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Expected mean square rate estimation of repeated measurements model

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Abstract

In this paper, we obtained the estimation corresponding to the expected mean square rate of repeated measurement model depend on maximum likelihood method (MLM), restricted maximum likelihood method (REMLM) and modified restricted maximum likelihood method (MREMLM), and got 8 cases that were classified into three types.

Keywords: Repeated Measurements Model (RMM), Maximum Likelihood Method (MLM), Restricted Maximum Likelihood Method (REMLM), Modified Restricted Maximum Likelihood Method (MREMLM)

1. Introduction

In several areas, such as health and life sciences, epidemiology, biomedical, environmental, manufacturing, psychological, educational studies, and so on, repeated measurement analysis is commonly used. Repeated measurements are a concept used to characterize data in which the result variable is observed several times and likely under various experimental conditions within each experimental device. The analysis of variance of repeated measurements, also referred to as randomized designs of blocks and split plots [1, 7, 13, 14].

The maximum likelihood estimation is a method of estimating the parameters of a model. This estimation method is one of the most widely used which selects the set of values of the model parameters that maximize the likelihood function. Intuitively, this maximizes the "agreement" of the selected model with the observed data. It is giving a unified approach to estimation [6, 8, 12]. Many

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studies have explored the repeated measurements model, for example: Vonesh and Chinchilli (1997) discussed the univariate repeated measurements model, analysis of variance model [13]. Keselman, Algina and Kowalchuk in (2001) Studied the designs of repeated measures The various approaches are presented with a discussion of their strengths and weaknesses, and recommendations are made regarding the 'best' choice of analysis, [9]. Moser and Macchiavelli in (2002) used model selection methods for repeated measurements of covariance structure have been explored, [10]. AL-Mouel and Mustafa, in (2014) studied the sphericity test for one-way Multivariate Repeated Measurements Analysis of variance mode, [3]. AL-Mouel and Naji, in (2014) devoted to study of one-way multivariate repeated measurements structure of covariance model, [4]. AL-Mouel and Hassan in (2016) estimated the repeated measurement model parameters by using maximum likelihood method, [5]. Özgür and et al., in (2020) discussed the non-Gaussian repeated measurement results, they analyzed the linear mixed-effects models, using maximum likelihood analysis using a general linear model for expected responses and arbitrary structural models for the covariances within the case, [11]. In this work, we obtained the estimation corresponding to the expected mean square rate of repeated measurement model depend on maximum likelihood method (MLM), restricted maximum likelihood method (REMLM) and modified restricted maximum likelihood method (MREMLM), and got 8 cases that were classified into three types.

2. Setting Up The Model

The repeated measurement model can be summarized as following:

$$h_{\mathrm{abc}} = \theta + \mathrm{A}_b + \pi_{a(b)} + \mathrm{B}_c + (\mathrm{AB})_{bc} + \epsilon_{abc} \tag{2.1}$$

where
$a = 1, \ldots, I$ "is an index for experimental unit within group (b)",
$b = 1, \ldots, J$ "is an index for levels of the between-units factor (Group)",
$c = 1, \ldots, K$ "is an index for levels of the within-units factor (Time)",
h_{abc} : "is the response measurement at time (c) for unit (a) within group (b)", θ : "is the overall mean",
A_b : "is the added effect for treatment group (b)",
$\pi_{a(b)}$: "is the random effect for due to experimental unit (a) within treatment group$(b)''$,
B_c : "is the added effect for time $(c)''$,
$(\mathrm{AB})_{bc}$: "is the added effect for the group $(b)\times$ time (c) interaction",
ϵ_{abc} : "is the random error on time (c) for unit (a) within group $(b)''$.
For the parameterization to be of full rank, we imposed the following set of conditions: $\sum_{b=1}^{J} A_b = 0; \sum_{c=1}^{K} \mathrm{B}_c = 0; \quad \sum_{b=1}^{J} (\mathrm{AB})_{bc} = 0$ for each c = 1, ..., K
$\sum_{c=1}^{K} (\mathrm{AB})_{bc} = 0$ for each b = 1, ..., J
and let, the ϵ_{abc} and $\pi_{a(b)}$ are independent with

$$\epsilon_{abc} i.i.d \sim N\left(0, \sigma_\epsilon^2\right) \text{ and } \pi_{a(b)} i.i.d \sim N\left(0, \sigma_\pi^2\right). \tag{2.2}$$

The (ANOVA) table of the repeated measurements model is:

