |\alpha| \leq 1 \Big\}$$

where $\alpha = (\alpha_1, ..., \alpha_N)$, $|\alpha| = |\alpha_1| + ... + |\alpha_N|$ and $D^\alpha u$ denote the distributional derivatives. The space $W^1L_\varphi(\Omega)$ is called the Musielak-Sobolev space. For $u \in W^1L_\varphi(\Omega)$, let

$$\overline{\varrho}_{\varphi,\Omega}(u) = \sum_{|\alpha| \leq 1} \varrho_{\varphi,\Omega}(D^\alpha u) \text{ and } \|u\|^1_{\varphi,\Omega} = \inf \Big\{ \lambda > 0 : \overline{\varrho}_{\varphi,\Omega}(\frac{u}{\lambda}) \leq 1 \Big\}$$

these functionals are convex modular and a norm on $W^1L_\varphi(\Omega)$ respectively. The pair $\langle W^1L_\varphi(\Omega), \|u\|^1_{\varphi,\Omega}\rangle$ is a Banach space if φ satisfy the following condition

$$\text{there exists a constant } c > 0 \text{ such that } \inf_{x\in\Omega} \varphi(x,1) > c.$$

The space $W^1L_\varphi(\Omega)$ is identified to a subspace of the product $\Pi_{|\alpha|\leq 1} L_\varphi(\Omega) = \Pi L_\varphi$; this subspace is $\sigma(\Pi L_\varphi, \Pi E_{\overline{\varphi}})$ closed.

We denote by $\mathfrak{D}(\Omega)$ the Schwartz space of infinitely smooth function with compact support in Ω and by $\mathfrak{D}(\overline{\Omega})$ the restriction of $\mathfrak{D}(\mathbb{R}^N)$ on Ω. The space $W^1_0L_\varphi(\Omega)$ is defined as the $\sigma(\Pi L_\varphi, \Pi E_{\overline{\varphi}})$ closure of $\mathfrak{D}(\Omega)$ in $W^1L_\varphi(\Omega)$ and the space $W^1_0E_\varphi(\Omega)$ as the (norm) closure of the Schwarz space $\mathfrak{D}(\Omega)$ in $W^1L_\varphi(\Omega)$.

For two complementary Musielak functions φ and $\overline{\varphi}$ we have

i)The Young inequality:

$$ts \leq \varphi(x,t) + \overline{\varphi}(x,s) \text{ for all } t, s \geq 0, x \in \Omega.$$

ii) The Hölder inequality:

$$\Big| \int_\Omega u(x)\, v(x)\, dx \Big| \leq 2\|u\|_{\varphi,\Omega}\, \|v\|_{\overline{\varphi},\Omega}, \text{ for all } u \in L_\varphi(\Omega), v \in L_{\overline{\varphi}}(\Omega).$$

We say that a sequence of function u_n converges to u for the modular convergence in $W^1L_\varphi(\Omega)$ (respectively in $W^1_0L_\varphi(\Omega)$) if we have

$$\lim_{n\to\infty} \overline{\varrho}_{\varphi,\Omega}\Big(\frac{u_n - u}{\lambda}\Big) = 0, \text{ for some } \lambda > 0.$$

Define the following space of distributions

$$W^{-1}L_{\overline{\varphi}}(\Omega)=\Big\{f\in\mathfrak{D}'(\Omega):f=\sum_{|\alpha|\le 1}(-1)^{|\alpha|}D^{\alpha}f_{\alpha}\text{ where }f_{\alpha}\in L_{\overline{\varphi}}(\Omega)\Big\}$$

and

$$W^{-1}E_{\overline{\varphi}}(\Omega)=\Big\{f\in\mathfrak{D}'(\Omega):f=\sum_{|\alpha|\le 1}(-1)^{|\alpha|}D^{\alpha}f_{\alpha}\text{ where }f_{\alpha}\in E_{\overline{\varphi}}(\Omega)\Big\}.$$

2.3. Inhomogeneous Musielak-Orlicz-Sobolev spaces

Let Ω be a bounded open subset of $\mathbb{R}^N$, $T>0$ and set $\Omega_T=\Omega\times(0,T)$. For each $\alpha\in\mathbb{N}^N$, denote by D_x^{α} the distributional derivative on Ω_T of order α with respect to the variable $x\in\Omega$. The inhomogeneous Musielak-Orlicz-Sobolev spaces are defined as follows,

$$W^{1,x}L_{\varphi}(\Omega_T)=\Big\{u\in L_{\varphi}(\Omega_T):D_x^{\alpha}u\in L_{\varphi}(\Omega_T)\quad\text{for all}\quad|\alpha|\le 1\Big\},$$

and

$$W^{1,x}E_{\varphi}(\Omega_T)=\Big\{u\in E_{\varphi}(\Omega_T):D_x^{\alpha}u\in E_{\varphi}(\Omega_T)\quad\text{for all}\quad|\alpha|\le 1\Big\}.$$

The last space is a subspace of the first one, and both are Banach spaces under the norm,

$$\|u\|=\sum_{|\alpha|\le 1}\|D_x^{\alpha}u\|_{\varphi,\Omega_T}.$$

We can easily show that they form a complementary system when Ω satisfies the segment property. These spaces are considered as subspaces of the product space $\Pi L_{\varphi}(\Omega_T)$ which have as many copies as there is α-order derivatives, $|\alpha|\le 1$. We shall also consider the weak topologies $\sigma(\Pi L_{\varphi},\Pi E_{\overline{\varphi}})$ and $\sigma(\Pi L_{\varphi},\Pi L_{\overline{\varphi}})$). If $u\in W^{1,x}L_{\varphi}(\Omega_T)$ then the function : $t\mapsto u(t)=u(t,\cdot)$ is defined on $(0,T)$ with values in $W^1L_{\varphi}(\Omega)$. If, further, $u\in W^{1,x}E_{\varphi}(\Omega_T)$ then the concerned function is a $W^1E_{\varphi}(\Omega)$-valued and is strongly measurable. Furthermore the following imbedding holds: $W^{1,x}E_{\varphi}(\Omega_T)\subset L^1(0,T;W^1E_{\varphi}(\Omega))$. The space $W^{1,x}L_{\varphi}(\Omega_T)$ is not in general separable, if $u\in W^{1,x}L_{\varphi}(\Omega_T)$, we can not conclude that the function $u(t)$ is measurable on $(0,T)$. However, the scalar function $t\mapsto\| u(t)\|_{\varphi,\Omega}$ is in $L^1(0,T)$. The space $W_0^{1,x}E_{\varphi}(\Omega_T)$ is defined as the (norm) closure in $W^{1,x}E_{\varphi}(\Omega_T)$ of $D(\Omega_T)$. It is proved that when Ω has the segment property, then each element u of the closure of $D(\Omega_T)$ with respect of the weak* topology $\sigma(\Pi L_{\varphi},\Pi E_{\overline{\varphi}})$ is a limit, in $W^{1,x}L_{\varphi}(\Omega_T)$, of some subsequence $(u_n)\subset D(\Omega_T)$ for the modular convergence; i.e., if, for some $\lambda>0$, such that for all $|\alpha|\le 1$;

$$\int_{\Omega_T}\varphi\Big(x,\frac{|D_x^{\alpha}u_n-D_x^{\alpha}u|}{\lambda}\Big)dx\,dt\longrightarrow 0\quad\text{as }n\longrightarrow\infty.$$

This implies that (u_n) converges to u in $W^{1,x}L_{\varphi}(\Omega_T)$ for the weak topology $\sigma(\Pi L_{\varphi},\Pi L_{\overline{\varphi}})$. Consequently,

$$\overline{D(\Omega_T)}^{\sigma(\Pi L_{\varphi},\Pi E_{\overline{\varphi}})}=\overline{D(\Omega_T)}^{\sigma(\Pi L_{\varphi},\Pi L_{\overline{\varphi}})}.$$

This space will be denoted by $W_0^{1,x}L_{\varphi}(\Omega_T)$. Furthermore,

$$W_0^{1,x}E_{\varphi}(\Omega_T)=W_0^{1,x}L_{\varphi}(\Omega_T)\cap\Pi E_{\varphi}.$$

We have then the following complementary system

$$\Big(W_0^{1,x}L_{\varphi}(\Omega_T),F,W_0^{1,x}E_{\varphi}(\Omega_T),F_0\Big)$$

F being the dual space of $W_0^{1,x}E_\varphi(\Omega_T)$. It is also, except for an isomorphism, the quotient of $\Pi L_{\overline{\varphi}}$ by the polar set $W_0^{1,x}E_\varphi(\Omega_T)^\perp$, and will be denoted by $F = W^{-1,x}L_{\overline{\varphi}}(\Omega_T)$ and it is shown that,

$$W^{-1,x}L_{\overline{\varphi}}(\Omega_T) = \Big\{ f = \sum_{|\alpha|\leq 1} D_x^\alpha f_\alpha : f_\alpha \in L_{\overline{\varphi}}(\Omega_T) \Big\},$$

this space will be equipped with the usual quotient norm

$$\|f\| = \inf \sum_{|\alpha|\leq 1} \|f_\alpha\|_{\overline{\varphi},\Omega_T},$$

where the infimum is taken on all possible decompositions

$$f = \sum_{|\alpha|\leq 1} D_x^\alpha f_\alpha, f_\alpha \in L_{\overline{\varphi}}(\Omega_T).$$

The space F_0 is then given by,

$$W^{-1,x}L_{\overline{\varphi}}(\Omega_T) = \Big\{ f = \sum_{|\alpha|\leq 1} D_x^\alpha f_\alpha : f_\alpha \in E_{\overline{\varphi}}(\Omega_T) \Big\},$$

and is denoted by $F_0 = W^{-1,x}E_{\overline{\varphi}}(\Omega_T)$.

2.4. Some technical lemmas

Definition 2.3. *[32] Recall that an open domain $\Omega \subset \mathbb{R}^N$ has the segment property if there exist a locally finite open covering O_i of the boundary $\partial\Omega$ of Ω and a corresponding vectors y_i such that if $x \in \overline{\Omega} \cap O_i$ for some i, then $x + ty_i \in \Omega$ for $0 < t < 1$.*

Lemma 2.4. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ and let φ and $\overline{\varphi}$ be two complementary Musielak functions which satisfy the following conditions*

(i) *There exists a constant $c > 0$ such that $\inf_{x\in\Omega} \varphi(x, 1) \geq c$;*

(ii) *There exists a constant $A > 0$ such that for all $x, y \in \Omega$ with $|x - y| \leq \frac{1}{2}$,*

$$\frac{\varphi(x,t)}{\varphi(y,t)} \leq t^{\left(\frac{A}{\log(\frac{1}{|x-y|})}\right)} \text{ for all } \quad t \geq 1;$$

(iii) $\int_\Omega \varphi(x,\lambda)\, dx < \infty$, *for all $\lambda > 0$;*

(iv) *There exists a constant $C > 0$ such that $\overline{\varphi}(x, 1) \leq C$ a. e. in Ω.*

Under these assumptions, $\mathfrak{D}(\Omega)$ is dense in $L_\varphi(\Omega)$ with respect to the modular topology, $\mathfrak{D}(\Omega)$ is dense in $W_0^1L_\varphi(\Omega)$ for the modular convergence and $\mathfrak{D}(\overline{\Omega})$ is dense in $W^1L_\varphi(\Omega)$ for the modular convergence.

Proof . For the convenience to the reader, the new proof of this previews Lemma is given by Benkirane et al. in [8]. □ Consequently, the action of a distribution in $W^{-1}L_{\overline{\varphi}}(\Omega)$ on an element u of $W_0^1L_\varphi(\Omega)$ is well defined.

