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New subclasses of meromorphic bi-univalent functions by associated with subordinate

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Abstract

In the present paper, we define two subclasses $\Sigma(\lambda,\alpha,\beta)$, $\Sigma_{\mathcal{C}}(\alpha,\beta)$ of meromorphic univalent functions and subclass $\Sigma_{\mathcal{B},\mathcal{C}}(\alpha,\beta,\lambda)$ of meromorphic bi-univalent functions. Furthermore, we obtain estimates on the general coefficients $|b_n|$ $(n \geq 1)$ for functions in the subclasses $\Sigma(\lambda,\alpha,\beta)$, $\Sigma_{\mathcal{C}}(\alpha,\beta)$ and estimates for the early coefficients of functions in subclass $\Sigma_{\mathcal{B},\mathcal{C}}(\alpha,\beta,\lambda)$ by associated subordination. The results obtained in this paper would generalize and improve those in related works of several earlier authors.

Keywords: Meromorphic univalent functions, Meromorphic bi-univalent functions, Coefficients estimates, Subordinate.
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1. Introduction

Let Σ denote the class of meromorphic univalent functions f of the form

$$f(z) = z + b_0 + \sum_{n=1}^{\infty} \frac{b_n}{z^n}, \tag{1.1}$$

defined on the domain $\Delta = \{z \in \mathbb{C} : 1 < |z| < \infty\}$. It is well known that every function $f \in \Sigma$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \qquad (z \in \Delta)$$

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and

$$f(f^{-1}(w)) = w \qquad (M < |w| < \infty,\ M > 0)\,.$$

Furthermore, for $f \in \Sigma$ given by (1.1), the inverse map $g = f^{-1}$ has the following expansion:

$$g(w) = f^{-1}(w) = w + \sum_{n=0}^{\infty} \frac{B_n}{w^n} = w - b_0 - \frac{b_1}{w} - \frac{b_2 + b_0 b_1}{w^2} + \cdots, \quad (M < |w| < \infty). \tag{1.2}$$

Function $f \in \Sigma$ is said to be meromorphic bi-univalent, if the inverse function f^{-1} also belongs to Σ. The class of all meromorphic bi-univalent functions will be denoted by $\Sigma_{\mathcal{B}}$.

Let $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in $\mathbb{C}$ and let $\mathbb{U}^* := \mathbb{U}/\{0\}$ be the punctured unit disk.

We say that f is subordinate to F in $\mathbb{U}$, written as $f \prec F$ $(z \in \mathbb{U})$, if and only if $f(z) = F(w(z))$ for some Schwarz function w(z) such that:

$$w(0) = 0 \text{ and } |w(z)| < 1 \ (z \in \mathbb{U}).$$

If F is univalent in $\mathbb{U}$, then the subordination $f \prec F$ is equivalent to $f(0) = F(0)$ and $f(\mathbb{U}) \subset F(\mathbb{U})$.

Recently, many subclasses of meromorphic bi-univalent functions were introduced by researchers. Also they obtain upper bounds for the coefficient of these subclasses. We mention refer to [2, 5, 6, 10, 11] for the precise arguments.

In the present paper, we introduce two subclasses $\Sigma(\lambda, \alpha, \beta)$, $\Sigma_{\mathcal{C}}(\alpha, \beta)$ of meromorphic univalent functions and a subclass $\Sigma_{\mathcal{B},\mathcal{C}}(\alpha, \beta, \lambda)$ of meromorphic bi-univalent functions. Also, for functions belonging to subclasses $\Sigma(\lambda, \alpha, \beta)$, $\Sigma_{\mathcal{C}}(\alpha, \beta)$, estimates on the general coefficients are obtained and for functions belonging to subclass $\Sigma_{\mathcal{B},\mathcal{C}}(\alpha, \beta, \lambda)$, estimates on the initial coefficients are found.

Moreover, the results presented would generalize recent work of Hamidi et al. [1], Panigrahi [6] and Salehian et al. [9].

2. Lemmas

For the proofs of theorems we need the following lemmas.

Lemma 2.1. *[8] Let* $q(z) = \sum\limits_{n=1}^{\infty} B_n z^n$ *be analytic and univalent in* $\mathbb{U}$ *and suppose that* $q(z)$ *maps* $\mathbb{U}$ *onto a convex domain. If* $p(z) = \sum\limits_{n=1}^{\infty} A_n z^n$ *is analytic in* $\mathbb{U}$ *and satisfies the following subordination:*

$$p(z) \prec q(z) \quad (z \in \mathbb{U}),$$

then

$$|A_n| \leq |B_1| \quad (n = 1, 2, \cdots).$$

Lemma 2.2. *[4] Let* α *and* β *be real nembers such that* $0 \leq \alpha < 1 < \beta$. *The function* p *defined by*

$$p(z) = 1 + \frac{\beta - \alpha}{\pi} i \ log \left(\frac{1 - e^{2\pi i \frac{(1-\alpha)}{\beta - \alpha}} z}{1 - z} \right) \tag{2.1}$$

maps the unit disk $\mathbb{U}$ *onto the strip domain* $\{w : \alpha < Re(w) < \beta\}$.

