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Existence result of solutions for a class of nonlinear differential systems

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Abstract

In this paper, we will discuss the existence of bounded positive solutions for a class of nonlinear differential systems. The objective will be achieved by applying some results and techniques of functional analysis such as Schauder's fixed point theorem and potential theory tools.

Keywords: nonlinear differential system, potential theory, Green's function, positive solution, Schauder's fixed point theorem
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1. Introduction

We are interested in the study of bounded positive solutions to the following nonlinear differential system

$$\left\{ \begin{array}{l} \dfrac{1}{\varphi_i}\left(\varphi_i u_i'\right)' = p_i u_i^{\alpha_i} \prod\limits_{\substack{j=1 \\ j\neq i}}^{m} u_j^{k_{ij}} \quad , \quad \text{on } (0,+\infty) \\ \\ \varphi_i u_i'(0) = 0 \quad \text{and} \quad u_i(+\infty) = b_i \end{array} \right. , \ 1 \leq i \leq m \tag{1.1}$$

where $u = (u_1, \ldots, u_m)$, and for all $1 \leq i, j \leq m$, we have $k_{ii} = \alpha_i \geq 1$, $k_{ij} \geq 0$, $b_i > 0$,

$$u_i(+\infty) = \lim_{x\to+\infty} u_i(x)$$

$$\varphi_i u_i'(0) = \lim_{x\to 0} \varphi_i(x)\, u_i'(x)$$

and the functions φ_i satisfies the following condition (H1) :

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(H1) φ_i is a continuous function on $[0,+\infty)$, differentiable and positive on $(0,+\infty)$, such that

$$\int_1^{+\infty} \frac{dt}{\varphi_i(t)} < +\infty \quad \text{and} \quad \int_0^1 \frac{1}{\varphi_i(t)} \left(\int_0^t \varphi_i(s)\, ds \right) dt < +\infty \tag{1.2}$$

We denote $B^+((0,+\infty))$ the set of nonnegative measurable functions on $(0,+\infty)$. We define $C([0,+\infty])$ the space of all continuous functions u in $[0,+\infty)$ such that $\lim\limits_{x\to+\infty} u(x)$ exists and $C_0([0,+\infty))$ the subspace of $C([0,+\infty])$ consisting of functions which vanish continuously at $+\infty$.

For any function φ satisfying (H1), we denote by G the Green's function of the operator L with Dirichlet conditions, i.e.

$$\begin{cases} Lu = \dfrac{1}{\varphi}(\varphi u')' \quad \text{on} \quad (0,+\infty) \\ \varphi u'(0) = 0 \quad , \quad u(+\infty) = 0 \end{cases}$$

that is,

$$G(x,t) = \varphi(t) \int_{\max\{x,t\}}^{+\infty} \frac{dr}{\varphi(r)} \quad , \quad \text{for } x,\ t \in (0,+\infty)$$

and we define the potential of a function $f \in B^+((0,+\infty))$ by

$$Vf(x) = \int_0^{+\infty} G(x,t) f(t) dt$$

We point out that for each $f \in B^+((0,+\infty))$ such that $Vf(0) < +\infty$, we have

$$\begin{cases} Vf \in C_0([0,+\infty)) \cap C^1((0,+\infty)) \\ L(Vf) = -f \quad , \text{a.e. on } (0,+\infty) \\ A(Vf)'(0) = 0 \quad , \quad Vf(+\infty) = 0 \end{cases}$$

Let us introduce the functions p_i, $1 \le i \le m$, that satisfies the following condition (H2) :

(H2) $p_i : (0,+\infty) \to [0,+\infty)$ is measurable function such that $V_i p_i(0) < +\infty$.

where for all $f \in B^+((0,+\infty))$ and $1 \le i \le m$:

$$V_i\ f(x) = \int_0^{+\infty} G_i(x,t) f(t) dt$$

and

$$G_i(x,t) = \varphi_i(t) \int_{\max\{x,t\}}^{+\infty} \frac{dr}{\varphi_i(r)} \quad , \quad \text{for } x,\ t \in (0,+\infty)$$

Before stating the main result of this work, it should be mentioned that several mathematicians have dealt with many problems of type (1.1) using various analytical and numerical techniques and methods under different hypotheses as appropriate. We refer to [1]-[3], [6], [7], [9]-[17], [19], [21], [22] and corresponding references therein for some recent results of the solutions to systems of this type. This is because they appear in the modeling of many physical phenomena such as the propagation of pulses in birefringent optical fibers and supports Kerr-like photorefractive, we also refer to [4] and [18].