Lemma 2.5. *[14] Suppose that Ω satisfies the segment property and let $u \in W_0^1 L_\varphi(\Omega)$. Then, there exists a sequence $(u_n) \subset \mathfrak{D}(\Omega)$ such that $u_n \to u$ for the modular convergence in $W_0^1 L_\varphi(\Omega)$. Furthermore, if $u \in W_0^1 L_\varphi(\Omega) \cap L^\infty(\Omega)$ then*

$$\|u_n\|_\infty \leq \ (N+1)\|u\|_\infty.$$

Lemma 2.6. *[6, Lemma 1] If $u_n \to u$ for the modular convergence (with every $\lambda > 0$) in $L_\varphi(\Omega_T)$, then $u_n \to u$ strongly in $L_\varphi(\Omega_T)$.*

Lemma 2.7. *Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly lipschitzian, with $F(0) = 0$. Let φ be a Musielak function and let $u \in W^1 L_\varphi(\Omega)$ (resp. $W^1 E_\varphi(\Omega)$). Then, $F(u) \in W^1 L_\varphi(\Omega)$ (resp. $W^1 E_\varphi(\Omega)$). Moreover, if the set of discontinuity points D of F' is finite, then*

$$\frac{\partial}{\partial x_i} F(u) = \begin{cases} F'(u)\dfrac{\partial u}{\partial x_i} & a.e. \ in \quad \{x \in \Omega : u(x) \notin D\} \\ 0 & a.e. \ in \quad \{x \in \Omega : u(x) \in D\}. \end{cases}$$

Lemma 2.8. *[9]* ***(The Nemytskii operator)*** *Let Ω be an open subset of $\mathbb{R}^N$ with finite measure and let φ and ψ be two Musielak functions. Let $f : \Omega \times \mathbb{R}^{p_1} \to \mathbb{R}^{p_2}$ be a Caratheodory function such that*

$$|f(x,s)| \leq c(x) + k_1 \psi_x^{-1}(\varphi(x, k_2|s|)),$$

for almost every $x \in \Omega$ and all $s \in \mathbb{R}^{p_1}$, where k_1, k_2 are real positive constant and $c \in E_\psi(\Omega)$. Then the Nemytskii operator N_f, defined by $N_f(u)(x) = f(x, u(x))$ is continuous from $\Big(\mathbf{P}(E_\varphi(\Omega), \frac{1}{k_2})\Big)^{p_1} = \prod \Big\{ u \in L_\varphi(\Omega) : d(u, E_\varphi(\Omega)) < \frac{1}{k_2} \Big\}$ into $\Big(L_\psi(\Omega)\Big)^{p_2}$ for the modular convergence.
Furthermore, if $c \in E_\gamma(\Omega)$ and $\gamma \prec\prec \psi$ then N_f is strongly continuous from $\Big(\mathbf{P}(E_\varphi(\Omega), \frac{1}{k_2})\Big)^{p_1}$ into $\Big(E_\gamma(\Omega)\Big)^{p_2}$.

Lemma 2.9. *Let $u_k, u \in L_\varphi(\Omega)$. If $u_k \to u$ for the modular convergence, then $u_k \to u$ for $\sigma(L_\varphi, L_{\overline{\varphi}})$.*

Lemma 2.10. *[5] Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, satisfying the segment property, then*

$$\Big\{ u \in W_0^{1,x} L_\varphi(\Omega_T) : \frac{\partial u}{\partial t} \in W^{-1,x} L_{\overline{\varphi}}(\Omega_T) + L^1(\Omega_T) \Big\} \subset C\Big([0,T], L^1(\Omega)\Big).$$

Lemma 2.11. *Let $w_n, w \in L_\varphi(\Omega)$ and Let $v_n, v \in L_{\overline{\varphi}}(\Omega)$. If $w_n \to w$ and $v_n \to v$ modularly in $L_\varphi(\Omega)$ and $L_{\overline{\varphi}}(\Omega)$ respectively, then*

$$\lim_{n\to\infty} \int_\Omega w_n v_n \, dx = \int_\Omega wv \, dx.$$

Proof . Since $w_n \to w$ and $v_n \to v$ modularly in $L_\varphi(\Omega)$ and $L_{\overline{\varphi}}(\Omega)$ respectively, then let $\lambda, \mu > 0$ such that

$$\lim_{n\to\infty} \int_\Omega \varphi\Big(x, \frac{w_n - w}{\lambda}\Big)\, dx = 0 \quad \text{ and } \lim_{n\to\infty} \int_\Omega \overline{\varphi}\Big(x, \frac{v_n - v}{\mu}\Big)\, dx = 0.$$

On the other hand, note that

$$w_n v_n - wv = (w_n - w)(v_n - v) + w_n v + w v_n - 2wv.$$

By Young's inequality we get

$$\frac{1}{\lambda\mu}\Big|\int_\Omega (w_n v_n - wv)\,dx\Big| \le \int_\Omega \varphi\Big(x, \frac{w_n - w}{\lambda}\Big)\,dx + \int_\Omega \overline{\varphi}\Big(x, \frac{v_n - v}{\mu}\Big)\,dx + \frac{1}{\lambda\mu}\Big|\int_\Omega (w_n v + w v_n - 2wv)\,dx\Big|.$$

Passing to the limit as $n \to \infty$, the desired result follows. □

Lemma 2.12. *[7]**(Integral Poincaré's type inequality in Musielak spaces).** Let Ω be a bounded open subset of $\mathbb{R}^N$. Under the assumptions of lemma 2.4, and by assuming that $\varphi(x,t)$ depends only on $N-1$ coordinates of x, there exists a positive constant $\delta > 0$ which depends only on Ω such that*

$$\int_\Omega \varphi(x, |u(x)|)\,dx \le \int_\Omega \varphi(x, \delta|\nabla u(x)|)\,dx \quad \forall u \in W_0^1 L_\varphi(\Omega).$$

Lemma 2.13. *If $f_n \subset L^1(\Omega)$ with $f_n \to f \in L^1(\Omega)$ a. e. in Ω, $f_n, f \ge 0$ a. e. in Ω and $\int_\Omega f_n(x)\,dx \to \int_\Omega f(x)\,dx$, then $f_n \to f$ in $L^1(\Omega)$.*

3. Basic assumptions and main result

Through this paper Ω is a bounded open subset of $\mathbb{R}^N$, $N \ge 2$, $\Omega_T = \Omega \times (0,T)$ where T is a positive real number and φ is a Musielak function. Consider $b : \Omega \times \mathbb{R} \to \mathbb{R}$ a Carathéodory function such that for every $x \in \Omega$, $b(x,s)$ is a strictly increasing C^1-function with $b(x,0) = 0$ and for any $k > 0$, there exists $\lambda_k > 0$, a function $A_k \in L^\infty(\Omega)$ and a function $\widetilde{A}_k \in L_\varphi(\Omega)$ such that,

$$\lambda_k \le \frac{\partial b(x,s)}{\partial s} \le A_k(x) \quad \text{and} \quad \Big|\nabla_x\Big(\frac{\partial b(x,s)}{\partial s}\Big)\Big| \le \widetilde{A}_k(x). \tag{3.1}$$

Let $A : D(A) \subset W_0^{1,x} L_\varphi(\Omega_T) \to W^{-1,x} L_{\overline{\varphi}}(\Omega_T)$ be an operator of Leray-Lions type of the form:

$$\mathrm{A}u := -\mathrm{div}\, \mathcal{A}(x,t,u,\nabla u),$$

Our main goal in this study is to prove existence of renormalized solutions in the setting of Musielak spaces for the nonlinear parabolic problem

$$\begin{cases} \dfrac{\partial b(x,u)}{\partial t} - \mathrm{div}\, \mathcal{A}(x,t,u,\nabla u) - \mathrm{div}\, \Phi(x,t,u) = f & \text{in } \Omega_T \\ b(x,u)(t=0) = b(x,u_0) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{3.2}$$

where $\mathcal{A} : \Omega_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying, for almost every $(x,t) \in \Omega_T$ and for all $s \in \mathbb{R}, \xi, \eta \in \mathbb{R}^N (\xi \ne \eta)$ the following conditions

(H_1) There exists a function $c(x,t) \in E_{\overline{\varphi}}(\Omega_T)$ and some positive constants k_1, k_2 and a Musielak function $\psi \prec\prec \varphi$ such that

$$|\mathcal{A}(x,t,s,\xi)| \le c(x,t) + \overline{\varphi}_x^{-1}(\psi(x, k_1|s|)) + \overline{\varphi}_x^{-1}(\varphi(x, k_2|\xi|)).$$

(H_2) The vector $\mathcal{A}$ is strictly monotone

$$\Big(\mathcal{A}(x,t,s,\xi) - \mathcal{A}(x,t,s,\eta)\Big) \cdot \Big(\xi - \eta\Big) > 0.$$

(H_3) $\mathcal{A}$ is coercive, there exists a constant $\alpha > 0$ such that

$$\mathcal{A}(x,t,s,\xi)\cdot\xi \geq \alpha\varphi(x,|\xi|).$$

For the lower order term, we assume $\Phi : \Omega_T \times \mathbb{R} \to \mathbb{R}^N$ be a non coercive Carathéodory function satisfying a natural growth:

(H_4) For all $s \in \mathbb{R}$ and for almost every $x \in \Omega$,

$$|\Phi(x,t,s)| \leq \gamma(x,t) + \overline{\varphi}_x^{-1}(\varphi(x,|s|)), \text{ with } \gamma \in E_{\overline{\varphi}}(\Omega_T).$$

For that concerns the right hand, $f \in L^1(\Omega_T)$, $u_0 \in L^1(\Omega)$.

Lemma 3.1. *[38] Under assumptions (H_1)-(H_3), let (Z_n) be a sequence in $W_0^{1,x}L_\varphi(\Omega_T)$ such that*

$$Z_n \rightharpoonup Z \quad \text{in } W_0^{1,x}L_\varphi(\Omega_T) \text{ for } \sigma(\Pi L_\varphi(\Omega_T), \Pi E_{\overline{\varphi}}(\Omega_T)), \tag{3.3}$$

$$\Big(\mathcal{A}(x,t,Z_n,\nabla Z_n)\Big)_n \quad \text{is bounded in } \Big(L_{\overline{\varphi}}(\Omega_T)\Big)^N, \tag{3.4}$$

$$\lim_{n,s\to\infty}\int_{\Omega_T}\Big(\mathcal{A}(x,t,Z_n,\nabla Z_n) - \mathcal{A}(x,t,Z_n,\nabla Z\mathbf{1}_s)\Big)\cdot\Big(\nabla Z_n - \nabla Z\mathbf{1}_s\Big)dxdt = 0, \tag{3.5}$$

where $\mathbf{1}_s$ denotes the characteristic function of the set $\Omega_s = \Big\{x \in \Omega : |\nabla Z| \leq s\Big\}$. Then,

$$\nabla Z_n \to \nabla Z \quad a.e. \text{ in } \Omega_T, \tag{3.6}$$

$$\lim_{n\to\infty}\int_{\Omega_T}\mathcal{A}(x,t,Z_n,\nabla Z_n)\nabla Z_n\,dx\,dt = \int_{\Omega_T}\mathcal{A}(x,t,Z,\nabla Z)\nabla Z\,dx\,dt, \tag{3.7}$$

$$\varphi(x,|\nabla Z_n|) \longrightarrow \varphi(x,|\nabla Z|) \quad \text{in } L^1(\Omega_T). \tag{3.8}$$

In what follows, we will use the following real function of a real variable, called the truncation at height $k > 0$,

$$T_k(s) = \max\Big(-k, \min(k,s)\Big) = \begin{cases} s & \text{if } |s| \leq k \\ k\dfrac{s}{|s|} & \text{if } |s| > k. \end{cases}$$

Now, we give the definition of a renormalized solution for problem (3.2).

Definition 3.2. *A measurable function u defined on Ω_T is said a renormalized solution for problem (3.2) if*

$$T_k(u) \in W_0^{1,x}L_\varphi(\Omega_T) \quad \forall k \geq 0, \quad \text{and} \quad b(x,u) \in L^\infty(0,T,L^1(\Omega)), \tag{3.9}$$

$$\lim_{m\to\infty}\int_{\{m\leq|u(x,t)|\leq m+1\}}\mathcal{A}(x,t,u,\nabla u)\nabla u\,dxdt = 0, \tag{3.10}$$

and if, for every function r (renormalization) in $W^{1,\infty}(\mathbb{R})$ with compact support, we have

$$\begin{aligned}&\tfrac{\partial B_r(x,u)}{\partial t} - div\,(r(u)\mathcal{A}(x,t,u,\nabla u)) + r'(u)\mathcal{A}(x,t,u,\nabla u)\nabla u\\ &-div\,(r(u)\Phi(x,t,u)) + r'(u)\Phi(x,t,u)\nabla u = fr(u),\\ &\quad in \quad D'(\Omega_T),\end{aligned} \tag{3.11}$$

where $B_r(x,\tau) = \displaystyle\int_0^\tau \frac{\partial b(x,s)}{\partial s}r'(s)\,ds$ and $B_r(x,u)(t=0) = B_r(x,u_0)$ in Ω.

Remark 3.3. *[38, 40] For every $r \in W^{2,\infty}(\mathbb{R})$ nondecreasing function such that supp$(r') \subset [-k,k]$ and (3.1), we have*

$$\lambda_k |r(s_1) - r(s_2)| \le |B_r(x,s_1) - B_r(x,s_2)| \le \|A_k\|_{L^\infty(\Omega)} |r(s_1) - r(s_2)|,$$

for almost every $x \in \Omega$ and for every $s_1, s_2 \in \mathbb{R}$.