Source of Variation	Degree of Freedom	Sum Square	Mean Square	Expected of Mean Square
Group	$J-1$	SS_A	$\frac{SS_A}{J-1}$	$\frac{IK}{J-1}\sum_{b=1}^{J} A_b^2 + K\sigma_\pi^2 + \sigma_\epsilon^2$
Unit (Group)	$J(I-1)$	SS_π	$\frac{SS_\pi}{J(I-1)}$	$\mathrm{K}\sigma_\pi^2 + \sigma_\epsilon^2$
Time	$K-1$	SS_B	$\frac{SS_B}{K-1}$	$\frac{1J}{K-1}\sum_{c=1}^{K} B_c^2 + \sigma_e^2$
Group $\times$ Time	$(K-1)(J-1)$	$SS_{A\times B}$	$\frac{SS_{A\times B}}{(K-1)(J-1))}$	$\frac{t}{(K-1)(J-1)}\sum_{a=1}^{I}\sum_{c=1}^{K}(AB)_{bc}^2 + \sigma_e^2$
Residual	$J(K-1)(I-1)$	SS_e	$\frac{SS_e}{J(K-1)(I-1)}$	σ_ϵ^2

The sum of squares due to groups, subjects group, time, group $\times$ time and residuals are then defined respectively as follows:

$SS_A = IK\sum_{b=1}^{K}\left(\bar{h}_{.b.} - \bar{h}_{.....}\right)^2, SS_\pi = K\sum_{a=1}^{I}\sum_{b=1}^{J}\left(\bar{h}_{ab.} - \bar{h}_{.b.}\right)^2$

$SS_B = IJ\sum_{c=1}^{K}\left(\bar{h}_{...c} - \bar{h}_{...}\right)^2, SS_{A\times B} = I\sum_{b=1}^{J}\sum_{c=1}^{K}\left(\bar{h}_{.bc} - \bar{h}_{.b.} - \bar{h}_{.c} + \bar{h}_{...}\right)^2$

$SS_\epsilon = \sum_{a=1}^{I}\sum_{b=1}^{J}\sum_{c=1}^{K}\left(\bar{h}_{abc} - \bar{h}_{.bc} - h_{ab.} + \bar{h}_{.b.}\right)^2$

where

$\bar{h}_{...} = \frac{1}{1JK}\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K} h_{abc}$: the overall mean.

$\bar{h}_{.b.} = \frac{1}{IJ}\sum_{i=1}^{I}\sum_{c=1}^{K} y_{abc}$: the mean for group (b).

$\bar{h}_{ab.} = \frac{1}{K}\sum_{c=1}^{K} h_{abc}$: the mean for a th subject within group (b).

$\bar{h}_{..c} = \frac{1}{IJ}\sum_{i=1}^{I}\sum_{j=1}^{J} h_{abc}$: the mean for time (c).

$\bar{h}_{.bc} = \frac{1}{I}\sum_{a=1}^{I} h_{abc}$: the mean for group (b) at time (c).

Let

$$\theta_{abc} = \theta + A_b + \pi_{a(b)} + B_c + (AB)_{bc} \tag{2.3}$$

represent the mean of time (c) for unit (a) within group (b) and, let

$$H = \ell_0\theta + \sum_{b=1}^{J}\ell_b A_b + \sum_{a=1}^{I}\sum_{b=1}^{J}\ell_a\ell_b\pi_{a(b)} + \sum_{c=1}^{K}\ell_c B_c + \sum_{b=1}^{J}\sum_{c=1}^{k}\ell_b\ell_c(AB)_{bc} \tag{2.4}$$

be an arbitrary linear combination of parameters $\theta, A_1, \ldots, A_q, \pi_{1(1)}, \ldots, \pi_{I(J)}, \mathrm{B}_1, \ldots, \mathrm{B}_K, (\mathrm{AB})_{11}, \ldots, (\mathrm{AB})_{JK}$ the best linear unbiased estimators (BLUE's) of the estimable parameters θ, A_b, $\pi_{a(b)}, B_c$, $(AB)_{bc}$ and θ_{abc} are $\hat{\theta} = \bar{h}_{...}, \hat{A}_b = \bar{h}_{.b.} - \bar{h}_{...}, \hat{\pi}_{a(b)} = (1-r)\left(\bar{h}_{ab.} - \bar{h}_{.b.}\right)$, $\hat{B}_c = \bar{h}_{..c} - \bar{h}_{....}, \widehat{(AB)}_{bc} = \bar{h}_{.bc} + \bar{h}_{....} - \bar{h}_{.b.} - \bar{h}_{..c}$ and $\hat{\theta}_{abc} = (1-r)\left(\bar{h}_{ab.} - \bar{h}_{.b.}\right) + \bar{h}_{.bc}$, [2] from the variance analysis (ANOVA) table, we have that