Concerning the problem (1.1) in the case of a single equation of the form

$$\frac{1}{\varphi}\left(\varphi u'\right)' - qu = -f\left(\cdot, u\right) \quad \text{on } \left]0, \omega\right[$$

with Dirichlet conditions, Maâgli and Masmoudi in [16] showed a result of existence and uniqueness of a strictly positive regular solution. In [15], Maâgli proved the existence and the uniqueness of a positive solution of this equation but this time with the following conditions

$$u\left(0\right) = u\left(1\right) \quad \text{and} \quad \varphi u'\left(0\right) = \varphi u'\left(1\right)$$

In 2003, Maâgli and Zeddini [14] generalized the result of Taliaferro [20], they studied (1.1) with Dirichlet conditions and a nonlinear term nonnegative continuous nonincreasing with respect to the second variable.

Ben Othman et *al.* in [6] studied some existence results for the nonlinear equation

$$\frac{1}{\varphi}\left(\varphi u'\right)' = u\psi\left(\cdot, u\right) \quad \text{on } \left]0, \omega\right[$$

with different boundary conditions, where ψ is a nonnegative function.

One of the main objectives of Ghanmi et *al.* in [8] is to establish the necessary and sufficient conditions for the existence of global solutions of a system of the form (1.1) with some hypothesis.

In 2009, Gontara [9] studied the existence and nonexistence of solutions for a system with two equations of the form (1.1).

This document is organized as follows : In the next section, we will give the main result of this work. The purpose of section 3 is to give some technical results and to recall some potential theoretical tools that are essential to prove our main result. The last section is devoted to prove the main result by applying Schauder's fixed point Theorem and tools of the theory of potentials.

2. The main result

Now, we give the main result of our work, it is the following existence result :

Theorem 2.1. *Let $\varphi_1, \ldots, \varphi_m$ be functions satisfying (H1) and let $p_1, \ldots, p_m$ be functions satisfying (H2). Then for each $b_1, \ldots, b_m > 0$, the system (1.1) has a positive solution $u = (u_1, \ldots, u_m)$ in $C([0, +\infty]) \cap C^1((0, +\infty))$. Moreover, there exists $c_1, \ldots, c_m > 0$ such that for each $x \in [0, +\infty)$, we have*

$$0 < b_i \exp(-c_i V_i p_i(0)) \leq u_i(x) \leq b_i \quad , \quad \textit{for all } 1 \leq i \leq m$$

3. Preliminary results

Let φ be a function satisfying (H1). The objective of this section is to give some technical results concerning the operator $Lu = \dfrac{1}{\varphi}(\varphi u')'$ and to recall some potential theory tools which are crucial to prove our main result. For the proof and more details, we refer to [5], [9], [15] and [16].

In particular, we give an existence and a uniqueness result to the problem

$$\begin{cases} Lu = p\left(x\right) u^{\alpha} \quad , \quad x \in (0, +\infty) \\ \varphi u'\left(0\right) = 0 \quad , \quad u\left(\infty\right) = b > 0 \end{cases} \tag{3.1}$$

where $\alpha \geq 1$ and $p \in B^{+}\left((0, +\infty)\right)$ such that $Vp(0) < +\infty$.

Proposition 3.1. *Let $q \in B^+((0,+\infty))$ such that $Vq(0) < +\infty$. Then the family of functions*

$$F_q = \left\{ x \mapsto Vf(x) = \int_0^{+\infty} G(x,t)f(t)dt, \ |f| \le q \right\} \tag{3.2}$$

is uniformly bounded and equicontinuous in $[0,+\infty]$. Consequently, F_q is relatively compact in $C_0([0,\infty))$.

