Int. J. Nonlinear Anal. Appl. 12 (2021) No. 2, 53-60
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.5014

Estimate survival function of the Topp-Leone exponential distribution with application

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(Communicated by Madjid Eshaghi Gordji)

Abstract

This is a new lifetime Exponential “distribution using the Topp-Leone generated family of distributions proposed by Rezaei et al. The new distribution is called the Topp-Leone Exponential (TLE) distribution.” What is done in this paper is an estimation of the “unlabeled two parameters for ToppLeone Exponential distribution model by using the maximum likelihood estimator method to get the derivation of the point estimators for all unlabeled parameters according to iterative techniques as Newton − Raphson method , then to derive Ordinary least squares estimator method”. “Applying all two methods to estimate related probability functions; death density function, cumulative distribution function, survival function and hazard function (rate function).” “When examining the numerical results for probability survival function by employing mean squares error measure and mean absolute percentage measure, this may lead to work on the best method in modeling a set of real data”.

Keywords: Topp-Leone, Maximum Likelihood estimator method, Newton-Raphson method, Ordinary least squares estimator method, survival function.
2010 MSC: 33E30

1. Introduction

Noman Rashed studied properties and applications to Topp-Leone compound Rayleigh distribution (1.1). Fatoki Olayode studied the new distribution was found to be more flexible in modeling data that exhibits increasing, decreasing non-monotone failure rate (1.2). Rayed and Othman (1.3) studied this distribution with different angle they offered Beta compound Rayleigh distribution; they also acquired its mathematical properties of the distribution. Reyad et al (1.4). Provided an extension in ”compound Rayleigh distribution” called Kumaraswamy ”compound Rayleigh distribution” (KwCR) and also acquired its mathematical properties.

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Received: August 2020 *Accepted:* September 2020

Properties Topp-Leone exponential distribution:

“The p.d.f for Topp-Leone exponential distribution. Is”:

$$f_{TL}(l;\zeta,\psi) = 2\zeta\psi e^{-2\psi l}\left(1-e^{-\psi l}\right)^{\zeta-1}\left(1+e^{-\psi l}\right)^{\zeta-1};\quad l\geq 0 \tag{1.1}$$

ψ : “is scale parameter”
ζ : “is shape parameter”
“The cumulative distribution function for this distribution is”:

$$F_{TL}(l;\zeta,\psi) = \left(1-e^{-\psi l}\right)^{\zeta}\left(1+e^{-\psi l}\right)^{\zeta} \tag{1.2}$$

“It’s survival function is given by”:

$$s(l;\zeta,\psi) = 1-\left(1-e^{-\psi l}\right)^{\zeta}\left(1+e^{-\psi l}\right)^{\zeta} \tag{1.3}$$

“The hazard rate function is given by”:

$$h(l;\zeta,\psi) = \frac{2\zeta\psi e^{-2\psi l}\left(1-e^{-\psi l}\right)^{\zeta-1}\left(1+e^{-\psi l}\right)^{\zeta-1}}{1-(1-e^{-\psi l})^{\zeta}(1+e^{-\psi l})^{\zeta}} \tag{1.4}$$

“Maximum likelihood estimator method(MLEM)”:

“The MLM is the most common procedure to estimate the parameter ϕ which specifies a p.d.f. $f(l:\varphi)$ based on the observations $l_1, l_2, \ldots\ldots, l_n$ which were independent. sample from the distribution”.

$$L = \left[\prod_{i=1}^{m} f_{TL}\left(l_i;\zeta,\psi\right)\right] = \prod_{i=1}^{m}\left[2\zeta\psi e^{-2\psi l_i}\left(1-e^{-\psi l_i}\right)^{\zeta-1}\left(1+e^{-\psi l_i}\right)^{\zeta-1}\right] \tag{1.5}$$

$$\ln L = m\ln 2 + m\ln\zeta + m\ln\psi - 2\psi\sum_{i=1}^{m} l_i + (\zeta-1)\sum_{i=1}^{m}\ln\left(1-e^{-\psi l_i}\right) + (\zeta-1)\sum_{i=1}^{m}\ln\left(1+e^{-\psi l_i}\right) \tag{1.6}$$