Remark 2.3.

$$p(z) = 1 + \frac{\beta-\alpha}{\pi} i \ log \left(\frac{1 - e^{2\pi i \frac{(1-\alpha)}{\beta-\alpha}} z}{1-z} \right) = 1 + \sum_{n=1}^{\infty} p_n z^n, \tag{2.2}$$

where

$$p_n = \frac{\beta-\alpha}{n\pi} i \ \left(1 - e^{2n\pi i \frac{(1-\alpha)}{\beta-\alpha}}\right) \qquad (n = 1, 2, \cdots). \tag{2.3}$$

Specially

$$\lim_{\beta\to+\infty} p_n = \lim_{\beta\to+\infty} \left\{ \frac{1 - e^{2n\pi i \frac{(1-\alpha)}{(\beta-\alpha)}}}{\frac{n\pi}{(\beta-\alpha)i}} \right\} = 2(1-\alpha), \tag{2.4}$$

a simple check gives us that

$$p(z) = 1 + \sum_{n=1}^{\infty} 2(1-\alpha) z^n = \frac{1 + (1-2\alpha)z}{1-z} \qquad (\beta \to +\infty),$$

which implies that $p(z)$ $(\beta \to +\infty)$ *maps* $\mathbb{U}$ *onto the right half-plane* w *with* $Re\ w > \alpha$.

Lemma 2.4. *[3] Let* $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ *be a function with positive real part in* $\mathbb{U}$. *Then, for any complex number* ν,

$$|c_2 - \nu c_1^2| \le 2 \max\{1, |1 - 2\nu|\}.$$

Lemma 2.5. *[7] If* $p \in P$, *then* $|c_k| \le 2$ *for each* k, *where* P *is the family of all functions* p *analytic in* $\Delta = \{z \in \mathbb{C} : 1 < |z| < +\infty\}$ *for which* $Re(p(z)) > 0$ *where* $p(z) = 1 + \frac{c_1}{z} + \frac{c_2}{z^2} + \frac{c_3}{z^3} + \cdots$.

3. Coefficient bounds for functions in $\Sigma(\lambda, \alpha, \beta)$ and $\Sigma_{\mathcal{C}}(\alpha, \beta)$

In this section, we define two subclasses of meromorphic univalent and obtain the general coefficient estimates for functions in these subclasses.

Definition 3.1. *Let* λ, α *and* β *be real nembers such that* $0 \le \alpha < 1 < \beta$ *and* $\lambda \ge 1$. *The meromorphic univalent function* f *given by* (1.1) *is said to be in the class* $\Sigma(\lambda, \alpha, \beta)$, *if the following condition is satisfied:*

$$\alpha < Re \left(\lambda f'(z) + (1-\lambda) \frac{f(z)}{z} \right) < \beta \qquad (z \in \Delta).$$

Remark 3.2. *By putting* $\lambda = 1$, *the class* $\Sigma(\lambda, \alpha, \beta)$ *reduces to the class* $\Sigma_c^0(\alpha, \beta)$ *introduced and studied by Sim et al. [10].*

Theorem 3.3. *Let* f *given by* (1.1) *be in the class* $\Sigma(\lambda, \alpha, \beta)$ $(0 \le \alpha < 1 < \beta,\ \lambda \ge 1)$. *Then*

$$|b_n| \le \frac{2(\beta-\alpha)}{((n+1)\lambda - 1)\pi} \sin\left(\frac{1-\alpha}{\beta-\alpha} \pi \right) \quad (n \in \mathbb{N}).$$

Proof . Define a function $g : \mathbb{U}^* \to \mathbb{C}$ by

$$g(z) = f(\frac{1}{z}) = \frac{1}{z} + \sum_{n=0}^{\infty} b_n z^n, \quad (z \in \mathbb{U}^*). \tag{3.1}$$

Since $f \in \Sigma(\lambda, \alpha, \beta)$, we have

$$\alpha < Re\{-\lambda z^2 g'(z) + (1-\lambda) z g(z)\} < \beta \;\; (z \in \mathbb{U}). \tag{3.2}$$

Let

$$q(z) = -\lambda z^2 g'(z) + (1-\lambda) z g(z) \;\; (z \in \mathbb{U}). \tag{3.3}$$

So $q(z)$ is an analytic function in $\mathbb{U}$ such that $q(0) = 1$. Also, from (3.2) and Lemma 2.2, we have

$$q(z) \prec p(z) \;\; (z \in \mathbb{U}), \tag{3.4}$$

where $p(z)$ is given by (2.1). On the other hand, the function $p(z)$ is convex in $\mathbb{U}$, and has the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \tag{3.5}$$

where p_n is given by (2.3). From (3.1) and (3.3), we have

$$q(z) = 1 + \sum_{n=0}^{\infty} (1 - (n+1)\lambda)\, b_n z^{n+1}. \tag{3.6}$$

From (3.4), (3.6) and Lemma 2.1, we have

$$|1 - (n+1)\lambda||b_n| \le |p_1|.$$

Therefore

$$|b_n| \le \frac{|p_1|}{|1-(n+1)\lambda|} = \frac{2(\beta-\alpha)}{((n+1)\lambda - 1)\pi} sin\left(\frac{1-\alpha}{\beta-\alpha}\pi\right) \quad (n \in \mathbb{N}).$$

□

Theorem 3.4. *Let f given by* (1.1) *be in the class* $\Sigma(\lambda, \alpha, \beta)$ $(0 \le \alpha < 1 < \beta,\ \lambda > 1)$ *and* $\mu \in \mathbb{C}$. *Then*

$$\begin{aligned} |b_1 - \mu b_0^2| \quad \le \quad & \frac{2(\beta-\alpha)}{\pi(2\lambda-1)} \sin\left(\frac{1-\alpha}{\beta-\alpha}\pi\right) \times \\ \max \quad & \left\{1, \left|\frac{\mu(1-2\lambda)(\beta-\alpha)i}{\pi(1-\lambda)^2} - \frac{1}{2} - \left(\frac{\mu(1-2\lambda)(\beta-\alpha)i}{\pi(1-\lambda)^2} + \frac{1}{2}\right) e^{2\pi i \frac{1-\alpha}{\beta-\alpha}}\right|\right\}. \end{aligned}$$