Proof . We can write

$$Vf(x) = \int_x^{\infty} \frac{1}{\varphi(t)} (\int_0^t \varphi(r)f(r)dr)dt$$

We deduce that for $x, y \in [0,\infty)$, we have

$$|Vf(x) - Vf(y)| \le \int_x^y \frac{1}{\varphi(t)} \int_0^t \varphi(r)q(r)drdt$$

Since $Vq(0) < \infty$, it follows by the dominated convergence Theorem the equicontinuity of F_q in $[0,\infty)$. Moreover, since

$$|Vf(x)| \le \int_x^{\infty} \frac{1}{\varphi(t)} \int_0^t \varphi(r)q(r)drdt$$

We deduce that $\lim\limits_{x\to\infty} Vf(x) = 0$, uniformly in f which proves that F_q is uniformly bounded in $[0,\infty]$. Using Ascoli's Theorem, we deduce that F_q is relatively compact in $C_0([0,\infty))$. $\square$

Lemma 3.2. *Let $q \in B^+(0,+\infty)$ such that $Vq(0) < +\infty$. Then the problem*

$$\begin{cases} \dfrac{1}{\varphi}(\varphi u')' - qu = 0 \quad , \ a.e. \ on \ (0,+\infty) \\ \varphi u'(0) = 0 \quad , \quad u(0) = 1 \end{cases} \tag{3.3}$$

has a unique solution $\psi \in C([0,+\infty)) \cap C^1((0,+\infty))$ satisfying for each $t \in [0,+\infty)$,

$$1 \le \psi(t) \le \exp\left[\int_0^t \frac{1}{\varphi(s)} \left(\int_0^s \varphi(r)q(r)dr\right) ds\right]$$

Proof . Let K be the operator defined on $C([0,\infty))$ by

$$Kf(t) = \int_0^t \frac{1}{\varphi(s)} \int_0^s \varphi(r)q(r)f(r)drds \quad , \quad t \in [0,\infty)$$

which leads to

$$0 \le K^n 1(t) \le \frac{(K1(t))^n}{n!} \quad , \quad \text{for } t \in [0,\infty) \text{ and } n \in \mathbb{N}$$

Then, the series $\sum\limits_{n\ge 0} K^n 1$ converges uniformly to a function $\psi \in C([0,\infty))$ satisfying

$$\psi(t) = 1 + \int_0^t \frac{1}{\varphi(s)} \int_0^s \varphi(r)q(r)\psi(r)drds \quad , \quad \text{for } t \in [0,\infty)$$

This implies that $\psi \in C^1((0,\infty))$ is a solution of the problem (3.3). Moreover, we have

$$1 \leq \psi(t) \leq \sum_{n\geq 0} \frac{[K1(t)]^n}{n!} = \exp\left[K1(t)\right] \quad , \quad \text{for } t \in [0,\infty)$$

Now, we assume that u and v are two solutions in $C([0,\infty)) \cap C^1((0,\infty))$ of problem (3.3) and $\omega = |u-v|$, then

$$0 \leq \omega(t) \leq K\omega(t) \quad , \quad \text{for } t \in [0,\infty)$$

It follows that for $t \in [0,\infty)$ and $n \in \mathbb{N}$

$$0 \leq \omega(t) \leq K^n\omega(t) \leq \|\omega\|_\infty K^n 1(t) \leq \|\omega\|_\infty \frac{[K1(t)]^n}{n!}$$

By letting $n \to \infty$, we deduce that $\omega(t) = 0$, for $t \in [0,\infty)$ and so $u = v$ on $[0,\infty)$. $\square$

We denote by G_q the Green's function of the operator

$$u \mapsto \frac{1}{\varphi}(\varphi u')' - qu$$

on $(0,\infty)$ with Dirichlet conditions $\varphi u'(0) = 0$, $u(+\infty) = 0$. Then

$$G_q(x,t) = \varphi(t)\psi(x)\psi(t)\int_{\max\{x,t\}}^{+\infty} \frac{dr}{\varphi(r)\psi^2(r)}, \quad \text{for } x,t \in (0,+\infty)$$