Lemma 3.4. *Let φ be a Musielak function log-Hölder continuous, then there exists two Orlicz functions $\mathbf{q}$ and $\mathbf{Q}$ such that*
(i) *For all $(x,t) \in \Omega \times \mathbb{R}^+$*

$$\mathbf{q}(t) \le \varphi(x,t) \le \mathbf{Q}(t),$$

(ii) *One has also*

$$\overline{\varphi}_x^{-1}(\varphi(x,t)) \le \overline{\mathbf{Q}}^{-1}(\mathbf{Q}(t)) \text{ for all } (x,t) \in \Omega \times \mathbb{R}^+,$$

where $\overline{\mathbf{Q}}$ and $\overline{\varphi}$ are the complementary functions of $\mathbf{Q}$ and φ respectively.
(iii) *[34] $\mathbf{q}(t) \le \varphi(x,t) \Longleftrightarrow \overline{\varphi}(x,t) \le \overline{\mathbf{q}}(t)$.*

Proof . **(i)** For the construction of $\mathbf{q}$, one can consult [34], for $\mathbf{Q}$, let $(\Omega_i)_{i=1}^N$ be a finite partition of Ω such that $diam\Omega_i \le \frac{1}{2}$. Let us fix an element x_i in each part Ω_i. Let $x \in \Omega$, there exists $i \in \{1,...,N\}$ such that $x \in \Omega_i$. We have for all $t \ge 1$ and a.e $x \in \Omega$

$$\varphi(x,t) \le \varphi(x_i,t) t^{\left(\frac{A}{\log(\frac{1}{|x-x_i|})}\right)} \le \varphi(x_i,t) t^{\frac{A}{\log 2}} \le \sum_{i=1}^N \varphi(x_i,t) t^{\frac{A}{\log 2}}.$$

Put $\mathbf{Q}(t) = \sum_{i=1}^N \varphi(x_i,t) t^{\frac{A}{\log 2}}$ which is an N-function.

(ii) Let $s,t \in \mathbb{R}^+$ and $x \in \Omega$. We have $\varphi(x,t) \le \mathbf{Q}(t)$, then

$$st - \varphi(x,t) \ge st - \mathbf{Q}(t).$$

Passing to the sup over $t \ge 0$

$$\sup_{t\ge 0}\{st - \varphi(x,t)\} \ge \sup_{t \ge 0}\{st - \mathbf{Q}(t)\}.$$

That means

$$\overline{\varphi}(x,s) := \overline{\varphi}_x(s) \ge \overline{\mathbf{Q}}(s), \text{ for all } s \in \mathbb{R}^+.$$

It follows that for all $s \in \mathbb{R}^+$,

$$\overline{\varphi}_x^{-1}(s) \le \overline{\mathbf{Q}}^{-1}(s)$$

Taking $s = \varphi(x,t)$, since $\overline{\mathbf{Q}}^{-1}$ is an increasing function, we have $\forall t \in \mathbb{R}^+$,

$$\overline{\varphi}_x^{-1}(\varphi(x,t)) \le \overline{\mathbf{Q}}^{-1}(\varphi(x,t)) \le \overline{\mathbf{Q}}^{-1}(\mathbf{Q}(t)).$$

□

Remark 3.5. (R_1) *Since Ω is bounded, condition (i) of the previous lemma implies condition (iii) of lemma 2.4.*
(R_2) *If we assume that $\int_\Omega \overline{\varphi}(x,c)\,dx < \infty$ for all constant c, we don't need to use the N-function $\mathbf{q}$.*

The following theorem is our main result.

Theorem 3.6. *Suppose that the modular function φ verifies the hypotheses (i) and (ii) of lemma 2.4 the assumptions $(H_1)-(H_4)$ hold true and $f \in L^1(\Omega_T)$, then there exists at least a renormalized solution for problem (3.2) in the sense of definition 3.2 .*

The proof of the above theorem is divided into five steps.

Step 1: Approximate problems.
Let f_n be a sequence of regular function in $C_0^\infty(\Omega_T)$ which converges strongly to f in $L^1(\Omega_T)$ and such that $\|f_n\|_{L^1} \leq \|f\|_{L^1}$ and for each $n \in I\!N^*$, put

$$b_n(x,s) = T_n(b(x,s)) + \frac{1}{n}s,$$

$$\mathcal{A}_n(x,t,s,\xi) = \mathcal{A}(x,t,T_n(s),\xi) \text{ a.e } (x,t) \in \Omega_T, \forall s \in \mathbb{R}, \forall \xi \in \mathbb{R}^N,$$

and

$$\Phi_n(x,t,s) = \Phi(x,t,T_n(s)) \text{ a.e } (x,t) \in \Omega_T, \forall s \in \mathbb{R},$$

And let $u_{0n} \in C_0^\infty(\Omega)$ such that

$$\| b_n(x,u_{0n}) \|_{L^1} \leq \| b(x,u_0) \|_{L^1} \text{ and } b_n(x,u_{0n}) \longrightarrow b(x,u_0) \text{ in } L^1(\Omega).$$

Considering the following approximate problem

$$\begin{cases} \dfrac{\partial b_n(x,u_n)}{\partial t} - \operatorname{div} \mathcal{A}(x,t,u_n,\nabla u_n) - \operatorname{div} \Phi_n(x,t,u_n) = f_n & \text{in } \Omega_T \\ b_n(x,u_n)(t=0) = b_n(x,u_0) & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{3.12}$$

Let $z_n(x,t,u_n,\nabla u_n) = \mathcal{A}_n(x,t,u_n,\nabla u_n) + \Phi_n(x,t,u_n)$, which satisfies (A_1), (A_2), (A_3) and (A_4) of [4]. Indeed, it remains to verify (A_4), to do this we use Young's inequality as follows

$$\begin{aligned} |\Phi_n(x,t,u_n)\nabla u_n| &\leq |\gamma(x)||\nabla u_n| + \overline{\varphi}_x^{-1}(\varphi(x,|T_n(u_n)|))|\nabla u_n| \\ &= \frac{\alpha^2}{\alpha+2}\frac{\alpha+2}{\alpha^2}|\gamma(x,t)||\nabla u_n| \\ &\quad + \frac{\alpha+1}{\alpha}\overline{\varphi}_x^{-1}(\varphi(x,|T_n(u_n)|))\frac{\alpha}{\alpha+1}|\nabla u_n| \\ &\leq \frac{\alpha^2}{\alpha+2}\Big(\overline{\varphi}\Big(x,\frac{\alpha+2}{\alpha^2}|\gamma(x)|\Big) + \varphi\Big(x,|\nabla u_n|\Big)\Big) \\ &\quad + \overline{\varphi}\Big(x,\frac{\alpha+1}{\alpha}\overline{\varphi}_x^{-1}(\varphi(x,|T_n(u_n)|))\Big) \\ &\quad + \varphi\Big(x,\frac{\alpha}{\alpha+1}|\nabla u_n|\Big). \end{aligned}$$

While $\dfrac{\alpha}{\alpha+1} < 1$, using the convexity of φ and the fact that $\overline{\varphi}$ and $\overline{\varphi}_x^{-1} \circ \varphi$ are increasing functions, one has

$$\begin{aligned} |\Phi_n(x,t,u_n)\nabla u_n| \leq{}& \frac{\alpha^2}{\alpha+2}\overline{\varphi}\Big(x,\frac{\alpha+2}{\alpha^2}|\gamma(x,t)|\Big) + \frac{\alpha^2}{\alpha+2}\varphi\Big(x,|\nabla u_n|\Big) \\ &+ \overline{\varphi}\Big(x,\frac{\alpha+1}{\alpha}\overline{\varphi}_x^{-1}(\varphi(x,n))\Big) + \frac{\alpha}{\alpha+1}\varphi\Big(x,|\nabla u_n|\Big). \end{aligned}$$

Since $\gamma \in E_{\overline{\varphi}}(\Omega_T)$, $\overline{\varphi}\Big(x, \dfrac{\alpha+2}{\alpha^2}|\gamma(x,t)|\Big) \in L^1(\Omega)$ and thanks to lemma 3.4,

$$\overline{\varphi}\Big(x, \frac{\alpha+1}{\alpha}\overline{\varphi}_x^{-1}(\varphi(x,n))\Big) \leq \overline{\mathbf{q}}\Big(\frac{\alpha+1}{\alpha}\overline{\mathbf{Q}}^{-1}(\mathbf{Q}(n))\Big),$$

then we get

$$\Phi_n(x,t,u_n)\nabla u_n \geq -\Big(\frac{\alpha^2}{\alpha+2} + \frac{\alpha}{\alpha+1}\Big)\varphi\Big(x, |\nabla u_n|\Big) - \text{fixed} \quad L^1 function.$$

Using this last inequality and $(H3)$ we obtain

$$\begin{aligned} z_n(x,t,u_n,\nabla u_n)\nabla u_n &\geq \Big(\alpha - \frac{\alpha^2}{\alpha+2} - \frac{\alpha}{\alpha+1}\Big)\varphi\Big(x, |\nabla u_n|\Big) - \text{fixed} \quad L^1 function \\ &\geq \frac{\alpha^2}{(\alpha+1)(\alpha+2)}\varphi\Big(x, |\nabla u_n|\Big) - \text{fixed} \quad L^1 function. \end{aligned}$$

Thus, from [4, 37], the approximate problem (3.12) has at least one weak solution $u_n \in W_0^{1,x}L_\varphi(\Omega_T)$.

Step 2: A Priori Estimates.

Proposition 3.7. *Suppose that the assumptions $(H_1) - (H_4)$ hold true and let $(u_n)_n$ be a solution of the approximate problem (3.12). Then, for all $k > 0$, there exists two constants C_k, $\widehat{C}_k$ (not depending on n), such that:*

$$\| T_k(u_n) \|_{W_0^{1,x}L_\varphi(\Omega_T)} \leq C_k, \tag{3.13}$$

$$\int_\Omega B_k^n(x,u_n)(\sigma)\,dx \leq \widehat{C}_k + k\Big(\|f\|_{L^1(\Omega_T)} + \|b(x,u_0\|_{L^1(\Omega)}\Big), \tag{3.14}$$

for almost any $\sigma \in (0,T)$ where $B_k^n(x,\tau) = \displaystyle\int_0^\tau T_k(s)\frac{\partial b_n(x,s)}{\partial s}\,ds$, and

$$\lim_{k\to\infty} meas\Big\{(x,t) \in \Omega_T : |u_n| > k\Big\} = 0. \tag{3.15}$$

Proof. Testing the approximate problem (3.12) by $T_k(u_n)\mathbf{1}_{(0,\sigma)}$, one has for every $\sigma \in (0,T)$

$$\begin{aligned} &\int_\Omega \Big(B_k^n(x,u_n)(\sigma) - B_k^n(x,u_{0n})\Big)\,dx + \int_{\Omega_\sigma} \mathcal{A}(x,t,u_n,\nabla u_n)\nabla T_k(u_n)\,dx\,dt \\ &+ \int_{\Omega_\sigma} \Phi_n(x,t,u_n)\nabla T_k(u_n)\,dx\,dt = \int_{\Omega_\sigma} f_n T_k(u_n)\,dx\,dt. \end{aligned} \tag{3.16}$$

First, let us remark that $\Phi_n(x,t,u_n)\nabla T_k(u_n)$ is different from zero only on the set $\{|u_n| \leq k\}$ where $T_k(u_n) = u_n$. From (H_4) and then Young's inequality for an arbitrary $\alpha > 0$ (the constant of coercivity), we have

$$
\begin{aligned}
&\int_{\Omega_\sigma} \Phi_n(x,t,u_n)\nabla T_k(u_n)\,dx\,dt \\
&\le \int_{\Omega_\sigma} |\gamma(x)||\nabla T_k(u_n)|\,dx\,dt \\
&\quad + \int_{\Omega_\sigma} \overline{\varphi}_x^{-1}(\varphi(x,|T_k(u_n)|))|\nabla T_k(u_n)|\,dx\,dt \\
&= \frac{\alpha^2}{\alpha+2}\int_{\Omega_\sigma} \frac{\alpha+2}{\alpha^2}|\gamma(x)||\nabla T_k(u_n)|\,dx\,dt \\
&\quad + \int_{\Omega_\sigma} \frac{\alpha+1}{\alpha}\overline{\varphi}_x^{-1}(\varphi(x,|T_k(u_n)|))\frac{\alpha}{\alpha+1}|\nabla T_k(u_n)|\,dx\,dt \\
&\le \frac{\alpha^2}{\alpha+2}\Big(\int_{\Omega_\sigma} \overline{\varphi}\Big(x,\frac{\alpha+2}{\alpha^2}|\gamma(x)|\Big)\,dx\,dt + \int_{\Omega_\sigma}\varphi\Big(x,|\nabla T_k(u_n)|\Big)\,dx\,dt\Big) \\
&\quad + \int_{\Omega_\sigma} \overline{\varphi}\Big(x,\frac{\alpha+1}{\alpha}\overline{\varphi}_x^{-1}(\varphi(x,|T_k(u_n)|)\Big)\,dx\,dt \\
&\qquad + \int_{\Omega_\sigma} \varphi\Big(x,\frac{\alpha}{\alpha+1}|\nabla T_k(u_n)|\Big)\,dx\,dt.
\end{aligned} \tag{3.17}
$$