$$E\left(MS_\pi\right) = \tau_\pi = K\sigma_\pi^2 + \sigma_\epsilon^2 \tag{2.5}$$

and

$$E\left(MS_\epsilon\right) = \tau_\epsilon = \sigma_\epsilon^2 \tag{2.6}$$

since, the ANOVA estimators of τ_π and τ_ϵ are

$$\hat{\tau}_\epsilon = MS_\epsilon \text{ and } \hat{\tau}_\pi = MS_\pi. \tag{2.7}$$

The rate of expected mean squares is denote

$$r = \frac{\tau_\epsilon}{\tau_\pi} = \frac{\sigma_\epsilon^2}{\mathrm{K}\sigma_\pi^2 + \sigma_\epsilon^2} \tag{2.8}$$

note that $0 < r \leq 1$ is known if and only if $\sigma_\epsilon^2/\sigma_\pi^2$ is known. And the corresponding estimator of r is

$$No.1 \quad \hat{r} = \frac{MS_\epsilon}{MS_\pi} = \frac{SS_\epsilon}{SS_\pi}\frac{1}{(K-1)}. \tag{2.9}$$

These estimates can be beyond the parameter space. To trim the estimated value of $\hat{r}$ by result (9), we put $r > 1$, thus obtaining the estimator

$$No.2 \quad \hat{r} = \min\left\{\frac{SS_\epsilon}{SS_\pi}\frac{1}{(K-1)}, 1\right\} \tag{2.10}$$

the trimmed version, No. 1 , is the usual ANOVA estimator.

3. Estimation of parameters

We consider maximum likelihood method (MLM), restricted maximum likelihood method (REMLM) and modified restricted maximum likelihood method (MREMLM) the rate of expected mean squares. The statistical $\bar{h}_{...}, \bar{h}_{.b.} - \bar{h}_{...}, \bar{h}_{.c} - \bar{h}_{...}, \bar{h}_{.bc} + \bar{h}_{...} - \bar{h}_{.b.} - \bar{h}_{.c}, SS_\epsilon$ and SS_π form a set of complete sufficient statistics for $\theta, A_b, B_c, (AB)_{bc}, \sigma_\epsilon^2$ and σ_π^2. These six statistics are distributed independently as:

$$\left.\begin{array}{l} \bar{h}_{...} \sim N\left(\theta, \frac{\tau_\pi}{IJK}\right) \\ \bar{h}_{.b.} - \bar{h}_{...} \sim N\left(A_b, \frac{(J+1)\tau_\pi}{IK}\right) \\ \bar{h}_{...c} - \bar{h}_{....} \sim N\left(B_c, \frac{\sigma_\epsilon^2+\sigma_\pi^2}{IJ} + \frac{\tau_\pi}{IJ}\right) \\ \bar{h}_{.bc} + \bar{h}_{...} - h_{.b.} - \bar{h}_{.c} \sim N\left((AB)_{bc}, (J+1)\left(\frac{\sigma_\epsilon^2+\sigma_\pi^2}{IJ} + \frac{\tau_\pi}{IJK}\right)\right) \\ SS_\epsilon \sim \tau_\epsilon \chi^2[(I-1)(K-1)] \\ SS_\pi \sim \tau_\pi \chi^2[J(I-1)] \end{array}\right\} \tag{3.1}$$

where $\chi^2(t)$ a central distribution of chi-square of (t) degrees of freedom indicates. The maximum likelihood estimators of $\theta, A_b, B_c, (AB)_{bc}, \tau_\epsilon$ and τ_π are determined by the parameter values, which maximize the function:

$$\begin{aligned} & L\left(\theta, A_b, B_c, (AB)_{bc}, \tau_\epsilon, \tau_\pi \mid h\right) \propto \\ & (\tau_\pi)^{-2} (\tau_\pi)^{-\frac{J(I-1)}{2}} (\tau_\epsilon)^{-\frac{J(I-1)(K-1)}{2}} \exp\left\{-\frac{1}{2}\left[\frac{IJK\left(h_{...} - \theta\right)^2}{\tau_\pi} + \frac{nqp\left(\bar{h}_{.b.} - \bar{h}_{...} - A_b\right)^2}{\tau_\pi}\right.\right. \\ & \left.\left. + \frac{IJK\left(\bar{h}_{.c} - \bar{h}_{...} - B_c\right)^2}{\tau_\pi} + \frac{IJK\left(\bar{h}_{.bc} + \bar{h}_{.....} - h_{.b.} - \bar{h}_{..c} - (AB)_{bc}\right)^2}{\tau_\pi} + \frac{SS_\pi}{\tau_\pi} + \frac{SS_\epsilon}{\tau_\epsilon}\right]\right\} \end{aligned}$$