So, we define the potential kernel V_q in $B^+\left((0,+\infty)\right)$ by

$$V_q f(x) = \int_0^{+\infty} G_q(x,t)f(t)dt$$

Note that V_q is the unique kernel which satisfies the resolvent equation

$$V = V_q + V_q(qV) = V_q + V(qV_q) \tag{3.4}$$

So, if $u \in B^+\left((0,+\infty)\right)$ such that $V(qu)(0) < +\infty$, we have

$$\left(I - V_q(q\cdot)\right)\left(I + V(q\cdot)\right)u = \left(I + V(q\cdot)\right)\left(I - V_q(q\cdot)\right)u = u\cdot \tag{3.5}$$

Now, we recall an existence result given in [6] for the nonlinear problem

$$\begin{cases} Lu = \dfrac{1}{\varphi}\left(\varphi u'\right)' = u\varphi\left(\cdot, u\right) \quad \text{on } \;]0,+\infty[\\ \varphi u'\left(0\right) = 0 \;\text{ and }\; u\left(+\infty\right) = a > 0 \end{cases} \tag{3.6}$$

with the nonlinear term φ satisfies the following hypothesis :

(Y1) φ is nonnegative measurable function in $[0,+\infty) \times (0,+\infty)$.

(Y2) For each $c > 0$, there exists $q_c \in B^+\left((0,+\infty)\right)$ such that $Vq_c(0) < +\infty$ and for each $x \in (0,+\infty)$, the function $t \mapsto t(q_c(x) - \varphi(x,t))$ is continuous and nondecreasing on $[0,c]$.

Proposition 3.3. *For each $a > 0$, problem* (3.6) *has a positive bounded solution $u \in C([0,+\infty]) \cap C^1((0,+\infty))$ satisfying for each $x \in [0,\infty)$,*

$$e^{-V_{q(0)}} a \leq u(x) \leq a$$

where $q = q_a$ is the function given in (Y2).

Proof . See [6]. □

Lemma 3.4. *Let $a > 0$ and φ be a function satisfying (Y1) and (Y2). Let u be a positive function in $C([0,+\infty]) \cap C^1((0,+\infty))$. Then u is a solution of* (3.6)*, if and only if, u satisfies*

$$u + V(u\varphi(\cdot,u)) = a \quad \text{on } [0,+\infty) \tag{3.7}$$

Proof . Let u be a positive function in $C([0,\infty]) \cap C^1((0,\infty))$ satisfying (3.7), then $u \leq a$. Let $q = q_a$ be the function given by (Y2), then we have

$$u\lambda(\cdot,u) \leq qu \leq aq$$

Since $Vq(0) < \infty$, it follows from Proposition 3.1 that the function $v = V(u\lambda(\cdot,u))$ is in $C_0([0,\infty))$ and so v satisfies

$$\begin{cases} Lv = -\lambda(\cdot,u) \quad \text{a.e. on } (0,+\infty) \\ \lambda v'(0) = 0 \quad \text{and} \quad v(+\infty) = 0 \end{cases} \tag{3.8}$$

This with (3.7) proves that u is a solution of (3.6).Now, let u be a positive function in $C([0,\infty]) \cap C^1((0,\infty))$ satisfying (3.6). Since $\lambda u'(0) = 0$, then $\lambda u'(x) \geq 0$ for $x \in (0,\infty)$. It follows by $u(\infty) = a$ that $u \leq a$. So, by hypothesis (Y2), we have

$$u\lambda(\cdot,u) \leq aq$$

Again, we use Proposition 3.1 to find $v = V(u\lambda(.,u))$ satisfies (3.8). Put $w = u + V(u\lambda(\cdot,u))$. Hence the function w is a solution of

$$Lw = 0 \quad \text{a.e. on } (0,\infty)$$
$$\lambda w'(0) = 0, \quad w(\infty) = a$$

It follows that $w = a$ and so u satisfies (3.7). □

Proposition 3.5. *Let $\alpha > 1$ and $p \in B^+((0,+\infty))$ such that $Vp(0) < +\infty$. Then for each $a > 0$, the problem* (3.1) *has a unique solution $u \in C([0,+\infty]) \cap C^1((0,+\infty))$ satisfying*

$$a \exp(-\alpha a^{\alpha-1} Vp(0)) \leq u(x) \leq a \tag{3.9}$$

Proof . Let $\lambda(x,t) = p(x)t^{\alpha-1}$, then it is clear that λ satisfies (Y1) and (Y2) where q_a is explicitly given by $q_a(x) = \alpha a^{\alpha-1} p(x)$ for $x \in (0,\infty)$. So using Proposition 3.3, the problem (3.1) has a solution u in the space $C([0,\infty]) \cap C^1((0,\infty))$ satisfying (3.9).