Proof . We consider functions $g(z)$, $q(z)$ and $p(z)$ given by (3.1), (3.3) and (2.1). Since $q(z) \prec p(z)$ $(z \in \mathbb{U})$, then there exists an analytic function $r : \mathbb{U} \to \mathbb{U}$, with $r(0) = 0,\ |r(z)| < 1, z \in \mathbb{U}$, such that:

$$q(z) = p(r(z)). \tag{3.7}$$

Next, define the function h by

$$h(z) = \frac{1 + r(z)}{1 - r(z)} = 1 + h_1 z + h_2 z^2 + \cdots. \tag{3.8}$$

Since $r(z)$ is Schwarz function, $h(z)$ is an analytic function in $\mathbb{U}$, with $h(0) = 1$ and which has positive real part in $\mathbb{U}$. From (3.8) one can derive

$$r(z) = \frac{h(z) - 1}{h(z) + 1} = \frac{1}{2}h_1 z + \frac{1}{2}(h_2 - \frac{h_1^2}{2})z^2 + \cdots .$$

So

$$p\left(\frac{h(z) - 1}{h(z) + 1}\right) = 1 + \frac{1}{2}p_1 h_1 z + \left(\frac{1}{2}p_1 h_2 - \frac{1}{4}p_1 h_1^2 + \frac{1}{4}p_2 h_1^2\right) z^2 + \cdots . \tag{3.9}$$

By comparing the coefficients in (3.6) and (3.9), we gain

$$(1 - \lambda)b_0 = \frac{1}{2}p_1 h_1$$

and

$$(1 - 2\lambda)b_1 = \frac{1}{2}\left(p_1 h_2 - \frac{1}{2}p_1 h_1^2 + \frac{1}{2}p_2 h_1^2\right).$$

From the above equations, we have

$$\begin{aligned} b_1 - \mu b_0^2 &= \frac{1}{2(1-2\lambda)}\left(p_1 h_2 - \frac{1}{2}p_1 h_1^2 + \frac{1}{2}p_2 h_1^2\right) - \frac{\mu}{4(1-\lambda)^2}h_1^2 p_1^2 \\ &= \frac{p_1}{2(1-2\lambda)}\left(h_2 - \frac{1}{2}h_1^2 + \frac{1}{2}\frac{p_2}{p_1}h_1^2 - \frac{\mu(1-2\lambda)}{2(1-\lambda)^2}h_1^2 p_1\right) \\ &= \frac{p_1}{2(1-2\lambda)}\left(h_2 - \frac{1}{2}\left\{1 - \frac{p_2}{p_1} + \frac{\mu(1-2\lambda)}{(1-\lambda)^2}p_1\right\} h_1^2\right). \end{aligned}$$

So

$$b_1 - \mu b_0^2 = \frac{p_1}{2(1-2\lambda)}\left(h_2 - \nu h_1^2\right),$$

where

$$\nu = \frac{1}{2}(1 - \frac{p_2}{p_1} + \frac{\mu(1-2\lambda)}{(1-\lambda)^2}p_1).$$

By using Lemma 2.4, we can get

$$\begin{aligned} |b_1 - \mu b_0^2| = \frac{|p_1|}{2|2\lambda - 1|}\left|h_2 - \nu h_1^2\right| &\leq \frac{|p_1|}{(2\lambda - 1)}\max\{1, |1 - 2\nu|\} \\ &= \frac{|p_1|}{(2\lambda - 1)}\max\{1, |\frac{\mu(1-2\lambda)}{(1-\lambda)^2}p_1 - \frac{p_2}{p_1}|\}. \end{aligned}$$

By substituting

$$p_1 = \frac{\beta - \alpha}{\pi}i\ \left(1 - e^{2\pi i\frac{(1-\alpha)}{\beta-\alpha}}\right)$$

and

$$p_2 = \frac{\beta - \alpha}{2\pi}i\ \left(1 - e^{4\pi i\frac{(1-\alpha)}{\beta-\alpha}}\right),$$

in above equation we obtain the desired result. □

Definition 3.5. *Let α and β be real nembers such that $0 \leq \alpha < 1 < \beta$. The meromorphic univalent function f given by* (1.1) *is said to be in the class $\Sigma_{\mathcal{C}}(\alpha, \beta)$, if the following condition is satisfied:*

$$\alpha < Re\left(1 + \frac{zf''(z)}{f'(z)}\right) < \beta \quad (z \in \Delta).$$

Theorem 3.6. *Let f given by* (1.1) *be in the class $\Sigma_{\mathcal{C}}(\alpha, \beta)$. Then*

$$|b_1| \leq \frac{\beta - \alpha}{\pi} \sin\left(\frac{1-\alpha}{\beta - \alpha}\pi\right)$$

and

$$|b_n| \leq \frac{2(\beta - \alpha)}{n(n+1)\pi} \sin\left(\frac{1-\alpha}{\beta - \alpha}\pi\right) \prod_{k=2}^{n} \left\{1 + \frac{2(\beta - \alpha)}{\pi k} \sin\left(\frac{1-\alpha}{\beta - \alpha}\pi\right)\right\} \ (n \geq 2).$$

Proof . Define a function $g : \mathbb{U}^* \to \mathbb{C}$ by

$$g(z) = f(\frac{1}{z}) = \frac{1}{z} + \sum_{n=0}^{\infty} b_n z^n. \tag{3.10}$$

Since $f \in \Sigma_{\mathcal{C}}(\alpha, \beta)$, it follows that

$$\alpha < Re\{1 - \frac{(z^2 g'(z))'}{zg'(z)}\} < \beta \ \ (z \in \mathbb{U}). \tag{3.11}$$

Let

$$L(z) = 1 - \frac{(z^2 g'(z))'}{zg'(z)} = 1 + 2b_1 z^2 + 6b_2 z^3 + \cdots \ \ (z \in \mathbb{U}). \tag{3.12}$$

Then $L(z)$ is an analytic function in $\mathbb{U}$ such that $L(0) = 1$. So, from (3.11) and Lemma 2.2, we get

$$L(z) \prec p(z) \ \ (z \in \mathbb{U}), \tag{3.13}$$

where $p(z)$ is given by (2.1). Note that the function $p(z)$ is convex in $\mathbb{U}$, and has the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \tag{3.14}$$

where p_n is given by (2.3).