$$\begin{aligned} & L\left(\theta, A_b, B_c, (AB)_{bc}, \tau_\epsilon, \tau_\pi \mid h\right) \propto (\tau_\pi)^{-\frac{J(I-1)-4}{2}} (\tau_\epsilon)^{-\frac{J(I-1)(K-1)}{2}} \exp\left\{-\frac{1}{2}\left[\frac{IJK\left(h_{...,} - \theta\right)^2}{\tau_\pi} + \right.\right. \\ & \frac{nqp\left(\bar{h}_{.b} - \bar{h}_{....} - A_b\right)^2}{\tau_\pi} + \frac{IJK\left(\bar{h}_{.c} - \bar{h}_{...} - B_c\right)^2}{\tau_\pi} + \frac{IJK\left(\bar{h}_{.bc} + \bar{h}_{....} - h_{.b.} - \bar{h}_{.c} - (AB)_{bc}\right)^2}{\tau_\pi} \\ & \left.\left. + \frac{SS_\pi}{\tau_\pi} + \frac{SS_\epsilon}{\tau_\epsilon}\right]\right\} \end{aligned} \tag{3.2}$$

subject to the restrictions $\tau_\pi \geq \tau_\epsilon > 0$ and the ML estimates of τ_π and τ_ϵ are these values of τ_π and τ_ϵ that maximize the function:

$$\log L\left(\tau_\pi, \tau_\epsilon \mid h\right) \propto -\frac{J(I-1)-4}{2}\log\left(\tau_\pi\right) - \frac{J(I-1)(K-1)}{2}\log\left(\tau_\epsilon\right) + \left\{-\frac{1}{2}\left[\frac{SS_\pi}{\tau_\pi} + \frac{SS_\epsilon}{\tau_\epsilon}\right]\right\} \tag{3.3}$$

$$\log L_A\left(\tau_\pi, \tau_\epsilon \mid h\right)=-\frac{1}{2}\left\{[J(I-1)+2] \log \left(\tau_\pi\right)-J(I-1)(K-1) \log \left(\tau_\epsilon\right)+\frac{SS_\pi}{\tau_\pi}+\frac{SS_\epsilon}{\tau_\epsilon}\right\} \tag{3.4}$$

subject to the restrictions $\tau_\pi \geq \tau_\epsilon > 0$
The maximize $\log L_A$ for $\tau_\pi > 0$ and $\tau_\epsilon > 0$, but the solution can violate the constraint $\tau_\pi \geq \tau_\epsilon$. If we neglected the restriction $\tau_\pi \geq \tau_\epsilon$, we would have the following estimator:

$$\hat{\tau}_\pi=\frac{SS_\pi}{J(I-1)+2}, \quad \hat{\tau}_\epsilon=\frac{SS_\epsilon}{J(I-1)(K-1)} \tag{3.5}$$

Then, the corresponding estimator of $r=\frac{\tau_\epsilon}{\tau_\pi}$ would be

$$No.3 \hat{r}=\frac{SS_\epsilon}{SS_\pi} \frac{J(I-1)+2}{J(I-1)(K-1)} \tag{3.6}$$

The REML estimator of τ_π and τ_ϵ are defined to be those values of these parameters that maximize $\log L_B\left(\tau_\pi, \tau_\epsilon\right)$ subject to the restrictions $\tau_\pi \geq \tau_\epsilon > 0$

$$\log L_B\left(\tau_\pi, \tau_\epsilon\right)=-\frac{1}{2}\left\{[J(I-1)+1] \log \left(\tau_\pi\right)-J(I-1)(K-1) \log \left(\tau_\epsilon\right)+\frac{SS_\pi}{\tau_\pi}+\frac{SS_\epsilon}{\tau_\epsilon}\right\} \tag{3.7}$$

The maximize $\log L_B$ for $\tau_\pi > 0$ and $\tau_\epsilon > 0$, but the solution can violate the constraint $\tau_\pi \geq \tau_\epsilon$. If we neglected the restriction $\tau_\pi \geq \tau_\epsilon$, we would have the following estimator:

$$\hat{\tau}_\pi=\frac{SS_\pi}{J(I-1)+1}, \quad \hat{\tau}_\epsilon=\frac{SS_\epsilon}{J(I-1)(K-1)} \tag{3.8}$$