$$\frac{\partial \ln L}{\partial\zeta} = \frac{m}{\zeta} + \sum_{i=1}^{m}\ln\left(1-e^{-\psi l_i}\right) + \sum_{i=1}^{m}\ln\left(1+e^{-\psi l_i}\right));\quad \frac{\partial\ln L}{\partial\zeta} = 0 \tag{1.7}$$

$$\frac{\partial \ln L}{\partial\psi} = \frac{m}{\psi} - 2\sum_{i=1}^{m} l_i + (\zeta-1)\sum_{i=1}^{m}\frac{l_i e^{-\psi l_i}}{1-e^{-\psi l_i}} - (\zeta-1)\sum_{i=1}^{m}\frac{l_i e^{-\psi l_i}}{1+e^{-\psi l_i}};\quad \frac{\partial\ln L}{\partial\psi} = 0. \tag{1.8}$$

Here, it is better to have ”an initial value of each unknown parameters (ζ,ψ) to get the estimate values and identify the number of iterations”.

$$\begin{bmatrix}\zeta_{i+1}\\ \psi_{i+1}\end{bmatrix} = \begin{bmatrix}\zeta_i\\ \psi_i\end{bmatrix} - J_i^{-1}\begin{bmatrix}b_1(\zeta)\\ b_2(\psi)\end{bmatrix} \tag{1.9}$$

$$b_1(\zeta) = \frac{m}{\zeta} + \sum_{i=1}^{m} \ln\left(1 - e^{-\psi l_i}\right) + \sum_{i=1}^{m} \ln\left(1 + e^{-\psi l_i}\right) \tag{1.10}$$

$$b_2(\psi) = \frac{m}{\psi} - 2\sum_{i=1}^{m} l_i + (\zeta - 1)\sum_{i=1}^{m} \frac{l_i e^{-\psi l_i}}{1 - e^{-\psi l_i}} - (\zeta - 1)\sum_{i=1}^{m} \frac{l_i e^{-\psi l_i}}{1 + e^{-\psi l_i}} \tag{1.11}$$

$$J_i^{-1} = \begin{bmatrix} \frac{\partial b_1(\zeta)}{\partial \zeta} & \frac{\partial b_1(\zeta)}{\partial \psi} \\ \frac{\partial b_2(\psi)}{\partial \zeta} & \frac{\partial b_2(\psi)}{\partial \psi} \end{bmatrix} \tag{1.12}$$

$$\frac{\partial b_1(\zeta)}{\partial \zeta} = -\frac{m}{\zeta^2} \tag{1.13}$$

$$\frac{\partial b_1(\zeta)}{\partial \psi} = \sum_{i=1}^{m} \frac{l_i e^{-\psi l_i}}{l_i - e^{-\psi l_i}} - \sum_{i=1}^{m} \frac{l_i e^{-\psi l_i}}{l_i + e^{-\psi l_i}} \tag{1.14}$$

$$\frac{\partial b_2(\psi)}{\partial \zeta} = \sum_{i=1}^{m} \frac{l_i e^{-\psi l_i}}{l_i - e^{-\psi l_i}} - \sum_{i=1}^{m} \frac{l_i e^{-\psi l_i}}{l_i + e^{-\psi l_i}} \tag{1.15}$$

$$\frac{\partial b_2(\psi)}{\partial \psi} = -\frac{m}{\psi^2} - (\zeta - 1)\sum_{i=1}^{m} \frac{l_i^2 e^{-\psi l_i}}{\left(l_i - e^{-\psi l_i}\right)^2} + (\zeta - 1)\sum_{i=1}^{m} \frac{l_i^2 e^{-\psi l_i}}{\left(l_i + e^{-\psi l_i}\right)^2} \tag{1.16}$$

$$\begin{bmatrix} E_{i+1}(\zeta) \\ E_{i+1}(\psi) \end{bmatrix} = \begin{bmatrix} \zeta_{i+1} \\ \psi_{i+1} \end{bmatrix} - \begin{bmatrix} \zeta_i \\ \psi_i \end{bmatrix} \tag{1.17}$$