If we let

$$L(z) = 1 + \sum_{n=1}^{\infty} l_n z^n, \tag{3.15}$$

then from (3.13), (3.14), (3.15) and Lemma 2.1, we imply

$$|l_n| \leq |p_1| = \frac{2(\beta - \alpha)}{\pi} \sin\left(\frac{1-\alpha}{\beta - \alpha}\pi\right) \ \ (n \in \mathbb{N}). \tag{3.16}$$

Now from (3.12), we have

$$\left(z^2g'(z)\right)' = \left(\sum_{n=1}^{\infty} l_n z^n\right) \times \left(-zg'(z)\right).$$

So, by comparing the coefficients of the above relation, we get

$$2b_1 = l_2$$

and

$$b_n = \frac{1}{n(n+1)}\left[l_{n+1} - b_1 l_{n-1} - 2b_2 l_{n-2} - \cdots - (n-1)b_{n-1}l_1\right].$$

Therefore

$$\begin{aligned}|b_n| &\leq \frac{1}{n(n+1)}\left\{|l_{n+1}| + |b_1||l_{n-1}| + 2|b_2||l_{n-2}| \cdots + (n-1)|b_{n-1}||l_1|\right\}\\ &\leq \frac{|p_1|}{n(n+1)}\left(1 + \sum_{k=1}^{n-1} k|b_k|\right).\end{aligned}$$

Now, in order to prove the claim of theorem, so we have to show that

$$|b_n| \leq \frac{|p_1|}{n(n+1)}\left(1 + \sum_{k=1}^{n-1} k|b_k|\right) \leq \frac{|p_1|}{n(n+1)} \prod_{k=2}^{n}\left(1 + \frac{|p_1|}{k}\right) \quad (n \in \mathbb{N}). \tag{3.17}$$

We now use the mathematical induction for the proof of (3.17). Since

$$\begin{aligned}|b_1| &= \frac{|l_2|}{2} \leq \frac{|p_1|}{2},\\ |b_2| &\leq \frac{|p_1|}{6}\left(1 + |b_1|\right) \leq \frac{|p_1|}{6}\left(1 + \frac{|p_1|}{2}\right)\end{aligned}$$

and

$$|b_3| \leq \frac{|p_1|}{12}\left(1 + |b_1| + 2|b_2|\right) \leq \frac{|p_1|}{12}\left[1 + \frac{|p_1|}{2} + \frac{|p_1|}{3}\left(1 + \frac{|p_1|}{2}\right)\right] = \frac{|p_1|}{12}(1 + \frac{|p_1|}{2})(1 + \frac{|p_1|}{3}).$$

It is clear that the claim holds true for $n = 1, 2, 3$. We suppose that the proposition is correct for $n \leq m - 1$. Therefore, according to the induction hypothesis, we get

$$\begin{aligned}&|b_m| \leq \frac{|p_1|}{m(m+1)}\left(1 + |b_1| + 2|b_2| + 3|b_3| + \cdots + (m-1)|b_{m-1}|\right) \leq\\ &\frac{|p_1|}{m(m+1)}\left[1 + \frac{|p_1|}{2} + \left\{\frac{|p_1|}{3}\left(1 + \frac{|p_1|}{2}\right)\right\} + \cdots + \left\{\frac{|p_1|}{m}\left(1 + \frac{|p_1|}{2}\right) \ldots \left(1 + \frac{|p_1|}{m-1}\right)\right\}\right]\\ &= \frac{|p_1|}{m(m+1)}\left(1 + \frac{|p_1|}{2}\right)\left(1 + \frac{|p_1|}{3}\right) \cdots \left(1 + \frac{|p_1|}{m}\right).\end{aligned}$$

□

Theorem 3.7. *Let f given by* (1.1) *be in the class $\Sigma_{\mathcal{C}}(\alpha, \beta)$ and $\mu \in \mathbb{C}$. Then*

$$|b_2 - \mu b_1^2| \leq \frac{2(\beta - \alpha)}{4\pi} \sin\left(\frac{1-\alpha}{\beta-\alpha}\pi\right)\left[\frac{2}{3} + |\mu|\frac{2(\beta - \alpha)}{\pi} \sin\left(\frac{1-\alpha}{\beta-\alpha}\pi\right)\right].$$

Proof . We consider functions $g(z)$, $L(z)$ and $p(z)$ given by (3.10), (3.12) and (2.1). Since $L(z) \prec p(z)$ $(z \in \mathbb{U})$, then there exists an analytic function $r : \mathbb{U} \to \mathbb{U}$, with $r(0) = 0$, $|r(z)| < 1, z \in \mathbb{U}$, such that:

$$L(z) = p(r(z)). \tag{3.18}$$

Define the function h by

$$h(z) = \frac{1 + r(z)}{1 - r(z)} = 1 + h_1 z + h_2 z^2 + \cdots . \tag{3.19}$$

Clearly, h is analytic in $\mathbb{U}$, $h(0) = 1$ and $Re\ h(z) > 0$. From (3.19) one can derive

$$r(z) = \frac{h(z) - 1}{h(z) + 1} = \frac{1}{2}h_1 z + \frac{1}{2}(h_2 - \frac{h_1^2}{2})z^2 + \frac{1}{2}(h_3 - h_1 h_2 + \frac{1}{4}h_1^3)z^3 + \cdots .$$