Then, the corresponding estimator of $r=\frac{\tau_\epsilon}{\tau_\pi}$ would be

$$No.4 \quad \hat{r}=\frac{SS_\epsilon}{SS_\pi} \frac{J(I-1)+1}{J(I-1)(K-1)} \tag{3.9}$$

For the modification REML technique of estimates of τ_π and τ_ϵ which depend on the Jeffreys non-information prior distribution, the Jeffreys prior distribution is:

$$\pi\left(\tau_\pi, \tau_\epsilon\right)=\left(\tau_\pi \tau_\epsilon\right)^{-1}\left(\tau_\pi \geq \tau_\epsilon > 0\right) \tag{3.10}$$

Therefore, the estimators of τ_π and τ_ϵ are obtained by maximizing the function

$$\log L_C\left(\tau_\pi, \tau_\epsilon\right)=-\frac{1}{2}\left\{[J(I-1)+6] \log \left(\tau_\pi\right)-[J(I-1)(K-1)+2] \log \left(\tau_\epsilon\right)+\frac{SS_\pi}{\tau_\pi}+\frac{SS_\epsilon}{\tau_\epsilon}\right\} \tag{3.11}$$

subject to the restrictions $(\tau_\pi \geq \tau_\epsilon > 0)$.
The maximize $\log L_C$ for τ_π and $\tau_\epsilon > 0$, but the solution can violate the constraint $\tau_\pi \geq \tau_\epsilon$. If we neglected the restriction $\tau_\pi \geq \tau_\epsilon$, we would have the following estimator:

$$\widehat{\tau_\pi}=\frac{SS_\pi}{J(I-1)+6}, \quad \hat{\tau}_\epsilon=\frac{SS_\epsilon}{J(I-1)(K-1)+2} \tag{3.12}$$

Then, the corresponding estimator of $r=\frac{\tau_\epsilon}{\tau_\pi}$ would be

$$No.5 \hat{r}=\frac{SS_\epsilon}{SS_\pi} \frac{J(I-1)+6}{J(I-1)(K-1)+2} \tag{3.13}$$

The three functions $\log L_A\left(\tau_\pi, \tau_\epsilon\right), \log L_B\left(\tau_\pi, \tau_\epsilon\right)$ and $\log L_C\left(\tau_\pi, \tau_\epsilon\right)$ are all of the form:

$$\log L=-\frac{1}{2}\left[\left(l_\pi+k_\pi\right) \log \left(\tau_\epsilon\right)+\left(l_\epsilon+k_\epsilon\right) \log \left(\tau_\pi\right)+\frac{SS_\pi}{\tau_\pi}+\frac{SS_\epsilon}{\tau_\epsilon}\right] \tag{3.14}$$

where $l_\pi = J(I-1)$ and $l_\epsilon = J(I-1)(K-1)$ are the degrees of freedom. The choices for k_π and k_ϵ that given $\log L_A(\tau_\pi,\tau_\epsilon), \log L_B(\tau_\pi,\tau_\epsilon)$ and $\log L_C(\tau_\pi,\tau_\epsilon)$ are:

$$\left.\begin{array}{l}\log L_A(\tau_\pi,\tau_\epsilon): k_\pi = 2, k_\epsilon = 0\\ \log L_B(\tau_\pi,\tau_\epsilon): k_\pi = 1, k_\epsilon = 0\\ \log L_C(\tau_\pi,\tau_\epsilon): k_\pi = 6, k_\epsilon = 2\end{array}\right\} \tag{3.15}$$

the corresponding estimators of $r = \frac{\tau_\epsilon}{\tau_\pi}$ would be
The parameter space limits are not taken into account by the estimators (3.5), (3.8) and (3.12) and are therefore not true ML, REML or modified REMLs. It can be seen that the maximum values of τ_π and τ_ϵ are subject to the constraints of ($\tau_\pi \geq \tau_\epsilon > 0$) are

$$\widehat{\tau}_\pi = \begin{cases} f_\pi, & \text{if } f_\pi \geq f_\epsilon \\ f_{\pi\epsilon}, & \text{if } f_\pi < f_\epsilon \end{cases} \tag{3.16}$$

and

$$\widehat{\tau}_\epsilon = \begin{cases} f_\epsilon, & \text{if } f_\pi \geq f_\epsilon \\ f_{\pi\epsilon}, & \text{if } f_\pi < f_\epsilon \end{cases} \tag{3.17}$$