"Ordinary Least Squares Estimator Method (OLSEM)":

"The OLSEM is the most used way to estimate parameters in linear or nonlinear model. Researchers make use of this method to lessen the sum squares differences concerning observed sample values and expected estimated values by linear approximation. (5,6)"

$$Q = \vartheta_0 + \vartheta_1 x + E \tag{1.18}$$

$$E = Q_i - \hat{\vartheta}_0 - \hat{\vartheta}_1 x_i \tag{1.19}$$

$$\sum_{i=1}^{m} E_i^2 = \sum_{i=1}^{m} \left[q_i - \hat{q}_i\right]^2 \tag{1.20}$$

$$\sum_{i=1}^{m} E_i^2 = \sum_{i=1}^{m} \left[q_i - \hat{\vartheta}_0 - \hat{\vartheta}_1 x_i \right]^2 \tag{1.21}$$

$$F(l_i) = \left(1 - e^{-\psi l_i}\right)^{\zeta} \left(1 + e^{-\psi l_i}\right)^{\zeta} \tag{1.22}$$

$$E = F(l_i) - \left(1 - e^{-\psi l_i}\right)^{\zeta} \left(1 - e^{-\psi l_i}\right)^{\zeta} \tag{1.23}$$

$$\sum_{i=1}^{m} E_i^2 = \sum_{i=1}^{m} \left[F(l_i) - \left(1 - e^{-\psi l_i}\right)^{\zeta} \left(1 + e^{-\psi l_i}\right)^{\zeta} \right]^2 ;\ Let \sum_{i=1}^{m} E_i^2 = k(\zeta, \psi) \tag{1.24}$$

$$k(\zeta, \psi) = \sum_{i=1}^{m} \left[F(l_i) - \left(1 - e^{-\psi l_i}\right)^{\zeta} \left(1 + e^{-\psi l_i}\right)^{\zeta} \right]^2 \tag{1.25}$$

$$\frac{\partial k}{\partial \zeta} = -\sum_{i=1}^{m} 2 \left[F(l_i) - \left(1 - e^{-\psi l_i}\right)^{\zeta} \left(1 + e^{-\psi l_i}\right)^{\zeta} \right] \left(1 - e^{-\psi l_i}\right)^{\zeta} \left(1 + e^{-\psi l_i}\right)^{\zeta} \left\{ \ln\left(1 + e^{-\psi l_i}\right) + \ln\left(1 - e^{-\psi l_i}\right) \right\} \tag{1.26}$$

$\frac{\partial k}{\partial \zeta} = 0$

$$\frac{\partial k}{\partial \psi} = \sum_{i=1}^{m} 2 \left[F(l_i) - \left(1 - e^{-\psi l_i}\right)^{\zeta} \left(1 + e^{-\psi l_i}\right)^{\zeta} \right] \times \zeta l_i e^{-\psi l_i} \left\{ \left(1 - e^{-\psi l_i}\right)^{\zeta} \left(1 + e^{-\psi l_i}\right)^{\zeta - 1} - \left(1 + e^{-\psi l_i}\right)^{\zeta} \left(1 - e^{-\psi l_i}\right)^{\zeta - 1} \right\} \tag{1.27}$$

$\frac{\partial k}{\partial \psi} = 0$

Here, it is better to have "an initial value of each unknown parameters (ζ, ψ) to get the estimate values and identify the number of iterations".