Let us prove uniqueness. We assume that u and v are two solutions in $C([0,\infty]) \cap C^1((0,\infty))$ of (3.1) and put $w = u - v$. Then using Lemma 3.4, the function w satisfies

$$w + V(hw) = 0 \text{ on } (0,\infty) \tag{3.10}$$

where the function $h \in B^+((0,\infty))$ is defined by

$$h(x) = \begin{cases} p(x)\frac{u^\alpha(x) - v^\alpha(x)}{u(x)-v(x)} & \text{if } u(x) \neq v(x) \\ 0 & \text{if } u(x) = v(x) \end{cases}$$

Now, since $Vh(0) \leq \alpha a^{\alpha-1} Vp(0) < \infty$, we apply the operator $(I - V_h(h.))$ on both sides of (3.10), we obtain with (3.5) that $w = 0$ on $(0,\infty)$. So the uniqueness is proved. □

4. Proof of the main result

We must first reformulate the problem in the form of a fixed point problem. For that, let $E = C([0, +\infty]^m)$ the Banach space endowed with the norm

$$\|u\|_E = \|(u_1, \ldots, u_m)\|_E = \sum_{i=1}^{m} \|u_i\|_\infty$$

Now, let a, $b > 0$, to apply a fixed point argument, we consider the set

$$\Gamma = \left\{ u = (u_1, \ldots, u_m) \in E \mid b_i e^{-V_i \tilde{p}_i(0)} \leq u_i \leq b_i \right\}$$

where

$$\tilde{p}_i = \alpha_i b_i^{\alpha_i - 1} p_i \prod_{\substack{1 \leq j \leq m \\ j \neq i}} b_j^{k_{ij}}$$

Then Γ is a convex closed subset of E.

We define the operator Ψ on Γ by $\Psi u = z$ where $z = (z_1, \ldots, z_2)$ is the unique solution of the problem

$$\begin{cases} \dfrac{1}{\varphi_i} (\varphi_i z_i')' = p_i z_i^{\alpha_i} \prod\limits_{\substack{j=1 \\ j \neq i}}^{m} u_j^{k_{ij}} \quad , \quad \text{on } (0, +\infty) \\ \varphi_i z_i'(0) = 0 \quad \text{and} \quad z_i(\infty) = b_i \end{cases} \quad , \ 1 \leq i \leq m$$

Note that if $\Psi u = u$ then u is a solution of (1.1). So we will use the Schauder's Theorem.

In the case of a Banach space, the Schauder's fixed point Theorem affirms that : if Γ is a nonempty convex closed subset of a Banach space E, and Ψ is a continuous mapping of Γ into itself such that $\Psi(\Gamma)$ is relatively compact, then Ψ has a fixed point in Γ.

We verify the hypotheses of Schauder's Theorem :

(i) We point out that Ψ is well defined and $\Psi\Gamma \subset \Gamma$. Indeed, if for all $0 \leq i \leq m$, we have got

$$u_j \leq b_j \quad , \quad \forall j \neq i$$

then using Proposition 3.5, the problem

$$\begin{cases} \dfrac{1}{\varphi_i} (\varphi_i z_i')'(x) = p_i z_i^{\alpha_i} \prod\limits_{\substack{j=1 \\ j \neq i}}^{m} u_j^{k_{ij}} \quad , \quad x \in (0, +\infty) \\ \varphi_i z_i'(0) = 0 \quad \text{and} \quad z_i(+\infty) = b_i \end{cases} \quad , \ 1 \leq i \leq m$$

has a unique solution z_i in $C([0, \infty])$ for all $j \neq i$ with $1 \leq i \leq m$, satisfying