where $f_{\pi\epsilon} = \frac{(g_\pi f_\pi + g_\epsilon f_\epsilon)}{g_\pi + g_\epsilon}$ and $g_i = l_i + k_i, (i = \pi, \epsilon)$.
The τ_π and τ_ϵ estimators are given by (26) and (27) result in the following r estimators:

$$\hat{r} = \min\left\{\frac{f_\epsilon}{f_\pi}, 1\right\} \tag{3.18}$$

Therefore, the ML, REML and modified REML trimmed-off estimators are:

$$\text{No. 6}\quad \hat{r} = \min\left\{\frac{SS_\epsilon}{SS_\pi}\frac{J(I-1)+2}{J(I-1)(K-1)}, 1\right\} \tag{3.19}$$

$$No.7\quad \hat{r} = \min\left\{\frac{SS_\epsilon}{SS_\pi}\frac{J(I-1)+1}{J(I-1)(K-1)}, 1\right\} \tag{3.20}$$

$$No.8\quad \hat{r} = \min\left\{\frac{SS_\epsilon}{SS_\pi}\frac{J(I-1)+6}{J(I-1)(K-1)+2}, 1\right\} \tag{3.21}$$

Notice that they are truncated forms, respectively, of estimators No.3, No.4 and No.5.
The corresponding estimators of $\theta, A_b, \pi_{a(b)}, B_c, (AB)_{bc}, \theta_{abc}$ and $H = \ell_0\theta + \sum_{b=1}^{J}\ell_b A_b$ $+ \sum_{a=1}^{I}\sum_{b=1}^{J}\ell_a\ell_b\pi_{a(b)} + \sum_{c=1}^{K}\ell_c B_c + \sum_{b=1}^{J}\sum_{c=1}^{k}\ell_b\ell_c(AB)_{bc}$ are given by [2]

$$\left.\begin{array}{l}\hat{\theta} = \bar{h}\\ \hat{\pi}_{a(b)} = (1-\hat{r})\left(\bar{h}_{ab.} - \bar{y}_{.b.}\right)\\ \hat{A}_b = \bar{h}_{.b.} - \bar{h}\\ \hat{B}_c = \bar{h}_{..c} - \bar{h}\\ (\widehat{AB})_{bc} = \bar{h}_{.bc} + \bar{h}_{...} - \bar{h}_{.b.} - \bar{h}_{.c}\\ \hat{\theta}_{abc} = \bar{h}_{.bc} + \hat{\pi}_{a(b)}\\ \widehat{H} = \ell_0\theta + \sum_{b=1}^{J}\ell_b\left(\bar{h}_{.b.} - \bar{h}_{...}\right) + \sum_{a=1}^{I}\sum_{b=1}^{J}\ell_b\ell_c\left[(1-\hat{r})\left(\bar{h}_{ab.} - \bar{y}_{.b.}\right)\right]\\ \quad + \sum_{c=1}^{K}\ell_c\left(\bar{h}_{.c} - \bar{h}_{...}\right) + \sum_{b=1}^{J}\sum_{c=1}^{K}\ell_b\ell_c\left(\bar{h}_{.bc} + \bar{h}_{...} - \bar{h}_{.b} - \bar{h}_{..c}\right)\end{array}\right\} \tag{3.22}$$

where $(a=1,\ldots,I; b=1,\ldots,J; c=1,\ldots,K)$, for estimators No.1-No.8.
There are three types of estimators of $\theta, A_b, \pi_{a(b)}, B_c, (AB)_{bc}, \theta_{abc}$ and H as follows:
Type 1: This type consists of estimators as follows:

$$\begin{aligned}
&\hat{\theta}_1 = \bar{h}_{....}\\
&\widehat{\pi}_{1;a(b)} = (1-\hat{r})\left(\bar{h}_{ab.} - \bar{y}_{.b.}\right)\\
&\hat{A_{1;b}} = \hat{A_{1,z;b}} = \bar{h}_{.b.} - \bar{h}_{...}\\
&\widehat{B}_{1;c} = \widehat{B}_{1,z;c} = \bar{h}_{.c} - \bar{h}_{....}
\end{aligned}$$