$$\begin{bmatrix} \zeta_{i+1} \\ \psi_{i+1} \end{bmatrix} = \begin{bmatrix} \zeta_i \\ \psi_i \end{bmatrix} - J_i^{-1} \begin{bmatrix} w_1(\zeta) \\ w_2(\psi) \end{bmatrix} \tag{1.28}$$

$$w_1(\zeta) = -\sum_{i=1}^{m} 2 \left[F(l_i) - \left(1 - e^{-\psi l_i}\right)^{\zeta} \left(1 + e^{-\psi l_i}\right)^{\zeta} \right] \left(1 - e^{-\psi l_i}\right)^{\zeta} \left(1 + e^{-\psi l_i}\right)^{\zeta} \left\{ \ln\left(1 + e^{-\psi l_i}\right) + \ln\left(1 - e^{-\psi l_i}\right) \right\} \tag{1.29}$$

$$w_2(\psi) = \sum_{i=1}^{n} 2 \left[F(l_i) - \left(1 - e^{-\psi l_i}\right)^{\zeta} \left(1 + e^{-\psi l_i}\right)^{\zeta} \right] \times \zeta l_i e^{-\psi l_i} \left\{ \left(1 - e^{-\psi l_i}\right)^{\zeta} \left(1 + e^{-\psi l_i}\right)^{\zeta - 1} - \left(1 + e^{-\psi l_i}\right)^{\zeta} \left(1 - e^{-\psi l_i}\right)^{\zeta - 1} \right\} \tag{1.30}$$

$$J_i^{-1} = \begin{bmatrix} \frac{\partial w_1(\zeta)}{\partial \zeta} & \frac{\partial w_1(\zeta)}{\partial \psi} \\ \frac{\partial w_2(\psi)}{\partial \zeta} & \frac{\partial w_2(\psi)}{\partial \psi} \end{bmatrix} \tag{1.31}$$

$$\frac{\partial w_1(\zeta)}{\partial \zeta} = -\sum_{i=1}^{m} 2F(l_i)\left(1-e^{-\psi l_i}\right)^{\zeta}\left(1+e^{\psi l_i}\right)^{\zeta}\left\{\ln\left(1+e^{-\psi l_i}\right)+\left(1-e^{-\psi l_i}\right)\right\}^2 \tag{1.32}$$

$$\begin{aligned}\frac{\partial w_1(\zeta)}{\partial \zeta} = -2\sum_{i=1}^{m} &\left[2l_i e^{-\psi l_i}F(l_i)(1+\zeta\{\ln(1+e^{-\psi l_i})+\ln(1-e^{-\psi l_i})\})\right. \\ &\left.\left\{\left(1-e^{-\psi l_i}\right)^{\zeta-1}\left(1+e^{-\psi l_i}\right)^{\zeta}-\left(1-e^{-\psi l_i}\right)^{\zeta}\ \left(1+e^{-\psi l_i}\right)^{\zeta-1}\right\}\right] \\ &+l_i e^{-\psi l_i}\left[\left(1-e^{-\psi l_i}\right)^{2\zeta-1}\left(1+e^{-\psi l_i}\right)^{2\zeta}\left(2\left\{\ln\left(1+e^{-\psi l_i}\right)+\ln\left(1-e^{-\psi l_i}\right)\right\}-2\right)\right. \\ &\left.+\left(1-e^{-\psi l_i}\right)^{2\zeta}\left(1+e^{-\psi l_i}\right)^{2\zeta-1}\left[2-\zeta\left\{\ln\left(1+e^{-\psi l_i}\right)+\ln\left(\left(1-e^{-\psi l_i}\right)\right\}\right)\right\}\right]\end{aligned} \tag{1.33}$$

$$\begin{aligned}\frac{\partial w_2(\psi)}{\partial \zeta} = -2\sum_{i=1}^{m} &\left[2l_i e^{-\psi l_i}F(l_i)(1+\varsigma\{\ln(1+e^{-\psi l_i})+\ln(1-e^{-\psi l_i})\})\right. \\ &\left.\left\{\left(1-e^{-\psi l_i}\right)^{\zeta-1}\left(1+e^{-\psi l_i}\right)^{\zeta}-\left(1-e^{-\psi l_i}\right)^{\zeta}\ \left(1+e^{-\psi l_i}\right)^{\zeta-1}\right\}\right] \\ &+l_i e^{-\psi l_i}\left[\left(1-e^{-\psi l_i}\right)^{2\zeta-1}\left(1+e^{-\psi l_i}\right)^{2\zeta}\left(2\left\{\ln\left(1+e^{-\psi l_i}\right)+\ln\left(1-e^{-\psi l_i}\right)\right\}\ -2\right)\right. \\ &+\left(1-e^{-\psi l_i}\right)^{2\zeta}\left(1+e^{-\psi l_i}\right)^{2\zeta-1}\left[2-\zeta\left\{\ln\left(1+e^{-\psi l_i}\right)+\ln\left(\left(1-e^{-\psi l_i}\right)\right\}\right)\right\}\end{aligned}$$