So

$$\begin{aligned} p(r(z)) = p\left(\frac{h(z)-1}{h(z)+1}\right) &= 1 + \frac{1}{2}p_1 h_1 z + \left(\frac{1}{2}p_1 h_2 - \frac{1}{4}p_1 h_1^2 + \frac{1}{4}p_2 h_1^2\right) z^2 \\ &+ \left(\frac{1}{2}p_1 h_3 + \frac{1}{2}(p_2 - p_1)h_1 h_2 + \frac{1}{8}(p_1 - 2p_2 + p_3)h_1^3\right) z^3 + \cdots . \end{aligned} \tag{3.20}$$

By equating the corresponding coefficients in (3.12) and (3.20), we arrive at

$$\begin{aligned} 0 &= \frac{1}{2}p_1 h_1, \\ 2b_1 &= \frac{1}{2}p_1 h_2 - \frac{1}{4}p_1 h_1^2 + \frac{1}{4}p_2 h_1^2 \end{aligned}$$

and

$$6b_2 = \frac{1}{2}p_1 h_3 + \frac{1}{2}(p_2 - p_1)h_1 h_2 + \frac{1}{8}(p_1 - 2p_2 + p_3)h_1^3.$$

From the above equations, we have

$$h_1 = 0, \quad b_1 = \frac{1}{4}p_1 h_2 \quad \textit{and} \quad b_2 = \frac{1}{12}p_1 h_3.$$

Hence

$$b_2 - \mu b_1^2 = \frac{1}{12}p_1 h_3 - \frac{\mu}{16}p_1^2 h_2^2 = \frac{1}{4}\left(\frac{1}{3}h_3 - \frac{\mu}{4}p_1 h_2^2\right) p_1.$$

By using Lemma 2.5, we get

$$|b_2 - \mu b_1^2| \leq \frac{|p_1|}{4}\left[\frac{1}{3}|h_3| + \frac{|\mu|}{4}|p_1||h_2|^2\right] \leq \frac{|p_1|}{4}\left[\frac{2}{3} + |\mu||p_1|\right].$$

□

4. Coefficient bounds for functions in $\Sigma_{\mathcal{B},\mathcal{C}}(\alpha,\beta,\lambda)$

In this section, we introduce new subclass of meromorphic bi-univalent and find the initial coefficients estimates for functions in this subclass.

Definition 4.1. *Let λ, α and β be real nembers such that $0 \le \alpha < 1 < \beta$ and $\lambda \ge 1$. The function f given by* (1.1) *is said to be in the class $\Sigma_{\mathcal{B},\mathcal{C}}(\alpha,\beta,\lambda)$, if the following conditions are satisfied:*

$$f \in \Sigma_{\mathcal{B}} \text{ and } \alpha < Re\left(\lambda\frac{zf'(z)}{f(z)} + (1-\lambda)(1+\frac{zf''(z)}{f'(z)})\right) < \beta \quad (z \in \Delta)$$

and

$$\alpha < Re\left(\lambda\frac{wg'(w)}{g(w)} + (1-\lambda)(1+\frac{wg''(w)}{g'(w)})\right) < \beta \quad (w \in \Delta),$$

where $g(w) = f^{-1}(w)$.

Remark 4.2. *If $f \in \Sigma_{\mathcal{B},\mathcal{C}}(\alpha,\beta,\lambda)$ and $\beta \to +\infty$, then the function f is said to be in the class $\mathcal{T}_{\Sigma_b}(\alpha,\lambda)$ introduced and studied by Panigrahi [6].*

If $f \in \Sigma_{\mathcal{B},\mathcal{C}}(\alpha,\beta,\lambda)$, $\lambda = 1$ and $\beta \to +\infty$, then the function f is said to be in the meromorphic bi-starlike of order $\alpha(0 \le \alpha < 1)$ presented and studied by Hamidi et al. [1].

Theorem 4.3. *Let f given by* (1.1) *be in the class $\Sigma_{\mathcal{B},\mathcal{C}}(\alpha,\beta,\lambda)$. Then*

$$|b_0| \le \min\left\{\frac{|p_1|}{\lambda}, \sqrt{\frac{|p_1| + |p_2 - p_1|}{\lambda}}\right\},$$

$$|b_1| \le \min\left\{\frac{|p_1|}{2(2\lambda-1)}, \frac{1}{2(2\lambda-1)}\sqrt{\left|\frac{p_1^4}{\lambda^2} - p_1^2 - p_2^2 + 2p_1p_2\right| + |p_1|^2}\right\}$$

and

$$|b_2| \le \frac{1}{3(3\lambda-2)}\left[2|p_2 - p_1| + |p_1| + |p_1 - 2p_2 + p_3 - \frac{p_1^3}{\lambda^2}|\right],$$

where p_1, p_2, p_3 given by (2.3).