Since $\gamma \in E_{\overline{\varphi}}(\Omega_T)$, then $\frac{\alpha^2}{\alpha+2}\int_{\Omega_\sigma}\overline{\varphi}\Big(x,\frac{\alpha+2}{\alpha^2}|\gamma(x,t)|\Big)\,dx\,dt = \gamma_0$ and while $\frac{\alpha}{\alpha+1} < 1$, using the convexity of φ and from lemma 3.4,

$$
\int_{\Omega_\sigma} \overline{\varphi}\Big(x,\frac{\alpha+1}{\alpha}\overline{\varphi}_x^{-1}(\varphi(x,|T_k(u_n)|)\Big)\,dx\,dt \le \int_{\Omega_\sigma} \overline{\mathbf{q}}\Big(\frac{\alpha+1}{\alpha}\overline{\mathbf{Q}}^{-1}(\mathbf{Q}(k)\Big)\,dx\,dt = C_k^\alpha < \infty
$$

Then (3.17) becomes

$$
\begin{aligned}
&\int_{\Omega_\sigma} \Phi_n(x,t,u_n)\nabla T_k(u_n)\,dx\,dt \\
&\le \gamma_0 + C_k^\alpha + \frac{\alpha^2}{\alpha+2}\int_{\Omega_\sigma}\varphi\Big(x,|\nabla T_k(u_n)|\Big)\,dx\,dt \\
&\qquad + \frac{\alpha}{\alpha+1}\int_{\Omega_\sigma}\varphi\Big(x,|\nabla T_k(u_n)|\Big)\,dx\,dt.
\end{aligned} \tag{3.18}
$$

On the other hand, we have $\|f_n\|_{L^1} \le \|f\|_{L^1}$, which implies that

$$
\int_{\Omega_T} f_n T_k(u_n)\,dx\,dt \le k\|f\|_{L^1}. \tag{3.19}
$$

Concerning the first integral in (3.16), we have by construction of $B_k^n(x,u_n)$,

$$
\int_\Omega B_k^n(x,u_n)(\sigma)\,dx \ge 0 \tag{3.20}
$$

and

$$
0 \le \int_\Omega B_k^n(x,u_{0n})\,dx \le k\int_\Omega |b_n(x,u_{0n})|\,dx \le k\|b(x,u_0)\|_{L^1(\Omega)}. \tag{3.21}
$$

Combining (3.16), (3.18), (3.19), (3.20) and (3.21) we get

$$\begin{aligned}
&\int_{\Omega_\sigma} \mathcal{A}(x,t,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)\,dx\,dt \\
&\leq \gamma_0 + kC_{b,f} + C_k^\alpha + \frac{\alpha^2}{\alpha+2}\int_{\Omega_\sigma}\varphi\Big(x,|\nabla T_k(u_n)|\Big)\,dx\,dt \\
&\quad + \frac{\alpha}{\alpha+1}\int_{\Omega_\sigma}\varphi\Big(x,|\nabla T_k(u_n)|\Big)\,dx\,dt,
\end{aligned} \tag{3.22}$$

where $C_{b,f} = \|f\|_{L^1(\Omega)} + \|b(x,u_0)\|_{L^1(\Omega)}$. Thanks to (H_3), we deduce

$$\int_{\Omega_\sigma}\Big(\alpha - \frac{\alpha^2}{\alpha+2} - \frac{\alpha}{\alpha+1}\Big)\varphi\Big(x,|\nabla T_k(u_n)|\Big)\,dx\,dt \leq \gamma_0 + kC_{b,f} + C_k^\alpha. \tag{3.23}$$

Since $\Big(\alpha - \frac{\alpha^2}{\alpha+2} - \frac{\alpha}{\alpha+1}\Big) = \frac{\alpha^2}{(\alpha+1)(\alpha+2)} > 0$, finally we have

$$\int_{\Omega_T}\varphi\Big(x,|\nabla T_k(u_n)|\Big)\,dx\,dt \leq (\gamma_0 + kC_{b,f} + C_k^\alpha)\frac{(\alpha+1)(\alpha+2)}{\alpha^2} = C_k. \tag{3.24}$$

To prove (3.14), we combine (3.16), (3.18), (3.19), (3.21), (3.22) and (3.24) with $\widehat{C}_k = \gamma_0 + C_k^\alpha + (\frac{\alpha^2}{\alpha+2} + \frac{\alpha}{\alpha+1})C_k$. Finally, we prove (3.15), to this end, since $T_k(u_n)$ is bounded in $W_0^{1,x}L_\varphi(\Omega_T)$ there exists $\lambda > 0$ and a constant C_0 such that

$$\int_{\Omega_T}\varphi\Big(x,\frac{|T_k(u_n)|}{\lambda}\Big)\,dx\,dt \leq C_0$$

By using young's inequality, we obtain

$$\begin{aligned}
\text{meas}\Big\{|u_n| > k\Big\} &= \frac{1}{k}\int_{\{|u_n|>k\}} k\,dx\,dt \leq \frac{1}{k}\int_{\Omega_T}|T_k(u_n)|\,dx\,dt \\
&\leq \frac{\lambda}{k}\Big(\int_{\Omega_T}\varphi\Big(x,\frac{|T_k(u_n)|}{\lambda}\Big)\,dx\,dt + \int_{\Omega_T}\overline{\varphi}(x,1)\,dx\,dt\Big) \\
&\leq \frac{\lambda}{k}\Big(C_0 + \overline{\mathbf{q}}(1)|\Omega_T|\Big) \qquad \forall n, \quad \forall k > 0, \\
&\longrightarrow 0 \quad \text{as } k \longrightarrow \infty.
\end{aligned} \tag{3.25}$$

Which implies (3.15).

Lemma 3.8. *Let u_n be a solution of the approximate problem (3.12), then:*
(i) $u_n \longrightarrow u$ *a.e. in* Ω_T,
(ii) $b_n(x,u_n) \longrightarrow b(x,u)$ *a.e. in* Ω_T,
(iii) $b(x,u) \in L^\infty(0,T;L^1(\Omega))$.

Proof. For (i) and (ii), we argue as in [40, Proposition 5.3], we take a $C^2(\mathbb{R})$ nondecreasing function Γ_k such that $\Gamma_k(s) = \begin{cases} s & \text{for } |s| \leq \frac{k}{2} \\ k & \text{for } |s| \geq k \end{cases}$ and multiplying the approximate problem (3.12) by $\Gamma_k'(u_n)$ we obtain

$$\begin{aligned}
\frac{\partial B_\Gamma^n(x,u_n)}{\partial t} &= \text{div}\Big(\mathcal{A}(x,t,u_n,\nabla u_n)\Gamma_k'(u_n)\Big) - \mathcal{A}(x,t,u_n,\nabla u_n)\Gamma_k''(u_n)\nabla u_n \\
&\quad + \text{div}\Big(\Gamma_k'(u_n)\Phi_n(x,t,u_n)\Big) - \Gamma_k''(u_n)\Phi_n(x,t,u_n)\nabla u_n + f_n\Gamma_k'(u_n),
\end{aligned} \tag{3.26}$$

where $B^n_\Gamma(x,\tau)=\displaystyle\int_0^\tau \frac{\partial b^n_k(x,s)}{\partial s}\Gamma'_k(s)\,ds$.
Remarking that $\overline{\varphi}_x^{-1}\circ\varphi$ is an increasing function, $\gamma\in E_{\overline{\varphi}}(\Omega_T)$, $supp(\Gamma'_k), supp(\Gamma''_k)\subset[-k,k]$ and using Young's inequality we get

$$\begin{aligned}
&\Big|\int_{\Omega_T}\Gamma'_k(u_n)\Phi_n(x,t,u_n)\,dx\,dt\Big|\\
&\leq \|\Gamma'_k\|_{L^\infty}\Big(\int_{\Omega_T}|\gamma(x,t)|\,dx\,dt+\int_{\Omega_T}\overline{\varphi}_x^{-1}(\varphi(x,|T_k(u_n)|)\,dx\,dt\Big)\\
&\leq \|\Gamma'_k\|_{L^\infty}\Big(\int_{\Omega_T}\Big(\overline{\varphi}(x,|\gamma(x,t)|)+\varphi(x,1)\Big)\,dx\,dt+\int_{\Omega_T}\overline{\varphi}_x^{-1}(\varphi(x,k)\,dx\,dt\Big)\\
&\leq \|\Gamma'_k\|_{L^\infty}\Big(\int_{\Omega_T}\Big(\overline{\varphi}(x,|\gamma(x,t)|)+\varphi(x,1)\Big)\,dx\,dt+\int_{\Omega_T}\overline{\mathbf{Q}}^{-1}(\mathbf{Q}(k)\,dx\,dt\Big)\\
&< C_{1,k},
\end{aligned}\tag{3.27}$$

and (here, we use also estimate (3.24))

$$\begin{aligned}
&\Big|\int_{\Omega_T}\Gamma''_k(u_n)\Phi_n(x,t,u_n)\nabla u_n\,dx\,dt\Big|\\
&\leq \|\Gamma''_k\|_{L^\infty}\Big(\int_{\Omega_T}|\gamma(x,t)|\,dx\,dt+\int_{\Omega_T}\overline{\varphi}_x^{-1}(\varphi(x,|T_k(u_n)|)|\nabla T_k(u_n)|\,dx\,dt\Big)\\
&\leq \|\Gamma''_k\|_{L^\infty}\Big[\int_{\Omega_T}\Big(\overline{\varphi}(x,|\gamma(x,t)|)+\varphi(x,1)\Big)\,dx\,dt+\int_{\Omega_T}\varphi(x,k)\,dx\,dt\\
&\qquad+\int_{\Omega_T}\varphi(x,|\nabla T_k(u_n)|)\,dx\,dt\Big]\\
&< C_{2,k},
\end{aligned}\tag{3.28}$$

where $C_{1,k}$ and $C_{2,k}$ are two positive constants independent of n. Then each term in the right-hand side of (3.26) is bounded either in $L^1(\Omega_T)$ or in $W^{-1,x}L_{\overline{\varphi}}(\Omega_T)$, which implies that

$$\frac{\partial B^n_\Gamma(x,u_n)}{\partial t}\text{ is bounded in }L^1(\Omega_T)+W^{-1,x}L_{\overline{\varphi}}(\Omega_T).\tag{3.29}$$

Moreover, due to the properties of Γ'_k and (3.1), we have

$$|\nabla B^n_\Gamma(x,u_n)|\leq\|A_k\|_{L^\infty(\Omega)}|\nabla T_k(u_n)|\|\Gamma'_k\|_{L^\infty(\Omega)}+k\|\Gamma'_k\|_{L^\infty(\Omega)}\widetilde{A}_k(x),$$

which implies by (3.13), that

$$B^n_\Gamma(x,u_n)\text{ is bounded in }W^{1,x}_0L_\varphi(\Omega_T).$$

Arguing as in [40, 18, 19], we get (i) and (ii) of lemma 3.8.

To prove (iii), using (ii), we pass to the limit inferior in (3.14) as $n\longrightarrow+\infty$, we get

$$\frac{1}{k}\int_\Omega B_k(x,u)(\sigma)\,dx\leq\frac{\widehat{C}_k}{k}+\Big(\|f\|_{L^1(\Omega_T)}+\|b(x,u_0\|_{L^1(\Omega)}\Big),$$

for almost any $\sigma\in(0,T)$. Tanks to the definition of $B_k(x,s)$ and the convergence of $\displaystyle\frac{1}{k}\int_\Omega B_k(x,u)$ to $b(x,u)$ as k goes to $+\infty$, this gives that $b(x,u)\in L^\infty(0,T;L^1(\Omega))$.
The next lemma will be used later, proving it now.