$$\begin{aligned}\frac{\partial w_2(\psi)}{\partial \psi} = 2\sum_{i=1}^{m}F\left(l_i\right)&\left[-\zeta(\zeta-1)l_i^2\left(e^{-\psi l_i}\right)^2\left\{\left(1-e^{-\psi l_i}\right)^{\zeta}\left(1+e^{-\psi l_i}\right)^{\zeta-2}+\left(1+e^{-\psi l_i}\right)^{\zeta}\left(1-e^{-\psi l_i}\right)^{\zeta-2}\right\}\right. \\ &+2\zeta^2 l_i^2\left(e^{-\psi l_i}\right)^2\left(1+e^{-\psi l_i}\right)^{\zeta-1}\left(1-e^{-\psi l_i}\right)^{\zeta-1}-\zeta l_i^2 e^{-\psi l_i}\left\{\left(1-e^{-\psi l_i}\right)^{\zeta}\left(1+e^{-\psi l_i}\right)^{\zeta-1}\right. \\ &\left.\left.-\left(1+e^{-\psi l_i}\right)^{\zeta}\left(1-e^{-\psi l_i}\right)^{\zeta-1}\right\}\right]+\zeta(\zeta-1)l_i^2\left(e^{-\psi l_i}\right)^2\left\{\left(1-e^{-\psi l_i}\right)^{2\zeta}\left(1+e^{-\psi l_i}\right)^{2(\zeta-1)}\right. \\ &\left.+\left(1+e^{-\psi l_i}\right)^{2\zeta}\left(1-e^{-\psi l_i}\right)^{2(\zeta-1)}\right\}-2\varsigma^2 l_i^2\left(e^{-\psi l_i}\right)^2\left(1+e^{-\psi l_i}\right)^{2\zeta-1}\left(1-e^{-\psi l_i}\right)^{2\zeta-1} \\ &+\varsigma l_i^2 e^{-\psi l_i}\left\{\left(1-e^{-\psi l_i}\right)^{2\zeta}\left(1+e^{-\psi l_i}\right)^{2\zeta-1}-\left(1+e^{-\psi l_i}\right)^{2\zeta}\left(1-e^{-\psi l_i}\right)^{2\zeta-1}\right\} \\ &+2l_i e^{-\psi l_i}\left\{\left(1-e^{-\psi l_i}\right)^{\zeta}\left(1+e^{-\psi l_i}\right)^{\zeta-1}-\left(1+e^{-\psi l_i}\right)^{\zeta}\left(1-e^{-\psi l_i}\right)^{\zeta-1}\right\}\end{aligned} \tag{1.35}$$

$$\begin{bmatrix} E_{i+1}(\zeta) \\ E_{i+1}(\psi) \end{bmatrix} = \| \begin{bmatrix} \zeta_{i+1} \\ \psi_{i+1} \end{bmatrix} - \begin{bmatrix} \zeta_i \\ \psi_i \end{bmatrix} \tag{1.36}$$

The practical section:

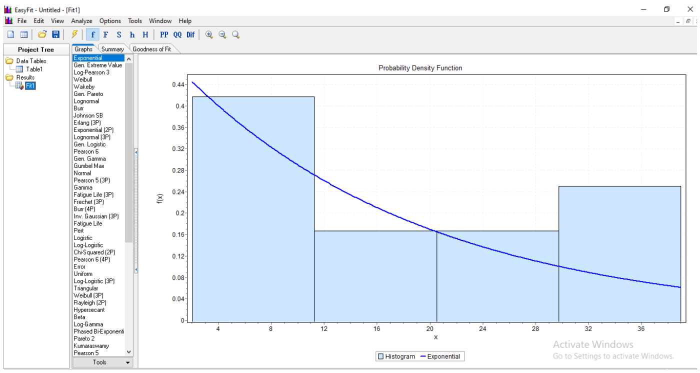
The Covid 19 data has been selected from the main hospital in Diwaniyah province in Iraq, so it is a real life data that shows how this pandemic has failure time(death time) for the duration of (120) days ; beginning from Jun to September 2020 .