Proof . For meromorphic function f of the form (1.1), we have:

$$\lambda\frac{zf'(z)}{f(z)} + (1-\lambda)(1+\frac{zf''(z)}{f'(z)}) =$$
$$1 - \frac{\lambda b_0}{z} + \frac{\lambda b_0^2 + 2(1-2\lambda)b_1}{z^2} - \frac{\lambda {b_0}^3 - 3\lambda b_0 b_1 - 3(2-3\lambda)b_2}{z^3} + \cdots \quad (4.1)$$

and for its inverse map, $g = f^{-1}$ of the form (1.2), we have:

$$\lambda\frac{wg'(w)}{g(w)} + (1-\lambda)(1+\frac{wg''(w)}{g'(w)}) =$$
$$1 + \frac{\lambda b_0}{w} + \frac{\lambda b_0^2 - 2(1-2\lambda)b_1}{w^2} + \frac{\lambda {b_0}^3 - 3(2-3\lambda)b_2 - 6(1-2\lambda)b_0 b_1}{w^3} + \cdots . \quad (4.2)$$

Define functions ϕ and ψ by

$$\phi(z) = f(\frac{1}{z}) \quad and \quad \psi(w) = g(\frac{1}{w}) \quad (z, w \in \mathbb{U}^*).$$

respectively. Therefore

$$-\lambda\frac{z\phi'(z)}{\phi(z)} + (1-\lambda)\left(1 - \frac{(z^2\phi'(z))'}{z\phi'(z)}\right) = \lambda\frac{f'(\frac{1}{z})}{zf(\frac{1}{z})} + (1-\lambda)(1 + \frac{f''(\frac{1}{z})}{zf'(\frac{1}{z})}), \ (z \in \mathbb{U}^*) \tag{4.3}$$

and

$$-\lambda\frac{w\psi'(w)}{\psi(w)} + (1-\lambda)\left(1 - \frac{(w^2\psi'(w))'}{w\psi'(w)}\right) = \lambda\frac{g'(\frac{1}{w})}{wg(\frac{1}{w})} + (1-\lambda)(1 + \frac{g''(\frac{1}{w})}{wg'(\frac{1}{w})}), \ (w \in \mathbb{U}^*). \tag{4.4}$$

Since $f \in \Sigma_{\mathcal{B},\mathcal{C}}(\alpha, \beta, \lambda)$, we have

$$\alpha \le Re\left\{-\lambda\frac{z\phi'(z)}{\phi(z)} + (1-\lambda)\left(1 - \frac{(z^2\phi'(z))'}{z\phi'(z)}\right)\right\} \le \beta \quad (z \in \mathbb{U}) \tag{4.5}$$

and

$$\alpha \le Re\left\{-\lambda\frac{w\psi'(w)}{\psi(w)} + (1-\lambda)\left(1 - \frac{(w^2\psi'(w))'}{w\psi'(w)}\right)\right\} \le \beta \quad (w \in \mathbb{U}). \tag{4.6}$$

Now, let

$$L(z) = -\lambda\frac{z\phi'(z)}{\phi(z)} + (1-\lambda)\left(1 - \frac{(z^2\phi'(z))'}{z\phi'(z)}\right) \tag{4.7}$$

and

$$T(w) = -\lambda\frac{w\psi'(w)}{\psi(w)} + (1-\lambda)\left(1 - \frac{(w^2\psi'(w))'}{w\psi'(w)}\right). \tag{4.8}$$

From (4.1), (4.2), (4.7) and (4.8), we get

$$L(z) = 1 - \lambda b_0 z + [\lambda b_0^2 + 2(1-2\lambda)b_1]z^2 - [\lambda {b_0}^3 - 3\lambda b_0 b_1 - 3(2-3\lambda)b_2]z^3 + \cdots \tag{4.9}$$

and

$$T(w) = 1 + \lambda b_0 w + [\lambda b_0^2 - 2(1-2\lambda)b_1]w^2 + [\lambda {b_0}^3 - 3(2-3\lambda)b_2 - 6(1-2\lambda)b_0 b_1]w^3 + \cdots \tag{4.10}$$

Also, from (4.5) and (4.6), we get

$$L(z) \prec p(z) \ \ and \ \ T(w) \prec p(w) \quad (z, w \in \mathbb{U}), \tag{4.11}$$

where $p(z)$ is given by (2.1) and has the series given by (2.2). Also, we imply from (4.11), there exists two analytic functions $r, s : \mathbb{U} \to \mathbb{U}$, with $r(0) = 0 = s(0)$, $|r(z)| < 1$, $|s(w)| < 1$, $z, w \in \mathbb{U}$, such that:

$$L(z) = p(r(z)) \ \ and \ \ T(w) = p(s(w)). \tag{4.12}$$

Define functions h and k by

$$h(z) = \frac{1 + r(z)}{1 - r(z)} = 1 + h_1 z + h_2 z^2 + \cdots \ and \ k(w) = \frac{1 + s(w)}{1 - s(w)} = 1 + k_1 w + k_2 w^2 + \cdots. \tag{4.13}$$

Clearly, h and k are analytic functions in $\mathbb{U}$, $h(0) = 1 = k(0)$ and which have positive real part in $\mathbb{U}$. From (4.13) one can derive

$$r(z) = \frac{h(z)-1}{h(z)+1} = \frac{1}{2}h_1 z + \frac{1}{2}(h_2 - \frac{h_1^2}{2})z^2 + \frac{1}{2}(h_3 - h_1h_2 + \frac{h_1^3}{4})z^3 + \cdots \tag{4.14}$$

and

$$s(w) = \frac{k(w)-1}{k(w)+1} = \frac{1}{2}k_1 w + \frac{1}{2}(k_2 - \frac{k_1^2}{2})w^2 + \frac{1}{2}(k_3 - k_1k_2 + \frac{k_1^3}{4})w^3 + \cdots t. \tag{4.15}$$