$$b_i e^{-V_i \tilde{p}_i(0)} \leq z_i \leq b_i$$

(ii) Now, we prove that $\Psi\Gamma$ is relatively compact in $C([0, \infty]^m)$. To arrive to this result, let $u \in \Gamma$ and put $(z_1, \ldots, z_m) = \Psi(u_1, \ldots, u_m)$. Using Lemma 3.4, the functions z_i satisfy

$$z_i + V_i \left(p_i z_i^{\alpha_i} \prod_{\substack{j=1 \\ j \neq i}}^{m} u_j^{k_{ij}} \right) = b_i \quad , \quad \text{on } [0, +\infty) \tag{4.1}$$

Then for $(x_1,\ldots,x_m)$, $(x_1',\ldots,x_m') \in [0,+\infty]^m$, we have

$$\begin{aligned}
& \left\|\Psi\left(u_1,\ldots,u_m\right)\left(x_1,\ldots,x_m\right)-\Psi\left(u_1,\ldots,u_m\right)\left(x_1',\ldots,x_m'\right)\right\| \\
= & \sum_{i=1}^{m}\left|z_i\left(x_i\right)-z_i\left(x_i'\right)\right| \\
= & \sum_{i=1}^{m}\left|V_i\left(p_i z_i^{\alpha_i}\prod_{\substack{j=1\\ j\neq i}}^{m}u_j^{k_{ij}}\right)\left(x_i\right)-V_i\left(p_i z_i^{\alpha_i}\prod_{\substack{j=1\\ j\neq i}}^{m}u_j^{k_{ij}}\right)\left(x_i'\right)\right|
\end{aligned}$$

Now, using that $(u_1,\ldots,u_m)$ and $(z_1,\ldots,z_m)$ are in Γ, it follows that

$$V_i\left(p_i z_i^{\alpha_i}\prod_{\substack{j=1\\ j\neq i}}^{m}u_j^{k_{ij}}\right)\in F_{\frac{b_i}{\alpha_i}\tilde{p}_i}\ ,\ \text{for all } 1\leq i\leq m$$

This implies, by Proposition 3.1, that $\Psi\Gamma$ is equicontinuous in $[0,+\infty]^m$. Now, since

$$\left\{\Psi\left(u_1,\ldots,u_m\right)\left(x_1,\ldots,x_m\right) \mid \left(u_1,\ldots,u_m\right)\in\Gamma\right\}$$

is uniformly bounded in $[0,+\infty]^m$, we deduce by Ascoli's Theorem that $\Psi\Gamma$ is relatively compact in $C\left([0,+\infty]^m\right)$.

(iii) Let us prove the continuity of Ψ in Γ. Let $(u_{1n},\ldots,u_{mn})$ be a sequence in Γ converging to $(u_1,\ldots,u_m)\in\Gamma$ with respect to $\|\cdot\|$. Put $(z_{1n},\ldots,z_{mn}) = \Psi\left(u_{1n},\ldots,u_{mn}\right)$ and $(z_1,\ldots,z_m) = \Psi\left(u_1,\ldots,u_m\right)$. Then

$$\left|\Psi\left(u_{1n},\ldots,u_{mn}\right)\left(x_1,\ldots,x_m\right)-\Psi\left(u_1,\ldots,u_m\right)\left(x_1,\ldots,x_m\right)\right| = \sum_{i=1}^{m}\left|z_{in}\left(x_i\right)-z_i\left(x_i\right)\right|$$

We denote by $Z_{in} = z_{in} - z_i$. We start by evaluating Z_{in}. By (4.1), we have for $x_i \in [0,+\infty]$

$$\begin{aligned}
Z_{in}\left(x_i\right) &= V_i\left(p_i z_i^{\alpha_i}\prod_{\substack{j=1\\ j\neq i}}^{m}u_j^{k_{ij}}\right)\left(x_i\right)-V_i\left(p_i z_i^{\alpha_i}\prod_{\substack{j=1\\ j\neq i}}^{m}u_j^{k_{ij}}\right)\left(x_i\right) \\
&= V_i\left(p_i z_i^{\alpha_i}\left[\prod_{\substack{j=1\\ j\neq i}}^{m}u_j^{k_{ij}}-\prod_{\substack{j=1\\ j\neq i}}^{m}u_{jn}^{k_{ij}}\right]\right)\left(x_i\right)-V_i\left(h_i Z_{in}\right)\left(x_i\right)
\end{aligned}$$