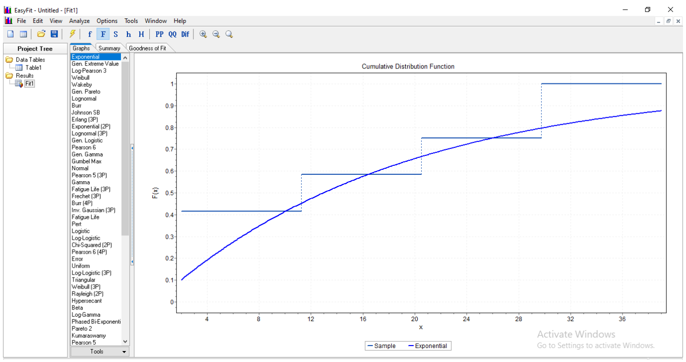
Number of infected people is (244), (170 male, 74 femal), twelve of them were deod and (232) patients survived .

"When applying the test statistic (Kolmogorov-Smirnov) depending upon statistical programming (EasyFit 5.5 Professional) in order to fit Topp-Leone Exponential distribution data, it is discovered that the calculated value is (0.12378), this means data is distributed according to Topp-Leone Exponential distribution".

"H_0 : The survival time data is distributed as Topp-Leone Exponential".

"H_1 : The survival time data is not distributed as Topp-Leone Exponential".





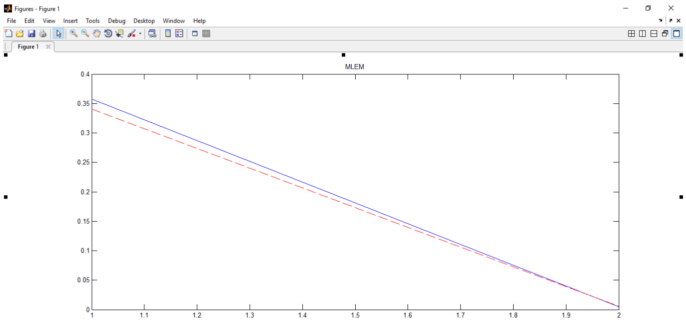
By MATLAB (R2014a) , below are the estimated parameters results

Table 1

MLEM	OLSEM
Initial values	
$\alpha_0 = 0.17$; $\theta_0 = 0.03$	$\alpha_0 = 17$; $\theta_0 = 0.03$
Estimated values	
$\hat{\alpha} = 0.3570$; $\hat{\theta} = 0.0048$	=4.4828 ; =0.1479

$$\text{MSE}\left[\hat{s}\left(l_i\right)\right] = \frac{1}{m}\sum_{i=1}^{m}\left[\hat{s}\left(l_i\right) - s\left(l_i\right)\right]^2 = 0.005488685$$

$$\text{MAPE}\left[\hat{s}\left(l_i\right)\right] = \frac{1}{m}\sum_{i=1}^{m}\left|\frac{\hat{s}\left(l_i\right) - s\left(l_i\right)}{s\left(l_i\right)}\right| = 0.276892408$$



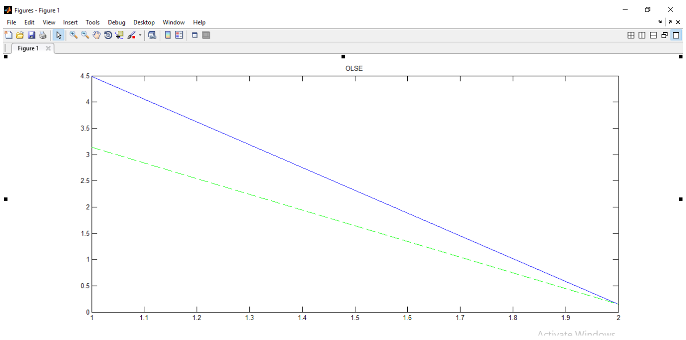


Table 2: "Estimated values for functions f(t), F(t), s(t), h(t) by MLEM"