Lemma 3.9. *Let u_n be a solution of the approximate problem (3.12), then:*
(i) $\{\mathcal{A}(x,t,T_k(u_n),\nabla T_k(u_n))\}_n$ *is bounded in* $(L_{\overline{\varphi}}(\Omega_T))^N$,
(ii) $\displaystyle\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{\{m\le|u_n|\le m+1\}}\mathcal{A}(x,t,u_n,\nabla u_n)\nabla u_n\,dx=0.$

Proof. (i) We will use the Banach-Steinhaus theorem. Let $\phi\in(E_\varphi(\Omega_T))^N$ be an arbitrary function. From (H_2) we can write

$$\Big(\mathcal{A}(x,t,T_k(u_n),\nabla T_k(u_n))-\mathcal{A}(x,t,T_k(u_n),\phi)\Big)\cdot\Big(\nabla T_k(u_n)-\phi\Big)\ge 0.$$

Which gives:

$$\begin{aligned}&\int_{\Omega_T}\mathcal{A}(x,t,T_k(u_n),\nabla T_k(u_n))\phi\,dx\\&\le\int_{\Omega_T}\mathcal{A}(x,t,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)\,dx\\&\quad+\int_{\Omega_T}\mathcal{A}(x,t,T_k(u_n),\phi)(\phi-\nabla T_k(u_n))\,dx.\end{aligned}\tag{3.30}$$

Let us denote by J_1 and J_2 the first and the second integral respectively in the right-hand side of (3.30), so that

$$J_1=\int_{\Omega_T}\mathcal{A}(x,t,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)\,dx.$$

Going back to (3.22), we obtain

$$\begin{aligned}J_1\le{}&\gamma_0+kC_{b,f}+C_k^\alpha+\frac{\alpha^2}{\alpha+2}\int_{\Omega_T}\varphi\Big(x,|\nabla T_k(u_n)|\Big)\,dx\,dt\\&+\frac{\alpha}{\alpha+1}\int_{\Omega_T}\varphi\Big(x,|\nabla T_k(u_n)|\Big)\,dx\,dt,\end{aligned}\tag{3.31}$$

And thanks to (3.13), there exists a positive constant C_{J_1} independent of n such that

$$J_1\le C_{J_1}.\tag{3.32}$$

Now we estimate the integral J_2, to this end, remark that

$$\begin{aligned}J_2&=\int_{\Omega_T}\mathcal{A}(x,t,T_k(u_n),\phi)(\phi-\nabla T_k(u_n))\,dx\,dt\\&\le\int_{\Omega_T}|\mathcal{A}(x,t,T_k(u_n),\phi)||\phi|\,dx\,dt+\int_{\Omega_T}|\mathcal{A}(x,t,T_k(u_n),\phi)||\nabla T_k(u_n)|\,dx\,dt.\end{aligned}$$

On the other hand, let η be large enough, from (H_1) and the convexity of $\overline{\varphi}$, we get:

$$\begin{aligned}&\int_{\Omega_T}\overline{\varphi}\Big(x,\frac{|\mathcal{A}(x,T_k(u_n),\phi)|}{\eta}\Big)\,dx\,dt\\&\le\int_{\Omega_T}\overline{\varphi}\Big(x,\frac{c(x)+\overline{\varphi}_x^{-1}(\Psi(x,k_1|T_k(u_n)|)+\overline{\varphi}_x^{-1}(\varphi(x,k_2|\phi|))}{\eta}\Big)\,dx\,dt\\&\le\frac{1}{\eta}\int_{\Omega_T}\overline{\varphi}(x,c(x))\,dx\,dt+\frac{1}{\eta}\int_{\Omega_T}\overline{\varphi}\Big(x,\overline{\varphi}_x^{-1}(\Psi(x,k_1|T_k(u_n)|))\Big)\,dx\,dt\\&\quad+\frac{1}{\eta}\int_{\Omega_T}\overline{\varphi}\Big(x,\overline{\varphi}_x^{-1}(\varphi(x,k_2|\phi|))\Big)\,dx\,dt\\&\le\frac{1}{\eta}\int_{\Omega_T}\overline{\varphi}(x,c(x))\,dx\,dt+\frac{1}{\eta}\int_{\Omega_T}\Psi(x,k_1k)\,dx\,dt+\frac{1}{\eta}\int_{\Omega_T}\varphi(x,k_2|\phi|)\,dx\,dt.\end{aligned}\tag{3.33}$$

Since $\phi \in (E_\varphi(\Omega_T))^N$, $c(x) \in E_{\overline{\varphi}}(\Omega_T)$, by remark 2.1 and lemma 3.4, we have

$$\int_{\Omega_T} \Psi(x, k_1 k)\, dx\, dt \leq k(\epsilon) \int_{\Omega_T} \varphi(x, \epsilon k_1 k)\, dx\, dt < \infty,$$

we deduce that $\{\mathcal{A}(x,t,T_k(u_n),\phi)\}$ is bounded in $(L_{\overline{\varphi}}(\Omega_T))^N$ and we have $\{\nabla T_k(u_n)\}$ is bounded in $(L_\varphi(\Omega_T))^N$, consequently, $J_2 \leq C_{J_2}$, where C_{J_2} is a positive constant not depending on n. And then we obtain

$$\int_{\Omega_T} \mathcal{A}(x, T_k(u_n), \nabla T_k(u_n))\phi\, dx\, dt \leq C_{J_1} + C_{J_2}, \quad \text{for all } \phi \in (E_\varphi(\Omega_T))^N. \tag{3.34}$$

Finally, $\{\mathcal{A}(x,t,T_k(u_n),\nabla T_k(u_n))\}_n$ is bounded in $(L_{\overline{\varphi}}(\Omega_T))^N$.

(*ii*) Testing (3.12) by $\theta_m(u_n) = T_{m+1}(u_n) - T_m(u_n)$, we have

$$\begin{aligned}
&\int_\Omega B_m(x,u_n)(T)\, dx + \int_{\Omega_T} \mathcal{A}(x,t,u_n,\nabla u_n)\nabla\theta_m(u_n)\, dx\, dt \\
&\quad + \int_{\Omega_T} \Phi_n(x,t,u_n)\nabla\theta_m(u_n)\, dx\, dt = \int_\Omega B_m(x,u_{0n})\, dx \\
&\quad + \int_{\Omega_T} f_n \theta_m(u_n)\, dx\, dt,
\end{aligned} \tag{3.35}$$

where $B_m(x,\tau) = \displaystyle\int_0^\tau \frac{\partial b(x,s)}{\partial s}\theta_m(s)\, ds$. Since $B_m(x,u_n)(T) \geq 0$, hence from (H_3) and (H_4), it follows

$$\begin{aligned}
&\alpha \int_{\Omega_T} \varphi(x, |\nabla\theta_m(u_n)|)\, dx\, dt \\
&\leq \int_{\Omega_T} \overline{\varphi}_x^{-1}(\varphi(x,|u_n|))|\nabla\theta_m(u_n)|\, dx\, dt + \int_{\Omega_T} |\gamma(x,t)||\nabla\theta_m(u_n)|\, dx\, dt \\
&\quad + \int_\Omega B_m(x,u_{0n})\, dx + \int_{\Omega_T} f_n\theta_m(u_n)\, dx\, dt.
\end{aligned} \tag{3.36}$$

That means, knowing that $\nabla\theta_m(u_n) = \nabla u_n \mathbf{1}_{E_{m,n}}$ a.e. in Ω_T where

$$E_{m,n} := \Big\{(x,t) \in \Omega_T : m \leq |u_n| \leq m+1\Big\},$$

and following the same argument as in the proof of (3.13) of proposition 3.7, we get

$$\begin{aligned}
&\alpha \int_{\Omega_T} \varphi(x, |\nabla\theta_m(u_n)|)\, dx\, dt \\
&\leq \int_{\Omega_T} \overline{\varphi}_x^{-1}(\varphi(x,|u_n|))|\nabla u_n|\mathbf{1}_{E_{m,n}}\, dx\, dt + \int_{E_{m,n}} |\gamma(x,t)||\nabla\theta_m(u_n)|\, dx\, dt \\
&\quad + \int_\Omega B_m(x,u_{0n})\, dx + \int_{\Omega_T} f_n\theta_m(u_n)\, dx\, dt \\
&\leq \int_{\Omega_T} \overline{\varphi}\Big(x, \frac{\alpha+1}{\alpha}\overline{\varphi}_x^{-1}(\varphi(x,|u_n|)\Big)\mathbf{1}_{E_{m,n}}\, dx\, dt + \int_{\Omega_T} \varphi\Big(x, \frac{\alpha}{\alpha+1}|\nabla\theta_m(u_n)|\Big)\, dx\, dt \\
&\quad + \frac{\alpha^2}{\alpha+2}\Big(\int_{E_{m,n}} \overline{\varphi}\Big(x, \frac{\alpha+2}{\alpha^2}|\gamma(x,t)|\Big)\, dx\, dt + \int_{\Omega_T} \varphi\Big(x, |\nabla\theta_m(u_n)|\Big)\, dx\, dt\Big) \\
&\quad + \int_\Omega B_m(x,u_{0n})\, dx + \int_{\Omega_T} f_n\theta_m(u_n)\, dx\, dt.
\end{aligned} \tag{3.37}$$

let $C^{\alpha}_{max} := \max\Big((\alpha+1), \dfrac{(\alpha+1)(\alpha+2)}{\alpha^2}\Big)$, it follows

$$\begin{aligned}&\int_{\Omega_T}\varphi(x,|\nabla\theta_m(u_n)|)\,dx\,dt\\ &\le C^{\alpha}_{max}\Big[\int_{E_{m,n}}\overline{\varphi}\Big(x,\frac{\alpha+2}{\alpha^2}|\gamma(x,t)|\Big)\,dx\,dt+\int_{\Omega}B_m(x,u_{0n})\,dx\\ &\qquad+\int_{E_{m,n}}\overline{\mathbf{q}}\Big(\frac{\alpha+1}{\alpha}\overline{\mathbf{Q}}^{-1}(\mathbf{Q}(|u_n|)\Big)\,dx\,dt+\int_{\Omega_T}f_n\theta_m(u_n)\,dx\,dt\Big].\end{aligned}\tag{3.38}$$

Now, let us concentrate on the convergence as $n\to\infty$ of each integral in (3.38), which can be treated by the same way (Lebesgue's dominated convergence theorem), take for example the first one:

$$\int_{\{m\le|u_n|\le m+1\}}\overline{\varphi}\Big(x,\frac{\alpha+2}{\alpha^2}|\gamma(x,t)|\Big)\,dx=\int_{\Omega}\overline{\varphi}\Big(x,\frac{\alpha+2}{\alpha^2}|\gamma(x,t)|\Big)\mathbf{1}_{\{m\le|u_n|\le m+1\}}\,dx\,dt$$

Put $g_n=\overline{\varphi}\Big(x,\dfrac{\alpha+2}{\alpha^2}|\gamma(x,t)|\Big)\mathbf{1}_{\{m\le|u_n|\le m+1\}}$, since $\mathbf{1}$ is continuous, then

$$g_n\longrightarrow g=\overline{\varphi}\Big(x,\frac{\alpha+2}{\alpha^2}|\gamma(x,t)|\Big)\mathbf{1}_{\{m\le|u|\le m+1\}}\quad\text{a.e. in }\Omega_T.$$

And we have $|g_n|\le\overline{\varphi}\Big(x,\dfrac{\alpha+2}{\alpha^2}|\gamma(x,t)|\Big)$ which is integrable on Ω_T, since $\gamma\in E_{\overline{\varphi}}(\Omega_T)$. From Lebesgue's dominated convergence theorem, we obtain

$$\lim_{n\to\infty}\int_{\Omega_T}g_n\,dx\,dt=\int_{\Omega_T}\lim_{n\to\infty}g_n\,dx\,dt=\int_{\Omega_T}\overline{\varphi}\Big(x,\frac{\alpha+2}{\alpha^2}|\gamma(x,t)|\Big)\mathbf{1}_{\{m\le|u|\le m+1\}}\,dx\,dt.$$

Passing to the limit as $n\to\infty$ in (3.38), we get

$$\begin{aligned}&\lim_{n\to\infty}\int_{\Omega_T}\varphi(x,|\nabla\theta_m(u_n)|)\,dx\,dt\\ &\le C^{\alpha}_{max}\Big[\int_{\{m\le|u|\le m+1\}}\overline{\mathbf{q}}\Big(\frac{\alpha+2}{\alpha^2}|\gamma(x)|\Big)\,dx\,dt+\int_{\Omega}B_m(x,u_0)\,dx\\ &\qquad+\int_{\{m\le|u|\le m+1\}}\overline{\mathbf{q}}\Big(\frac{\alpha+1}{\alpha}\overline{\mathbf{Q}}^{-1}(\mathbf{Q}(|u|)\Big)\,dx\,dt\\ &\qquad+\int_{\Omega_T}f\theta_m(u)\,dx\,dt\Big].\end{aligned}\tag{3.39}$$