where $h_i \in B^+((0,\infty))$ for all $1\leq i\leq m$, and

$$h_i\left(x_i\right)=\begin{cases} p_i\left(x_i\right)\left(\prod\limits_{\substack{j=1\\ j\neq i}}^{m}u_{jn}^{k_{ij}}\left(x_i\right)\right)\dfrac{z_{in}^{\alpha_i}\left(x_i\right)-z_i^{\alpha_i}\left(x_i\right)}{z_{in}\left(x_i\right)-z_i\left(x_i\right)} & ,\ \text{if }\ z_{in}\left(x_i\right)\neq z_i\left(x_i\right) \\ 0 & ,\ \text{if }\ z_{in}\left(x_i\right)=z_i\left(x_i\right)\end{cases}$$

Since $V_i h_i(0) < \infty$, applying the operator $(I - V_{ih}(h\cdot))$ on both side of

$$Z_{in} + V_i\left(h_i Z_{in}\right) = V_i\left(p_i z_i^{\alpha_i}\left[\prod_{\substack{j=1\\ j\neq i}}^{m} u_j^{k_{ij}} - \prod_{\substack{j=1\\ j\neq i}}^{m} u_{jn}^{k_{ij}}\right]\right)$$

we obtain by (3.4) and (3.5) that

$$Z_{in} = V_{ih}\left(p_i z_i^{\alpha_i}\left[\prod_{\substack{j=1\\ j\neq i}}^{m} u_j^{k_{ij}} - \prod_{\substack{j=1\\ j\neq i}}^{m} u_{jn}^{k_{ij}}\right]\right)$$

So,

$$|Z_{in}| \leq V_i\left(p_i z_i^{\alpha_i}\left[\prod_{\substack{j=1\\ j\neq i}}^{m} u_j^{k_{ij}} - \prod_{\substack{j=1\\ j\neq i}}^{m} u_{jn}^{k_{ij}}\right]\right)$$

Now, since

$$p_i z_i^{\alpha_i}\left[\prod_{\substack{j=1\\ j\neq i}}^{m} u_j^{k_{ij}} - \prod_{\substack{j=1\\ j\neq i}}^{m} u_{jn}^{k_{ij}}\right] \leq 2 p_i z_i^{\alpha_i}\left(\prod_{\substack{j=1\\ j\neq i}}^{m} b_j^{k_{ij}}\right)$$

and $V_i p_i(0) < \infty$, we deduce by the dominated convergence Theorem, that

$$V_i\left(p_i z_i^{\alpha_i}\left[\prod_{\substack{j=1\\ j\neq i}}^{m} u_j^{k_{ij}} - \prod_{\substack{j=1\\ j\neq i}}^{m} u_{jn}^{k_{ij}}\right]\right) \to 0\ , \quad \text{as } n \to +\infty$$

It follows that Z_{in} converge to 0 as $n \to +\infty$.

Analogously, we have $Z_{in}(x_i)$ converge to 0 as $n \to +\infty$. This proves that for each $(x_1, \ldots, x_m) \in [0, +\infty)^m$,

$$\Psi\left(u_{1n}, \ldots, u_{mn}\right)\left(x_1, \ldots, x_m\right) \to \Psi\left(u_1, \ldots, u_m\right)\left(x_1, \ldots, x_m\right)\ , \quad \text{as } n \to +\infty$$

Now, since $\Psi\Gamma$ is relatively compact in $C([0, +\infty]^m)$, we deduce that

$$\left\|\Psi\left(u_{1n}, \ldots, u_{mn}\right)\left(x_1, \ldots, x_m\right) - \Psi\left(u_1, \ldots, u_m\right)\left(x_1, \ldots, x_m\right)\right\| \to 0\ , \quad \text{as } n \to +\infty$$

According to the above, all the hypotheses of the Schauder's Theorem are verified, then there exists $u = (u_1, \ldots, u_m) \in \Gamma$ such that $\Psi\left(u_1, \ldots, u_m\right) = (u_1, \ldots, u_m)$. So (u, v) is the desired solution. This completes the proof.