T	pdf	cdF	Survival	hazard
2	0.042965001	0.24302587	0.75697413	0.014651639
3	0.03288925	0.280398609	0.719601391	0.014651639
5	0.023373859	0.335348965	0.664651035	0.015052911
7	0.018581755	0.37686912	0.62313088	0.016248702
11	0.013535577	0.439879552	0.560120448	0.018039818
14	0.011364886	0.4770197	0.5229803	0.020017678
17	0.009834688	0.508699837	0.491300163	0.021731002
22	0.008061362	0.553135106	0.446864894	0.024165476
29	0.006442881	0.603483348	0.396516652	0.029819987
36	0.005350701	0.644540479	0.355459521	0.035167115
39	0.00498117	0.660026415	0.339973585	0.045704817
39	0.00498117	0.660026415	0.339973585	0.056758877

$$\mathrm{MSE}\left[s\left(l_i\right)\right]=\frac{1}{m}\sum_{i=1}^{m}\left[\hat{s}\left(l_i\right)-s\left(l_i\right)\right]^2=0.000578096$$

$$\mathrm{MAPE}\left[\hat{s}\left(l_i\right)\right]=\frac{1}{m}\sum_{i=1}^{m}\left|\frac{\hat{s}\left(l_i\right)-s\left(l_i\right)}{s\left(l_i\right)}\right|=0.085534946$$

Conclusions:

It is observed that the probability survival function for the estimated values of both methods decreases with increasing failure times while the risk function increases.

Table 3: "Estimated values for functions f(t), F(t), s(t), h(t) by MLEM"

T	pdf	cdF	Survival	hazard
2	0.044280929	0.026944901	0.973055099	0.045507114
3	0.086029935	0.092699094	0.907300906	0.094819629
5	0.122767425	0.313725733	0.686274267	0.178889739
7	0.104568976	0.54646481	0.45353519	0.230564195
11	0.044653541	0.83812271	0.16187729	0.275848086
14	0.019943367	0.930655265	0.069344735	0.287597427
17	0.008486283	0.970979649	0.029020351	0.29242525
22	0.001968223	0.993328767	0.006671233	0.295031334
29	0.000249343	0.999156778	0.000843222	0.295703074
36	3.14627 E-05	0.999893631	0.000106369	0.295787777
39	1.29546 E-05	0.999956204	4.3796 E-05	0.295794967
39	1.29546 E-05	0.999956204	4.3796 E-05	0.295794967

It is advisable to employ (OLSEM) of Topp-Leone Exponential distribution of Covid 19 with the help of MSE criterion.

References

[1] N. Rasheed, *Topp–Leone compound Rayleigh distribution: properties and applications*, Res. J. Math. Stat. Sci. 7(3) (2019) 51–58.

[2] F. Olayode, *The Topp-Leone Rayleigh distribution with application*, Amer. J. Math. Stat. 9(6) (2019) 215–220.

[3] H. Reyad and S. Othman, *The beta compound Rayleigh distribution: Properties and applications*, Int. J. Adv. Stat. Prob. 5 (2017) 57–64.

[4] H. M. Reyad, S. A. Othman , A. M. Younis and A. M. Hashis, *The Kumaraswamy compound Rayleigh distribution: properties and estimation*. Int. J. Adv. Math. Sci. 5(1) (2017) 36–42.

[5] C. M. Douglas and C. R. George, *Applied Statistics and Probability for Engineering*, Third Edition, John Wiley and Sons Inc, 2003.

[6] G. D. Hutcheson and N. Sofroniou, *The Multivariate Social Scientist*, Sage Publications, London, 1999.