From (2.2), (4.14) and (4.15), we have

$$\begin{aligned} p(r(z)) = p\left(\frac{h(z)-1}{h(z)+1}\right) &= 1 + \frac{1}{2}p_1h_1z + \left(\frac{1}{2}p_1h_2 - \frac{1}{4}p_1h_1^2 + \frac{1}{4}p_2h_1^2\right)z^2 \\ &+ \left(\frac{1}{8}p_1h_1^3 - \frac{1}{2}p_1h_1h_2 + \frac{1}{2}p_1h_3 + \frac{1}{2}p_2h_1h_2 - \frac{1}{4}p_2h_1^3 + \frac{1}{8}p_3h_1^3\right)z^3 + \cdots \end{aligned} \tag{4.16}$$

and

$$\begin{aligned} p(s(w)) = p\left(\frac{k(w)-1}{k(w)+1}\right) &= 1 + \frac{1}{2}p_1k_1w + \left(\frac{1}{2}p_1k_2 - \frac{1}{4}p_1k_1^2 + \frac{1}{4}p_2k_1^2\right)w^2 \\ &+ \left(\frac{1}{8}p_1k_1^3 - \frac{1}{2}p_1k_1k_2 + \frac{1}{2}p_1k_3 + \frac{1}{2}p_2k_1k_2 - \frac{1}{4}p_2k_1^3 + \frac{1}{8}p_3k_1^3\right)w^3 + \cdots . \end{aligned} \tag{4.17}$$

It follows from (4.9), (4.10), (4.16) and (4.17) that

$$-\lambda b_0 = \frac{1}{2}p_1h_1, \tag{4.18}$$

$$\lambda b_0^2 + 2(1-2\lambda)b_1 = \frac{1}{2}p_1h_2 - \frac{1}{4}p_1h_1^2 + \frac{1}{4}p_2h_1^2, \tag{4.19}$$

$$\begin{aligned} &-\lambda {b_0}^3 + 3\lambda b_0b_1 + 3(2-3\lambda)b_2 = \\ &\qquad \frac{1}{8}p_1h_1^3 - \frac{1}{2}p_1h_1h_2 + \frac{1}{2}p_1h_3 + \frac{1}{2}p_2h_1h_2 - \frac{1}{4}p_2h_1^3 + \frac{1}{8}p_3h_1^3, \end{aligned} \tag{4.20}$$

$$\lambda b_0 = \frac{1}{2}p_1k_1, \tag{4.21}$$

$$\lambda b_0^2 - 2(1-2\lambda)b_1 = \frac{1}{2}p_1k_2 - \frac{1}{4}p_1k_1^2 + \frac{1}{4}p_2k_1^2 \tag{4.22}$$

and

$$\begin{aligned} &\lambda {b_0}^3 - 6(1-2\lambda)b_0b_1 - 3(2-3\lambda)b_2 = \\ &\qquad \frac{1}{8}p_1k_1^3 - \frac{1}{2}p_1k_1k_2 + \frac{1}{2}p_1k_3 + \frac{1}{2}p_2k_1k_2 - \frac{1}{4}p_2k_1^3 + \frac{1}{8}p_3k_1^3. \end{aligned} \tag{4.23}$$

From equations (4.18) and (4.21), we obtain

$$h_1 = -k_1\ , \quad 2\lambda^2 b_0^2 = \frac{1}{4}p_1^2(h_1^2 + k_1^2).$$

Applying Lemma 2.5 for the above equation, we obtain

$$|b_0|^2 \le \frac{|p_1|^2(|h_1|^2 + |k_1|^2)}{8\lambda^2} \le \frac{|p_1|^2}{\lambda^2}.$$

Now, from (4.19) and (4.22), we get that

$$b_0^2 = \frac{1}{4\lambda}p_1(h_2 + k_2) + \frac{1}{8\lambda}(p_2 - p_1)(h_1^2 + k_1^2).$$

By using Lemma 2.5 once again, we readily get

$$|b_0|^2 \leq \frac{|p_1| + |p_2 - p_1|}{\lambda}.$$

Also, from (4.19) and (4.22), we obtain

$$\lambda^2 b_0^4 - 4(1-2\lambda)^2 b_1^2 = \frac{1}{4}p_1^2 h_2 k_2 - \frac{1}{8}p_1(p_1 - p_2)h_1^2(h_2 + k_2) + \frac{1}{16}h_1^4(p_1 - p_2)^2.$$

Therefore, we find that

$$4(1-2\lambda)^2 b_1^2 = \frac{1}{16}\left(\frac{p_1^4}{\lambda^2} - p_1^2 - p_2^2 + 2p_1p_2\right)h_1^4 + \frac{1}{8}p_1^2 k_2\left(\frac{p_1 - p_2}{p_1}h_1^2 - h_2\right) + \frac{1}{8}p_1^2 h_2\left(\frac{p_1 - p_2}{p_1}k_1^2 - k_2\right).$$

Now taking the absolute value of both sides of above equation, we obtain

$$4|1-2\lambda|^2|b_1|^2 \leq \frac{1}{16}\left|\frac{p_1^4}{\lambda^2} - p_1^2 - p_2^2 + 2p_1p_2\right||h_1|^4 + \frac{1}{8}|p_1|^2|k_2|\left|\frac{p_1 - p_2}{p_1}h_1^2 - h_2\right| + \frac{1}{8}|p_1|^2|h_2|\left|\frac{p_1 - p_2}{p_1}k_1^2 - k_2\right|. \tag{4.24}$$

By applying Lemma 2.4, we obtain

$$\frac{1}{8}|p_1|^2|k_2|\left|\frac{p_1 - p_2}{p_1}h_1^2 - h_2\right| \leq \frac{1}{4}|p_1|^2|k_2|\max\left\{1; |2\frac{p_2}{p_1} - 1|\right\} = \frac{1}{4}|p_1|^2|k_2|\max\left\{1; |e^{2\pi i(1-\alpha)/(\beta-\alpha)}|\right\} = \frac{1}{4}|p_1|^2|k_2|. \tag{4.25}$$