$$\begin{aligned}
&(\widehat{AB})_{1;bc} = (\widehat{AB})_{1,z;bc} = \bar{h}_{.bc} + \bar{h}_{...} - \bar{h}_{.b.} - \bar{h}_{..c}\\
&\hat{\theta}_{1;abc} = \hat{\theta}_{1,z;abc} = \bar{h}_{...} + \widehat{A}_{1,z;b} + \hat{\pi}_{1,z;a(b)} + \widehat{\mathrm{B}}_{1,z;c} + (\widehat{\mathrm{AB}})_{1,z;bc},\\
&\hat{\theta}_{1,z;abc} = \bar{h}_{.bc} + \widehat{\pi}_{1,z;a(b)}
\end{aligned}$$

and

$$\begin{aligned}
\widehat{H}_1 =& \ell_0\theta + \sum_{b=1}^{J}\ell_b\left(\bar{h}_{.b.} - \bar{h}_{...}\right) + \sum_{a=1}^{I}\sum_{b=1}^{J}\ell_b\ell_c\left[(1-\hat{r})\left(\bar{h}_{ab.} - \bar{y}_{.b.}\right)\right]\\
&+ \sum_{c=1}^{K}\ell_c\left(\bar{h}_{.c} - \bar{h}_{....}\right) + \sum_{b=1}^{J}\sum_{c=1}^{K}\ell_b\ell_c\left(\bar{h}_{.bc} + \bar{h}_{...} - \bar{h}_{.b.} - \bar{h}_{.c}\right)
\end{aligned}$$

with

$$\hat{r}_1 = \hat{r}_{1,z} = z\frac{SS_\epsilon}{SS_\pi}$$

where $(a=1,\ldots,I; b=1,\ldots,J; c=1,\ldots,K)$ and z is an arbitrary positive constant. Let $\widehat{\boldsymbol{\theta}}_{\mathbf{1}} = \widehat{\boldsymbol{\theta}}_{\mathbf{1,z}}$ denote the vector of dimensions $n\times 1$ whose ath component is $\hat{\theta}_{1,z;a}$.
Type 1 estimators will be called untruncated estimators. This type contains No.1, No.3, No.4, No.5and No.7.
Tvpe 2: This type consists of estimators as follows:

$$\begin{aligned}
&\hat{\theta}_2 = \bar{h}_{....}\\
&\hat{\pi}_{2;a(b)} = (1-\hat{r})\left(\bar{h}_{ab.} - \bar{y}_{.b.}\right),\\
&\hat{A}_{2;b} = \hat{A}_{2,z;b} = \bar{h}_{.b.} - \bar{h}_{....}\\
&\hat{B}_{2;c} = \hat{B}_{2,z;c} = \bar{h}_{.c} - \bar{h}_{...},\\
&(\widehat{AB})_{2;bc} = (\widehat{AB})_{2,z;bc} = \bar{h}_{.bc} + \bar{h}_{...} - \bar{h}_{.b.} - \bar{h}_{..c},\\
&\hat{\theta}_{2;abc} = \hat{\theta}_{2,z;abc} = \bar{h}_{...} + \widehat{A}_{2,z;b} + \hat{\pi}_{2,z;a(b)} + \widehat{\mathrm{B}}_{2,z;c} + (\widehat{\mathrm{AB}})_{2,z;bc},\\
&\hat{\theta}_{2,z;abc} = \bar{h}_{.bc} + \hat{\pi}_{2,z;a(b)},
\end{aligned}$$

and

$$\begin{aligned}
\widehat{H}_1 =& \ell_0\theta + \sum_{b=1}^{J}\ell_b\left(\bar{h}_{.b.} - \bar{h}_{...}\right) + \sum_{a=1}^{I}\sum_{b=1}^{J}\ell_b\ell_c\left[(1-\hat{r})\left(\bar{h}_{ab.} - \bar{y}_{.b.}\right)\right]\\
&+ \sum_{c=1}^{K}\ell_c\left(\bar{h}_{.c} - \bar{h}_{...}\right) + \sum_{b=1}^{J}\sum_{c=1}^{K}\ell_b\ell_c\left(\bar{h}_{.bc} + \bar{h}_{...} - \bar{h}_{.b.} - \bar{h}_{..c}\right)
\end{aligned}$$

with

$$\hat{r}_2 = \hat{r}_{2,z} = \min\left\{z\frac{SS_\epsilon}{SS_\pi}, 1\right\}$$

where $(a = 1, \ldots, I; b = 1, \ldots, J; c = 1, \ldots, K)$ and z is an arbitrary positive constant. Let $\widehat{\boldsymbol{\theta}}_2 = \widehat{\boldsymbol{\theta}}_{2,\mathrm{z}}$ denote the vector of dimensions $n \times 1$ whose ath component is $\hat{\theta}_{2,z;a}$.
Type 2 estimators will be called truncated estimators. This type contains No.2, No.6, No. 7 and No.8.
Type 3: This type consists of estimators as follows:

$$\begin{aligned}
&\hat{\theta}_3 = \bar{h}\\
&\hat{\pi}_{3;a(b)} = (1-\hat{r})\left(\bar{h}_{ab.} - \bar{y}_{.b.}\right),\\
&\hat{A}_{3;b} = \hat{A}_{3,z;b} = \bar{h}_{.b.} - \bar{h}_{....}\\
&\hat{B}_{3;c} = \hat{B}_{3,z;c} = \bar{h}_{.c} - \bar{h}_{...}\\
&(\widehat{AB})_{3;bc} = (\widehat{AB})_{3,z;bc} = \bar{h}_{.bc} + \bar{h}_{...} - \bar{h}_{.b.} - \bar{h}_{..c}\\
&\hat{\theta}_{3;abc} = \hat{\theta}_{3,z;abc} = \bar{h}_{.....} + \widehat{A}_{3,z;b} + \hat{\pi}_{3,z;a(b)} + \widehat{\mathrm{B}}_{3,z;c} + (\widehat{\mathrm{AB}})_{3,z;bc},\\
&\hat{\theta}_{3,z;abc} = \bar{h}_{.bc} + \hat{\pi}_{3,z;a(b)}
\end{aligned}$$

and

$$\begin{aligned}
\widehat{H}_3 = \ell_0\theta &+ \sum_{b=1}^{J}\ell_b\left(\bar{h}_{.b.} - \bar{h}_{...}\right) + \sum_{a=1}^{I}\sum_{b=1}^{J}\ell_b\ell_c\left[(1-\hat{r})\left(\bar{h}_{ab.} - \bar{y}_{.b.}\right)\right]\\
&+ \sum_{c=1}^{K}\ell_c\left(\bar{h}_{.c} - \bar{h}_{.....}\right) + \sum_{b=1}^{J}\sum_{c=1}^{K}\ell_b\ell_c\left(\bar{h}_{.bc} + \bar{h}_{...} - \bar{h}_{.b.} - \bar{h}_{.c}\right)
\end{aligned}$$

with

$$\hat{r}_3 = \hat{r}_{3,z} = \mathrm{f}_3\left(SS_\pi, SS_\epsilon\right)$$

where $(a = 1, \ldots, I; b = 1, \ldots, J; c = 1, \ldots, K)$ and $\mathrm{f}_3(x, y)$ is an arbitrary function of $x, y > 0$. Let $\widehat{\theta}_3 = \widehat{\theta}_{3,z}$ denote the vector of dimensions $n \times 1$ whose ath component is $\hat{\theta}_{3,z;a}$
This type contains No.1, No.2, No.3, No.4, No.5, No.6.No.7, No.8.

Summary and Classification of Estimators

A full list of the estimators viewed in this paper is as follows:

$$\begin{aligned}
&\text{No.1 } \hat{r} = \frac{MS_\epsilon}{MS_\pi} = \frac{SS_\epsilon}{SS_\pi}\frac{1}{(K-1)}, \quad (K > 1)\\
&\text{No.2 } \hat{r} = \min\left\{\frac{SS_\epsilon}{SS_\pi}\frac{1}{(K-1)}, 1\right\}\\
&No.3 \ \hat{r} = \frac{SS_\epsilon}{SS_\pi}\frac{J(I-1)+2}{J(I-1)(K-1)}, \quad (K > 1 \text{ and } I > 1).\\
&\text{No.4 } \hat{r} = \frac{SS_\epsilon}{SS_\pi}\frac{J(I-1)+1}{J(I-1)(K-1)}, \quad (K > 1 \text{ and } I > 1).\\
&\text{No.5 } \hat{r} = \frac{SS_\epsilon}{SS_\pi}\frac{J(I-1)+6}{J(I-1)(K-1)+2},
\end{aligned}$$

$$\text{No.6 } \hat{r} = \min\left\{\frac{SS_\epsilon}{SS_\pi}\frac{J(I-1)+2}{J(I-1)(K-1)}, 1\right\}$$
$$\text{No.7 } \hat{r} = \min\left\{\frac{SS_\epsilon}{SS_\pi}\frac{J(I-1)+1}{J(I-1)(K-1)}, 1\right\},$$
$$\text{No.8 } \hat{r} = \min\left\{\frac{SS_\epsilon}{SS_\pi}\frac{J(I-1)+6}{J(I-1)(K-1)+2}, 1\right\}$$

These estimators were classified into three types: 1,2 and 3.

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