Now, we will pass to the limit as $m\to\infty$, by Lebesgue's theorem each integral in (3.39) goes to zero as m goes to ∞, which gives

$$\lim_{m\to\infty}\lim_{n\to\infty}\int_{\Omega_T}\varphi(x,|\nabla\theta_m(u_n)|)\,dx\,dt=0.\tag{3.40}$$

Our aim here is to prove that $\lim\limits_{m\to\infty}\lim\limits_{n\to\infty}\int_{\Omega_T}\Phi_n(x,t,u_n)\nabla\theta_m(u_n)\,dx\,dt=0$, to this end, Young's inequality allows us to get

$$\begin{aligned}\int_{\Omega_T}\Phi_n(x,t,u_n)\nabla\theta_m(u_n)\,dx\,dt\ &\le\int_{\Omega_T}\varphi(x,|\nabla\theta_m(u_n)|)\,dx\,dt\\ &\quad+\int_{\{m\le|u_n|\le m+1\}}\overline{\varphi}(x,\Phi_n(x,t,u_n))\,dx\,dt.\end{aligned}\tag{3.41}$$

We have already proved that the first integral in the right-hand side of (3.41) goes to zero as m and n go to ∞, it remains to show that the second one goes to zero again. indeed, note that, for $n \geq m+1 \geq |u_n|$ we have $T_n(u_n) = T_{m+1}(u_n) = u_n$, then, from (H_4) and the convexity of $\overline{\varphi}$ we obtain

$$\begin{aligned} &\int_{\{m\leq|u_n|\leq m+1\}} \overline{\varphi}(x, \Phi_n(x,t,u_n))\,dx\,dt \\ &= \int_{\{m\leq|u_n|\leq m+1\}} \overline{\varphi}(x, |\Phi(x,t,T_{m+1}(u_n))|)\,dx\,dt \\ &\leq \int_{\{m\leq|u_n|\leq m+1\}} \overline{\varphi}(\overline{\varphi}_x^{-1}(\varphi(x,|T_{m+1}(u_n)|))\,dx\,dt \\ &\leq \int_{\{m\leq|u_n|\leq m+1\}} \varphi(x,|T_{m+1}(u_n)|)\,dx\,dt \\ &\leq \int_{\Omega_T} \mathbf{Q}(m+1)\,dx\,dt. \end{aligned} \tag{3.42}$$

We deduce that

$$\begin{aligned} &\int_{\{m\leq|u_n|\leq m+1\}} \overline{\varphi}(x, |\Phi(x,t,T_{m+1}(u_n))|)\,dx\,dt \\ &= \int_{\Omega_T} \overline{\varphi}(x, |\Phi(x,t,T_{m+1}(u_n)|)\,\mathbf{1}_{\{m\leq|u_n|\leq m+1\}}dx\,dt \leq C_{0,m}. \end{aligned} \tag{3.43}$$

Let us denote $G_n^m = \overline{\varphi}(x, |\Phi(x,t,T_{m+1}(u_n)|)\,\mathbf{1}_{\{m\leq|u_n|\leq m+1\}} \longrightarrow G^m$ a.e. in Ω where

$$G^m = \overline{\varphi}(x, |\Phi(x,t,T_{m+1}(u)|)\,\mathbf{1}_{\{m\leq|u|\leq m+1\}},$$

since $\overline{\varphi}$ is continuous and Φ is a Carathéodory function. From (3.43), G_n^m is bounded independently of n, using Lebesgue's theorem, it follows, as $n \longrightarrow \infty$

$$\int_{\{m\leq|u_n|\leq m+1\}} \overline{\varphi}(x, |\Phi_n(x,t,u_n)|)\,dx\,dt \longrightarrow \int_{\{m\leq|u|\leq m+1\}} \overline{\varphi}(x, |\Phi(x,t,u)|)\,dx\,dt. \tag{3.44}$$

And then

$$\lim_{m\to\infty}\lim_{n\to\infty} \int_{\{m\leq|u_n|\leq m+1\}} \overline{\varphi}(x, |\Phi_n(x,t,u_n)|)\,dx\,dt = 0 \tag{3.45}$$

Combining (3.40), (3.41) and (3.45) we get

$$\lim_{m\to\infty}\lim_{n\to\infty} \int_{\Omega_T} \Phi_n(x,t,u_n)\nabla\theta_m(u_n)\,dx\,dt = 0 \tag{3.46}$$

At the end, let $m, n \longrightarrow \infty$ in (3.35), we find

$$\lim_{m\to\infty}\lim_{n\to\infty} \int_{\{m\leq|u_n|\leq m+1\}} \mathcal{A}(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt = 0. \tag{3.47}$$

Step 3: Almost everywhere convergence of the gradients.
In this step, most parts of the proof of the following proposition are the same argument as in [38].

Proposition 3.10. *Let u_n be a solution of the approximate problem (3.12). Then, for all $k \geq 0$ we have (for a subsequence still denoted by u_n): as $n \to +\infty$,*
(i) $\nabla u_n \to \nabla u$ *a.e. in* Ω_T,
(ii) $\mathcal{A}(x,t,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup \mathcal{A}(x,t,T_k(u),\nabla T_k(u))$ *weakly in* $(L_{\overline{\varphi}}(\Omega_T))^N$,
(iii) $\varphi(x,|\nabla T_k(u_n)|) \to \varphi(x,|\nabla T_k(u)|)$ *strongly in* $L^1(\Omega_T)$.

Proof. Let $\theta_j \in D(\Omega_T)$ be a sequence such that $\theta_j \longrightarrow u$ in $W_0^{1,x}L_\varphi(\Omega_T)$ for the modular convergence and let $\psi_i \in D(\Omega)$ be a sequence which converges strongly to u_0 in $L^1(\Omega)$.

Put $Z_{i,j}^\mu = T_k(\theta_j)_\mu + \mathrm{e}^{-\mu t}T_k(\psi_i)$ where $T_k(\theta_j)_\mu$ is the mollification with respect to the time of $T_k(\theta_j)$, notice that $Z_{\mu,j}^i$ is a smooth function having the following properties:

$$\frac{\partial Z_{i,j}^\mu}{\partial t} = \mu(T_k(\theta_j) - Z_{i,j}^\mu), \quad Z_{i,j}^\mu(0) = T_k(\psi_i) \quad \text{and } |Z_{i,j}^\mu| \leq k,$$

$$Z_{i,j}^\mu \longrightarrow T_k(u)_\mu + \mathrm{e}^{-\mu t}T_k(\psi_i), \quad \text{in } W_0^{1,x}L_\varphi(\Omega_T) \quad \text{modularly as } j \longrightarrow \infty,$$

$$T_k(u)_\mu + \mathrm{e}^{-\mu t}T_k(\psi_i) \longrightarrow T_k(u), \quad \text{in } W_0^{1,x}L_\varphi(\Omega_T) \quad \text{modularly as } \mu \longrightarrow \infty.$$

Let now the function h_m defined on $\mathbb{R}$ for any $m \geq k$ by:

$$h_m(r) = \begin{cases} 1 & \text{if } |r| \leq m \\ -|r| + m + 1 & \text{if } m \leq |r| \leq m+1 \\ 0 & \text{if } |r| \geq m+1. \end{cases}$$

Put $E_{m,n} = \Big\{(x,t) \in \Omega_T : m \leq |u_n| \leq m+1\Big\}$ and testing the approximate problem (3.12) by the test function $\varphi_{n,j,m}^{\mu,i} = (T_k(u_n) - Z_{i,j}^\mu)h_m(u_n)$, we get

$$\begin{aligned}
&\Big\langle \frac{\partial b_n(x,u_n)}{\partial t}, \varphi_{n,j,m}^{\mu,i} \Big\rangle + \int_{\Omega_T} \mathcal{A}(x,t,u_n,\nabla u_n)(\nabla T_k(u_n) - \nabla Z_{i,j}^\mu)h_m(u_n)\,dx\,dt \\
&\quad + \int_{\Omega_T} \mathcal{A}(x,t,u_n,\nabla u_n)(T_k(u_n) - Z_{i,j}^\mu)\nabla u_n h_m'(u_n)\,dx\,dt \\
&\quad\quad + \int_{E_\varphi} \Phi_n(x,t,u_n)\nabla u_n h_m'(u_n)(T_k(u_n) - Z_{i,j}^\mu)\,dx\,dt \\
&\quad\quad\quad + \int_{\Omega_T} \Phi_n(x,t,u_n)h_m(u_n)(\nabla T_k(u_n) - \nabla Z_{i,j}^\mu)\,dx\,dt \\
&\quad\quad\quad = \int_{\Omega_T} f_n \varphi_{n,j,m}^{\mu,i}\,dx\,dt.
\end{aligned} \tag{3.48}$$

For to be simple, we will denote by $\epsilon(n,j,\mu,i)$ and $\epsilon(n,j,\mu)$ any quantities such that

$$\lim_{i\to+\infty}\lim_{\mu\to+\infty}\lim_{j\to+\infty}\lim_{n\to+\infty} \epsilon(n,j,\mu,i) = 0,$$

$$\lim_{\mu\to+\infty}\lim_{j\to+\infty}\lim_{n\to+\infty} \epsilon(n,j,\mu) = 0.$$

We have the following lemma which can be found in [38, 40].

Lemma 3.11. *(cf. [38, 40]) Let* $\varphi_{n,j,m}^{\mu,i} = (T_k(u_n) - Z_{i,j}^\mu)h_m(u_n)$*, then for any* $k \geq 0$ *we have:*

$$\Big\langle \frac{\partial b_n(x,u_n)}{\partial t}, \varphi_{n,j,m}^{\mu,i} \Big\rangle \geq \epsilon(n,j,\mu,i), \tag{3.49}$$

where $<,>$ *denotes the duality pairing between* $L^1(\Omega_T) + W^{-1,x}L_{\overline{\varphi}}(\Omega_T)$ *and* $L^\infty(\Omega_T) \cap W_0^{1,x}L_\varphi(\Omega_T)$.

To complete the proof of proposition 3.10, we establish the results below, for any fixed $k \geq 0$, we have:

(r_1) $\displaystyle\int_{\Omega_T} f_n \varphi^{\mu,i}_{n,j,m}\, dx\, dt = \epsilon(n, j, \mu)$.

(r_2) $\displaystyle\int_{\Omega_T} \Phi_n(x, t, u_n) h_m(u_n)(\nabla T_k(u_n) - \nabla Z^{\mu}_{i,j})\, dx\, dt = \epsilon(n, j, \mu)$.

(r_3) $\displaystyle\int_{E_{m,n}} \Phi_n(x, t, u_n)\nabla u_n h'_m(u_n)(T_k(u_n) - Z^{\mu}_{i,j})\, dx\, dt = \epsilon(n, j, \mu)$.

(r_4) $\displaystyle\int_{\Omega_T} \mathcal{A}(x, t, u_n, \nabla u_n)(T_k(u_n) - Z^{\mu}_{i,j})\nabla u_n h'_m(u_n)\, dx\, dt \leq \epsilon(n, j, \mu, m)$.

(r_5) $\displaystyle\int_{\Omega_T} [\mathcal{A}(x, t, T_k(u_n), \nabla T_k(u_n)) - \mathcal{A}(x, t, T_k(u_n), \nabla T_k(u)\mathbf{1}_s)] \times[\nabla T_k(u_n) - \nabla T_k(u)\mathbf{1}_s]\, dx\, dt \leq \epsilon(n, j, \mu, m, s)$.

The proofs of (r_1), (r_4) and (r_5) are the same as in [38, 40].