Similarly, we have

$$\frac{1}{8}|p_1|^2|h_2|\left|\frac{p_1 - p_2}{p_1}k_1^2 - k_2\right| \leq \frac{1}{4}|p_1|^2|h_2|. \tag{4.26}$$

By applying Lemma 2.5 in equations (4.24), (4.25) and (4.26), we find that

$$4|1-2\lambda|^2|b_1|^2 \leq \left|\frac{p_1^4}{\lambda^2} - p_1^2 - p_2^2 + 2p_1p_2\right| + |p_1|^2.$$

On the other hand, by subtracting (4.22) from (4.19), we get

$$4(1-2\lambda)b_1 = \frac{1}{2}p_1(h_2 - k_2).$$

Therefore, we get

$$|b_1| = \frac{|p_1||h_2 - k_2|}{8|1-2\lambda|} \leq \frac{|p_1|}{2(2\lambda - 1)}.$$

Next, to find the bound on b_2, consider the sum of (4.20) and (4.23) with $h_1 = -k_1$, we have

$$b_0 b_1 = \frac{1}{6(5\lambda-2)}[(h_1h_2 + k_1k_2)(p_2 - p_1) + p_1(h_3 + k_3)]. \tag{4.27}$$

Subtracting (4.23) from (4.20) with $h_1 = -k_1$, we obtain

$$6(2-3\lambda)b_2 = 2\lambda b_0^3 + 3(3\lambda-2)b_0b_1 + \frac{1}{4}(p_1 - 2p_2 + p_3)h_1^3 + \frac{1}{2}(h_1h_2 - k_1k_2)(p_2 - p_1) + \frac{1}{2}p_1(h_3 - k_3). \tag{4.28}$$

By using (4.18) and (4.27) in (4.28) gives

$$6(2-3\lambda)b_2 = (p_2 - p_1)\left(\frac{4\lambda-2}{5\lambda-2}h_1h_2 - \frac{\lambda}{5\lambda-2}k_1k_2\right) + p_1\left(\frac{4\lambda-2}{5\lambda-2}h_3 - \frac{\lambda}{5\lambda-2}k_3\right) + \frac{1}{4}(p_1 - 2p_2 + p_3 - \frac{p_1^3}{\lambda^2})h_1^3. \tag{4.29}$$

Applying Lemma 2.5 once again for the coefficients h_1, h_2, k_1 and k_2,we get

$$|b_2| \leq \frac{1}{3(3\lambda-2)}\left[2|p_2 - p_1| + |p_1| + |p_1 - 2p_2 + p_3 - \frac{p_1^3}{\lambda^2}|\right].$$

□

5. Corollaries and Consequences

By putting $\lambda = 1$ in Theorem 3.3, we obtain the following result.

Corollary 5.1. *Let f given by (1.1) be in the class $\Sigma_c^0(\alpha, \beta)$ $(0 \leq \alpha < 1 < \beta)$. Then*

$$|b_n| \leq \frac{|p_1|}{n} = \frac{2(\beta-\alpha)}{n\pi} \sin\left(\frac{1-\alpha}{\beta-\alpha}\pi\right) \quad (n \in \mathbb{N}).$$

If $\beta \to +\infty$ in Theorem 4.3, we obtain the following result.

Corollary 5.2. *Let $f(z) \in \Sigma$ given by (1.1) be in the class $\mathcal{T}_{\Sigma_b'}(\alpha, \lambda)$, then*

$$|b_0| \leq \begin{cases} \sqrt{\frac{2(1-\alpha)}{\lambda}} & ;\lambda + 2\alpha \leq 2 \\ \frac{2(1-\alpha)}{\lambda} & ;\lambda + 2\alpha \geq 2, \end{cases}$$

$$|b_1| \leq \min\left\{\frac{1-\alpha}{2\lambda-1}, \frac{1-\alpha}{2\lambda-1}\sqrt{\frac{4(1-\alpha)^2}{\lambda^2} + 1}\right\} = \frac{1-\alpha}{2\lambda-1}$$

and

$$|b_2| \leq \frac{2(1-\alpha)}{3(3\lambda-2)}\left[1 + \frac{4(1-\alpha)^2}{\lambda^2}\right].$$

Remark 5.3. *Corollary 5.2 provides the estimates of $|b_0|, |b_1|$ and $|b_2|$ obtained previously by Salehian et al. [9, Corollary 3.3]. Furthermore, the bounds on $|b_0|$ and $|b_1|$ given in Corollary 5.2 are better than those given by Panigrahi [6, Theorem 3.2].*

By putting $\lambda = 1$ in Corollary 5.2, we obtain the following result.

Corollary 5.4. *Let* $f(z)$ *given by* (1.1) *be meromorphic bi-starlike of order* $\alpha(0 \leq \alpha < 1)$ *in* Δ. *Then*

$$|b_0| \leq \begin{cases} \sqrt{2(1-\alpha)} & ; \alpha \leq \frac{1}{2} \\ 2(1-\alpha) & ; \alpha \geq \frac{1}{2}, \end{cases}$$

$$|b_1| \leq \min\left\{(1-\alpha), (1-\alpha)\sqrt{4(1-\alpha)^2+1}\right\} = 1-\alpha$$

and

$$|b_2| \leq \frac{2(1-\alpha)}{3}\left[1+4(1-\alpha)^2\right].$$

Remark 5.5. *Corollary 5.4 provides the estimates of* $|b_0|$ *and* $|b_1|$ *obtained previously by Salehian et al. [9, Corollary 3.5]. Also, the bound on* $|b_0|$ *given in Corollary 5.4 is better than that given by Hamidi et al. [1, Theorem 2]. Also we find estimate of coefficient* $|b_2|$ *of functions in this subclass.*

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