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References

[1] M. A. Abdellaoui, Z. Dahmani and N. Bedjaoui, *New existence results for a coupled system of nonlinear differential equations of arbitrary order*, Int. J. Nonlinear Anal. Appl. 6 (2015) 65–75.
[2] R. P. Agarwal and D. O'Regan, *Existence theory for single and multiple solutions to singular positone boundary value problems*, J. Differential Equations 175 (2001) 393–414.
[3] R. P. Agarwal and D. O'Regan, *Twin solutions to singular Dirichlet problems*, J. Math. Anal. Appl. 240 (1999) 433–445.
[4] N. Akhmediev and A. Ankiewicz, *Partially coherent solitons on a finite background*, Phys. Rev. Lett. 82 (1999) 2661–2664.
[5] D. H. Armitage and S. J. Gardiner, *Classical potential theory*, Springer-Verlag London, 2001.
[6] S. Ben Othman, H. Maâgli and N. Zeddini, *On the existence of positive solutions of nonlinear differential equation*, International Journal of Mathematical Sciences 2007 (2007) Article ID 58658, 12 pages.
[7] J. Damirchi and T. Rahimi, *Differential transform method for a nonlinear system of differential equations arising in HIV infection of CD4$^+$T cell*, Int. J. Nonlinear Anal. Appl. 7 (2016) 269–277.
[8] A. Ghanmi, H. Mâagli, V. Radulescu and N. Zeddini, *Large and bounded solutions for a class of nonlinear Schrödinger stationary systems*, Anal. Appl. 7 (2009) 391–404.
[9] S. Gontara, *Existence of bounded positive solutions of a nonlinear differential system*, Electron. J. Diff. Eq. 2012 (2012) 1–9.
[10] R. Kannan and D. O'Regan, *A note on singular boundary value problems with solutions in weighted spaces*, Nonlinear Anal. 37 (1999) 791–796.
[11] A. V. Lair and A. W. Wood, *Existence of entire large positive solutions of semilinear elliptic systems*, J. Diff. Eq. 164 (2000) 380–394.
[12] H. Li and M. Wang, *Existence and uniqueness of positive solutions to the boundary blow-up problem for an elliptic system*, J. Diff. Eq. 234 (2007) 246–266.
[13] R. Ma, *Existence of positive radial solutions for elliptic systems*, J. Math. Anal. Appl. 201 (1996) 375–386.
[14] H.Maâgli and N. Zeddini, *Positive solutions for a singular nonlinear Dirichlet problem*, Nonlinear Stud. 10 (2003) 295–306.
[15] H. Maâgli, *On the solution of a singular nonlinear periodic boundary value problem*, Poten. Anal. 14 (2001) 437–447.
[16] H. Maâgli and S. Masmoudi, *Sur les solutions d'un opérateur différentiel singulier semi-linéaire*, Poten. Anal. 10 (1999) 289–304.
[17] S. Masmoudi and N. Yazidi, *On the existence of positive solutions of a singular nonlinear differential equation*, J. Math. Anal. Appl. 268 (2002) 53–66.
[18] C. R. Menyuk, *Pulse propagation in an elliptically birefringent Kerr medium*, IEEE J. Quantum Elec. 25 (1989) 2674–2682.
[19] Y. Peng and Y. Song, *Existence of entire large positive solutions of a semilinear elliptic system*, Appl. Math. Comput. 155 (2004) 687–698.
[20] S. D. Taliaferro, *A nonlinear singular boundary value problem*, Nonlinear Anal. 3 (1979) 897–904.
[21] J. Velin, *A criterion for existence of a positive solution of a nonlinear elliptic system*, Anal. Appl. (Singap.) 6 (2008) 299–321.
[22] X. Wang and A. W. Wood, *Existence and nonexistence of entire positive solutions of semilinear elliptic systems*, J. Math. Anal. Appl. 267 (2002) 361–368.