To prove (r_2) and (r_3) to this end, we must have the strong convergence of $\Phi_n(x, t, T_{m+1}(u_n))$ in $(E_{\overline{\varphi}}(\Omega_T))^N$, for $n \geq m + 1$, we have

$$\Phi_n(x, t, u_n)h_m(u_n) = \Phi(x, t, T_{m+1}(u_n))h_m(T_{m+1}(u_n)) \text{ a.e in } \Omega_T.$$

put $P_n = \overline{\varphi}\Big(x, \dfrac{|\Phi(x, t, T_{m+1}(u_n)) - \Phi(x, t, T_{m+1}(u))|}{\eta}\Big)$. Since Φ is continuous with respect to its third argument and $u_n \longrightarrow u$ a.e in Ω_T, then $\Phi(x, t, T_{m+1}(u_n)) \to \Phi(x, t, T_{m+1}(u))$ a.e in Ω_T as n goes to infinity, besides $\overline{\varphi}(x, 0) = 0$, it follows

$$P_n \longrightarrow 0, \quad \text{a.e in } \Omega_T \text{ as } n \to \infty. \tag{3.50}$$

Using now the convexity of $\overline{\varphi}$ and (H_4), we have for every $\eta > 0$ and $n \geq m + 1$:

$$\begin{aligned}
P_n &= \overline{\varphi}\Big(x, \frac{|\Phi(x, t, T_{m+1}(u_n)) - \Phi(x, t, T_{m+1}(u))|}{\eta}\Big)\\
&\leq \overline{\varphi}\Big(x, \frac{|\Phi(x, t, T_{m+1}(u_n))| + |\Phi(x, t, T_{m+1}(u))|}{\eta}\Big)\\
&\leq \overline{\varphi}\Big(x, \frac{2}{\eta}|\gamma(x, t)| + \frac{2}{\eta}\overline{\mathbf{Q}}^{-1}(\mathbf{Q}((m + 1)))\Big)\\
&= \overline{\varphi}\Big(x, \frac{1}{2}\frac{4}{\eta}|\gamma(x, t)| + \frac{1}{2}\frac{4}{\eta}\overline{\mathbf{Q}}^{-1}(\mathbf{Q}((m + 1)))\Big)\\
&\leq \frac{1}{2}\overline{\varphi}(x, \frac{4}{\eta}|\gamma(x, t)|) + \frac{1}{2}\overline{\mathbf{q}}(\frac{4}{\eta}\overline{\mathbf{Q}}^{-1}(\mathbf{Q}((m + 1)))).
\end{aligned} \tag{3.51}$$

We put $Z^{\eta}_k(x) = \frac{1}{2}\overline{\varphi}(x, \frac{4}{\eta}|\gamma(x, t)|) + \frac{1}{2}\overline{\mathbf{q}}(\frac{4}{\eta}\overline{\mathbf{Q}}^{-1}(\mathbf{Q}((m + 1))))$, we have $Z^{\eta}_k \in L^1(\Omega_T)$, since $\gamma \in E_{\overline{\varphi}}(\Omega_T)$. Then by Lebesgue's dominated convergence theorem we get

$$\lim_{n\to\infty}\int_{\Omega_T} P_n\, dx\, dt = \int_{\Omega_T} \lim_{n\to\infty} P_n\, dx\, dt = 0. \tag{3.52}$$

This implies that $\{\Phi(x, t, T_{m+1}(u_n))\}$ converges modularly to $\Phi(x, t, T_{m+1}(u))$ as $n \to \infty$ in $(L_{\overline{\varphi}}(\Omega_T))^N$. Moreover, $\Phi(x, t, T_{m+1}(u_n))$ and $\Phi(x, t, T_{m+1}(u))$ lie in $(E_{\overline{\varphi}}(\Omega_T))^N$, indeed, from (H_4) we have for every $\eta > 0$

$$\begin{aligned}
&\int_{\Omega_T} \overline{\varphi}\Big(x, \frac{|\Phi(x,t,T_{m+1}(u_n))|}{\eta}\Big)\, dx\, dt \\
&\leq \int_{\Omega_T} \overline{\varphi}\Big(x, \frac{1}{\eta}|\gamma(x,t)| + \frac{1}{\eta}\overline{\varphi}_x^{-1}(\varphi(x,|T_{m+1}(u_n)|))\Big)\, dx\, dt \\
&\leq \int_{\Omega_T} \overline{\varphi}\Big(x, \frac{1}{2}\frac{2}{\eta}|\gamma(x,t)| + \frac{1}{2}\frac{2}{\eta}\overline{\mathbf{Q}}^{-1}(\mathbf{Q}((m+1)))\Big)\, dx\, dt \\
&\leq \int_{\Omega_T} \frac{1}{2}\overline{\varphi}(x, \frac{2}{\eta}|\gamma(x,t)|)\, dx\, dt + \int_{\Omega_T} \frac{1}{2}\overline{\varphi}\Big(x, \frac{2}{\eta}\overline{\mathbf{Q}}^{-1}(\mathbf{Q}((m+1)))\Big)\, dx\, dt \\
&\leq \int_{\Omega_T} \frac{1}{2}\overline{\varphi}(x, \frac{2}{\eta}|\gamma(x,t)|)\, dx\, dt + \int_{\Omega_T} \frac{1}{2}\overline{\mathbf{q}}\Big(\frac{2}{\eta}\overline{\mathbf{Q}}^{-1}(\mathbf{Q}((m+1)))\Big)\, dx\, dt \\
&< \infty \text{ since } \gamma \in E_{\overline{\varphi}}(\Omega_T) \text{ and } \Omega \text{ is bounded,}
\end{aligned}$$

the same for $\Phi(x,t,T_{m+1}(u))$. Thanks to lemma 2.6, we deduce that $\Phi(x,t,T_{m+1}(u_n)) \longrightarrow \Phi(x,t,T_{m+1}(u))$ strongly in $(E_{\overline{\varphi}}(\Omega_T))^N$. On the other hand, $\nabla T_k(u_n) \rightharpoonup \nabla T_k(u)$ weakly in $(L_\varphi(\Omega_T))^N$ as n goes to infinity, it follows that

$$\begin{aligned}
&\lim_{n\to\infty} \int_{\Omega_T} \Phi(x,t,u_n)h_m(u_n)[\nabla T_k(u_n) - \nabla Z^\mu_{i,j}]\, dx\, dt \\
&= \int_{\Omega_T} \Phi(x,t,u)h_m(u)[\nabla T_k(u) - \nabla Z^\mu_{i,j}]\, dx\, dt.
\end{aligned} \tag{3.53}$$

Using the modular convergence of $Z^\mu_{i,j}$ as $j \longrightarrow \infty$ and then $\mu \longrightarrow \infty$, we get (r_2). Now we prove (r_3), remark that for $n \geq m+1$, we have

$$\nabla u_n h'_m(u_n) = \nabla T_{m+1}(u_n) \text{ a.e in } \Omega_T.$$

By the almost everywhere convergence of u_n, we have $T_k(u_n) - Z^\mu_{i,j}$ converges to $T_k(u) - Z^\mu_{i,j}$ in $L^\infty(\Omega_T)$ weak-$*$ and we have already proved that $\Phi(x,t,T_{m+1}(u_n)) \longrightarrow \Phi(x,t,T_{m+1}(u))$ strongly in $(E_{\overline{\varphi}}(\Omega_T))^N$ then,

$$\Phi(x,t,T_{m+1}(u_n))\left(T_k(u_n) - Z^\mu_{i,j}\right) \longrightarrow \Phi(x,t,T_{m+1}(u))\left(T_k(u) - Z^\mu_{i,j}\right),$$

strongly in $E_{\overline{\varphi}}(\Omega_T)$ as $n \longrightarrow \infty$. Using again the fact that, $\nabla T_{m+1}(u_n) \rightharpoonup \nabla T_{m+1}(u)$ weakly in $(L_\varphi(\Omega_T))^N$ as n tends to $+\infty$ we obtain

$$\begin{aligned}
&\int_{E_{m,n}} \Phi_n(x,t,u_n)\nabla u_n h'_m(u_n)(T_k(u_n) - Z^\mu_{i,j})\, dx\, dt \\
&\longrightarrow \int_{E_m} \Phi(x,t,u)\nabla u(T_k(u) - Z^\mu_{i,j})\, dx\, dt \text{ as } n \longrightarrow \infty.
\end{aligned}$$

Using the modular convergence of $Z^\mu_{i,j}$ as $j \longrightarrow +\infty$ and letting μ tends to infinity, we get (r_3). As a consequence of lemma 3.1, the results of proposition 3.10 follow.

Step 4: Passing to the limit.

The limit u of the approximate solution u_n of (3.12) satisfies:

$$\lim_{m\to\infty} \int_{\{m\leq |u|\leq m+1\}} \mathcal{A}(x,t,u,\nabla u)\nabla u\, dx\, dt = 0. \tag{3.54}$$

Proof. Fix $m > 0$ and we can write

$$\begin{aligned}&\int_{\{m\leq|u_n|\leq m+1\}} \mathcal{A}(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt\\ &= \Big(\int_{\Omega_T} \mathcal{A}(x,t,u_n,\nabla u_n)(\nabla T_{m+1}(u_n) - \nabla T_m(u_n))\,dx\,dt\Big)\\ &= \Big(\int_{\Omega_T} \mathcal{A}(x,t,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\nabla T_{m+1}(u_n)\,dx\,dt\\ &\quad - \int_{\Omega_T} \mathcal{A}(x,t,T_m(u_n),\nabla T_m(u_n))\nabla T_m(u_n))\,dx\,dt\Big).\end{aligned}$$

Using (ii), (iii) of proposition 3.10 and passing to the limit as n goes to infinity for fixed m, we get

$$\begin{aligned}&\lim_{n\to\infty} \int_{\{m\leq|u_n|\leq m+1\}} \mathcal{A}(x,t,u_n,\nabla u_n)\nabla u_n\,dx\\ &= \int_{\{m\leq|u|\leq m+1\}} \mathcal{A}(x,t,u,\nabla u)\nabla u\,dx.\end{aligned}$$

Finally, we pass to the limit as m goes to infinity and then we use (3.47), it follows

$$\begin{aligned}&\lim_{m\to\infty}\lim_{n\to\infty} \int_{\{m\leq|u_n|\leq m+1\}} \mathcal{A}(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt\\ &= \lim_{m\to\infty} \int_{\{m\leq|u|\leq m+1\}} \mathcal{A}(x,t,u,\nabla u)\nabla u\,dx\,dt = 0.\end{aligned}$$

Which give the desired result.

Now, we will pass to the limit. Testing the approximate problem (3.12) by $r(u_n)$ with $r \in W^{1,\infty}(\mathbb{R})$ having a compact support such that for $k > 0$, $supp(r) \subset [-k, k]$ we get

$$\begin{aligned}&\frac{\partial B_r^n(x,u_n)}{\partial t} - \operatorname{div}\left(r(u_n)\mathcal{A}(x,t,u_n,\nabla u_n)\right) + r'(u_n)\mathcal{A}(x,t,u_n,\nabla u_n)\nabla u_n\\ &-\operatorname{div}\left(r(u_n)\Phi(x,t,u_n)\right) + r'(u_n)\Phi(x,t,u_n)\nabla u_n = fr(u_n) \quad \text{in} \quad D'(\Omega_T),\end{aligned} \tag{3.55}$$

where $B_r^n(x,\tau) = \displaystyle\int_0^\tau \frac{\partial b_n(x,s)}{\partial s} r'(s)\,ds$.

Our aim here is to pass to the limit in each term in the previous equality, let us start by the terms of the left-hand side:

Limit of the first term $\dfrac{\partial B_r^n(x,u_n)}{\partial t}$, since r is bounded and $B_r^n(x,u_n) \longrightarrow B_r(x,u)$ a.e in Ω_T and in $L^\infty(\Omega_T)$ weak*, then

$$\frac{\partial B_r^n(x,u_n)}{\partial t} \longrightarrow \frac{\partial B_r(x,u)}{\partial t} \quad \text{in} \quad D'(\Omega_T) \quad \text{as} \quad n\to\infty.$$

Remark that, since r and r' have a compact support in $\mathbb{R}$, there exists $k > 0$ such that $supp(r), supp(r') \subset [-k, k]$. For n large enough, we have:

$$r(u_n)\mathcal{A}(x,t,u_n,\nabla u_n) = r(u_n)\mathcal{A}(x,t,T_k(u_n),\nabla T_k(u_n)) \quad \text{a.e. in } \Omega_T,$$

$$\begin{aligned}r'(u_n)\mathcal{A}(x,t,u_n,\nabla u_n)\nabla u_n &= r'(u_n)\mathcal{A}(x,t,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n) \quad \text{a.e. in } \Omega_T,\\ r(u_n)\Phi_n(x,t,u_n) &= r(T_k(u_n))\Phi_n(x,t,T_k(u_n)),\end{aligned}$$

$$r'(u_n)\Phi_n(x,t,u_n)\nabla u_n = r'(T_k(u_n))\Phi_n(x,t,T_k(u_n))\nabla T_k(u_n).$$

For the second term of (3.55), Since $r(u_n) \to r(u)$ a.e in Ω_T as $n \to \infty$, r is bounded and $(ii), (iii)$ of proposition 3.10 we have

$$r(u_n)\mathcal{A}(x,t,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup r(u)\mathcal{A}(x,t,T_k(u),\nabla T_k(u))$$

weakly in $(L_{\overline{\varphi}}(\Omega_T))^N$ for $\sigma(\Pi L_{\overline{\varphi}}, \Pi E_{\varphi})$,
then

$$r(u_n)\mathcal{A}(x,t,u_n,\nabla u_n) \rightharpoonup r(u)\mathcal{A}(x,t,u,\nabla u) \text{ weakly in } (L_{\overline{\varphi}}(\Omega_T))^N.$$

Concerning the third term of (3.55), Since $r'(u_n) \to r'(u)$ a.e in Ω_T as $n \to \infty$, r' is bounded and $(ii), (iii)$ of proposition 3.10 we obtain, as $n \to \infty$

$$r'(u_n)\mathcal{A}(x,t,u_n,\nabla u_n)\nabla u_n \rightharpoonup r'(u)\mathcal{A}(x,t,T_k(u),\nabla T_k(u))\nabla T_k(u) \quad \text{weakly in } L^1(\Omega_T).$$

And then

$$r'(u)\mathcal{A}(x,t,T_k(u),\nabla T_k(u))\nabla T_k(u) = r'(u)\mathcal{A}(x,t,u,\nabla u)\nabla u \quad \text{a.e. in } \Omega_T.$$

Arguing similarly, we get the limit of the fourth term of (3.55),

$$r(u_n)\Phi_n(x,t,u_n) \to r(u)\Phi(x,t,u) \text{ strongly in } (E_{\varphi}(\Omega_T))^N.$$

For the remaining term of the left-hand side, we have $r'(u_n)$ converges to $r'(u)$ and $\nabla T_k(u_n) \rightharpoonup \nabla T_k(u)$ weakly in $(L_{\varphi}(\Omega_T))^N$ as $n \to +\infty$, while $\Phi_n(x, T_k(u_n))$ is uniformly bounded with respect to n and converges a.e. in Ω_T to $\Phi(x, T_k(u))$ as n tends to $+\infty$. Therefore

$$r'(u_n)\Phi_n(x,t,u_n)\nabla u_n \rightharpoonup r'(u)\Phi(x,t,u)\nabla u \text{ weakly in } L_{\varphi}(\Omega_T).$$

Concerning the right-hand side of (3.55), due to (i) of lemma 3.8 and the fact that f_n converges strongly to f in $L^1(\Omega_T)$, we have

$$f_n r(u_n) \longrightarrow f r(u) \quad \text{strongly in } L^1(\Omega_T) \text{ as } n \to \infty.$$

Now, we are ready to pass to the limit as $n \to \infty$ in each term of (3.55) to conclude that u satisfies (3.11). It remains to show that $B_r(x,u)$ satisfies the initial condition of (3.12). To do this, recall that, r' has a compact support, we have $B_r^n(x,u_n)$ is bounded in $L^\infty(\Omega_T)$. Moreover, (3.55) and the above considerations on the behavior of the terms of this equation show that $\dfrac{\partial B_r^n(x,u_n)}{\partial t}$ is bounded in $L^1(\Omega_T) + W^{-1,x}L_{\overline{\varphi}}(\Omega_T)$. As a consequence, an Aubin's type Lemma (cf [41, Corollary 4]) and (lemma 2.10) imply that $B_r^n(x,u_n)$ is in a compact set of $C^0([0,T]; L^1(\Omega))$. It follows that, $B_r^n(x,u_n)(t=0)$ converges to $B_r(x,u)(t=0)$ strongly in $L^1(\Omega)$. Due to remark 3.3 and the fact that $b_n(x,u_{0n}) \longrightarrow b(x,u_0)$ in $L^1(\Omega)$, we conclude that $B_r^n(x,u_n)(t=0) = B_r^n(x,u_{0n})$ converges to $B_r(x,u)(t=0)$ strongly in $L^1(\Omega)$. Then we conclude that $B_r(x,u)(t=0) = B_r(x,u_0)$ in Ω.

That is the full proof of the main result.

References

[1] A. Aberqi, J. Bennouna and H. Redwane, *A nonlinear parabolic problems with lower order terms and measure data*, Thai J. Math. 14(1) (2016) 115–130.

[2] A. Aberqi, J. Bennouna and M. Elmassoudi, *Existence and uniqueness of renormalized solution for nonlinear parabolic equations in Musielak-Orlicz spaces*, Bol. Soc. Paran. Mat. (2019) doi:10.5269/bspm.45234

[3] R. Adams, *Sobolev spaces*, Academic Press Inc, New York. 1975.

[4] M.L. Ahmed Oubeid, A. Benkirane and M. Sidi El Vally, *Strongly nonlinear parabolic prob- lems in Musielak-Orlicz-Sobolev spaces*, Bol. Soc. Paran. Mat. (2015) 191–223.

[5] M.L. Ahmed Oubeid, A. Benkirane and M. Sidi El Vally, *Parabolic equations in Musielak-Orlicz-Sobolev spaces*, Int. J. Anal. Appl. 4(2) (2014) 174–191.

[6] Y. Ahmida, I. Chlebicka, P. Gwiazda and A. Youssfi, *Gossez's approximation theorems in Musielak-Orlicz-Sobolev spaces*, J. Funct. Anal. (2018), https://doi.org/10.1016/j.jfa.2018.05.015

[7] M. Ait Khellou and A. Benkirane, *Elliptic inequalities with L^1 data in Musielak-Orlicz spaces*, Monat. Math. 183 (2017) 1–33.

[8] M. Ait Khellou and A. Benkirane, *Correction to: elliptic inequalities with L^1 data in Musielak-Orlicz spaces*, Monat. für Math. 187 (2018) 181–187.

[9] M. Ait Khellou, A. Benkirane and S. M. Douiri, *Existence of solutions for elliptic equations having natural growth terms in Musielak-Orlicz spaces*, J. Math. Comput. Sci. 4(4) (2014) 665–688.

[10] M. Ait Khellou and A. Benkirane, *Renormalized solution for nonlinear elliptic problems with lower order terms and L^1 data in Musielak-Orlicz spaces*, Ann. University of Craiova, Math. Comput. Sci. Series, 43(2)(2016) 164–187.

[11] A. Alvino, L. Boccardo, V. Ferone, L. Orsina and G. Trombetti, *Existence results for nonlinear elliptic equations with degenerate coercivity*, Ann. Mat. Pura Appl., IV. Ser. 182(1) (2003) 53–79.

[12] A. Alvino, V. Ferone and G. Trombetti, *A priori estimates for a class of non uniformly elliptic equations*, Atti Semin. Mat. Fis. Univ. Modena **46**-suppl., (1998) 381–391.

[13] E. Azroul, H. Redwane and M. Rhoudaf, *Existence of a renormalized solution for a class of nonlinear parabolic equations in Orlicz spaces*, Port. Math. 66(1) (2009) 29–63.

[14] A. Benkirane and M. Sidi El Vally(Ould Mohameden Val), *Variational inequalities in Musielak-Orlicz-Sobolev spaces*, Bull. Belg. Math.Soc. Simon Stevin, 21(5) (2014) 787–811.

[15] A. Benkirane and M. Sidi El Vally(Ould Mohameden Val), *An existence result for nonlinear elliptic equations in Musielak-Orlicz-Sobolev spaces*, Bull. Belg. Math.Soc. Simon Stevin 20 (2013) 57–75.

[16] C. Bennett and R. Sharpley, *Interpolation of operators*, Academic press, Boston. 1988.

[17] J. Bennouna, M. Hammoumi and A. Aberqi, *Nonlinear degenerated parabolic equations with lower order terms*, Elec. J Math. Anal. Appl. (2016) 234–253.

[18] D. Blanchard and F. Murat, *Renormalized solutions of nonlinear parabolic problems with L^1 data, existence and uniqueness*, Proc. R. Soc. Edinburgh Sect. A 127 (1997) 1137–1152.

[19] D. Blanchard, F. Murat and H. Redwane, *Existence et unicité de la solution renormalisée d'un problème parabolique assez général*, C. R. Acad. Sci. Paris Sér. 1329 (1999) 575–580.

[20] D. Blanchard and A. Porretta, *A Stefan problems with diffusion and convection*, Differ. Equ. 210 (2005) 383–428.

[21] L. Boccardo, A. Dall'Aglio and L. Orsina, *Existence and regularity results for some elliptic equations with degenerate coercivity.* Atti Semin. Mat. Fis. Univ. Modena **46**-suppl. (1998) 51–81.

[22] M. Bourahma, A. Benkirane and J. Bennouna, *Existence of renormalized solutions for some nonlinear elliptic equations in Orlicz spaces*, J. Rend. Circ. Mat. Palermo, II. Ser (2019) https://doi.org/10.1007/s12215-019-00399-z

[23] M. Bourahma, A. Benkirane and J. Bennouna, *An existence result of entropy solutions to elliptic problems in generalized Orlicz-Sobolev spaces*, J. Rend. Circ. Mat. Palermo, II. Ser (2020) https://doi.org/10.1007/s12215-020-00506-5

[24] M. Bourahma, J. Bennouna and M. El Moumni, *Existence of a weak bounded solutions for a nonlinear degenerate elliptic equations in Musielak spaces*, Moroccan J. of Pure and Appl. Anal. (MJPAA) 6(1) (2020) 16–33.

[25] G. Dal Maso, F. Murat, L. Orsina and A. Prignet, *Renormalized solution of elliptic equation with general measure data*. Ann. Scuola Norm. Sup. Pisa CI. Sci. 28(4) (1999) 741–808.

[26] M. S. B. Elemine Vall, A. Ahmed, A. Touzani and A. Benkirane, *Existence of entropy solutions for nonlinear elliptic equations in Musielak framework with L^1 data*, Bol. Soc. Paran. Mat.(3s.) v. 36 1 (2018) 125–150.

[27] B. El Haji, M. El Moumni and K. Kouhaila, *On a nonlinear elliptic problems having large monotonocity with L^1-data in weighted Orlicz-Sobolev spaces*, Moroccan J. Pure Appl. Anal. 5 (1) 104–116

[28] A. Elmahi and D. Meskine, *Strongly nonlinear parabolic equations with natural growth terms and L^1 data in Orlicz spaces* , Port. Math. Nova 62 (2005) 143–183.

[29] A. Elmahi and D. Meskine, *Strongly nonlinear parabolic equations with natural growth terms and L^1 data in Orlicz spaces.* Portugaliae Mathematica. Nova 62 (2005) 143–183.

[30] J.P. Gossez, *A strongly nonlinear elliptic problem in Orlicz-Sobolev spaces*, Proc. Sympos. Pure Math. 45 (1986) 455–462.

[31] J.P. Gossez, *Surjectivity results for pseudo-monotone mappings in complementary systems*, J. Math. Anal. Appl. 53, (1976) 484–494.

[32] J. P. Gossez, *Nonlinear elliptic boundary value problems for equations with rapidly (or slowly) increasing coefficients*, Trans. Amer. Math. Soc. 190. 1974 163-205.

[33] J. P. Gossez and V. Mustonen, *Variationnal inequalities in Orlicz-Sobolev spaces.*, Nonlinear Anal. 11 (1987) 317–492.

[34] P. Gwiazda I. Skrzypczak and A. Zatorska.Goldstein, *Existence of renormalized solutions to elliptic equation in Musielak-Orlicz space*, J. Diff. Equ. 264 (2018) 341–377.

[35] P. Harjulehto and P. Hästö, *Orlicz spaces and generalized Orlicz spaces*, Lecture Notes in Mathematics, vol. 2236, Springer, Cham. 2019.

[36] M. Krasnosel'skii and Ya. Rutikii, *Convex functions and Orlicz spaces*, Groningen, Nordhooff. 1969.

[37] R. Landes and V. Mustonen, *A strongly nonlinear parabolic initial-boundary value problem*, Ask. F. Mat. 25 (1987) 29–40.

[38] H. Moussa and M. Rhoudaf, *Renormalized solution for a nonlinear parabolic problems with noncoersivity in divergence form in Orlicz Spaces*, Appl. Math. Comput. 249 (2014) 253–264.

[39] J. Musielak, *Orlicz spaces and modular spaces*, Lecture Note in Mahtematics, 1034, Springer, Berlin. 1983.

[40] H. Redwane, *Existence results for a class of onlinear parabolic equations in orlicz spaces*, Elec. J. Qual. Theo. Diff. Equ. 2 (2010) 1–19.

[41] J. Simon, *Compact sets in* $L^p(0,T;B)$, Ann. Mat. Pura Appl. 146 (1987) 65–96.