Int. J. Nonlinear Anal. Appl. 12 (2021) No. 2, 85–97
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2020.20826.2203

On the Cauchy dual and complex symmetric of composition operators

Morteza Sohrabi

[a] Department of Mathematics, Lorestan University, Khorramabad, Iran

(Communicated by Madjid Eshaghi Gordji)

Abstract

In this paper, firstly we show that some classical properties for Cauchy dual and Moore-Penrose inverse of composition operators, such as complex symmetric and Aluthge transform on $L^2(\Sigma)$. Secondly we give a characterization for some operator classes of weak p-hyponormal via Moore-Penrose inverse of composition operators. Finally, some examples are then presented to illustrate that, the Moore-Penrose inverse of composition operators lie between these classes.

Keywords: Cauchy dual, Moore-Penrose inverse, polar decomposition, Aluthge transform, complex symmetric, p-hyponormal, p-paranormal.
2010 MSC: Primary 47B20; Secondary 47B38.

1. Introduction

Let (X, Σ, μ) be a complete σ-finite measure space. For any sub-σ-finite algebra $\mathcal{A} \subset \Sigma$, the L^2-space $L^2(X, \mathcal{A}, \mu_{|\mathcal{A}})$ is abbreviated by $L^2(\mathcal{A})$, and its norm is denoted by $\|.\|_2$. All comparisons between two functions or two sets are to be interpreted as holding up to a μ-null set. The support of a measurable function f is defined by $\sigma(f) = \{x \in X : f(x) \neq 0\}$. We denote the linear space of all complex-valued Σ-measurable functions on X by $f \in L^0(\Sigma)$. Let φ be a non-singular measurable transformation from X into X; that is, $\mu \circ \varphi^{-1}$ is absolutely continuous with respect to μ and write $\mu \circ \varphi^{-1} \ll \mu$. Let h be the Radon-Nikodym derivative $d\mu \circ \varphi^{-1}/d\mu$. The pair (X, Σ) is said to be normal invariant if $\varphi(\Sigma) \subset \Sigma$ and $\mu \ll \mu \circ \varphi^{-1}$. The composition operator $C_\varphi : L^2(\Sigma) \to L^0(\Sigma)$ induced by φ is given by $C_\varphi(f) = f \circ \varphi$, for each $f \in L^2(\Sigma)$. Here, the non-singularity of φ guarantees that C_φ is well defined. It is well known fact that for $u \in L^0(\Sigma)$, the multiplication

*Corresponding author
Email address: sohrabi.mo@lu.ac.ir (Morteza Sohrabi)

Received: July 2020 *Accepted:* August 2020

operator $M_u : L^2(\Sigma) \to L^0(\Sigma)$ is bounded if and only if $u \in L^\infty(\Sigma)$, and in this case, $\|M_u\| = \|u\|_\infty$. Now, by the change of variables formula; $\int_X |f \circ \varphi|^2 d\mu = \int_X h|f|^2 d\mu$, $\|C_\varphi f\|_2 = \|M_{\sqrt{h}} f\|_2$ for each $f \in L^2(\Sigma)$. It follows that C_φ maps $L^2(\Sigma)$ boundedly into itself, if and only if $h \in L^\infty(\Sigma)$, and in this case, $\|C_\varphi\| = \|h\|_\infty^{\frac{1}{2}}$. Some other basic facts about composition operators can be found in [17, 28, 30].

For each $f \in L^2(\Sigma)$ there is a unique function in $L^2(\mathcal{A})$, denoted $E^{\mathcal{A}}(f)$, such that, for every set $A \in \mathcal{A}$ of finite measure, $\int_A f d\mu = \int_A E(f) d\mu$. $E^{\mathcal{A}}(f)$ is called the conditional expectation of f with respect to $\mathcal{A}$, and $E^{\mathcal{A}}$ is the conditional expectation operator. As an operator on $L^2(\Sigma)$, $E^{\mathcal{A}}$ is the contractive orthogonal projection onto $L^2(\mathcal{A})$. Take $\mathcal{A} = \varphi^{-1}(\Sigma)$. So for each function f in $L^2(\Sigma)$ there is a Σ-measurable function F such that $E^{\varphi^{-1}(\Sigma)} f = F \circ \varphi$. Moreover, F is uniquely determined in $\sigma(h)$ (see [7]). Therefore, even though φ is not invertible the expression $F = (E^{\varphi^{-1}(\Sigma)} f) \circ \varphi^{-1}$ is well defined. Note that domain of $E^{\mathcal{A}}$ contains $L^2(\Sigma) \cup \{f \in Ł^0(\Sigma) : f \geq 0\}$. For further discussion of the conditional expectation operator see [23] and [26]. A result, Lambert and Hoover [19] shows that the adjoint ${C_\varphi}^*$ of C_φ on $L^2(\Sigma)$ is given by ${C_\varphi}^*(f) = hE^{\varphi^{-1}(\Sigma)}(f) \circ \varphi^{-1}$. From this it easily follows that ${C_\varphi}^* C_\varphi = M_h$ and $C_\varphi {C_\varphi}^* = M_{h\circ\varphi} E^{\varphi^{-1}(\Sigma)}$. The product $M_u \circ \varphi$ of M_u and C_φ is called a weighted composition operator,denoted by W, with

$$\|Wf\|_2 = \|\sqrt{hE(|u|^2) \circ \varphi^{-1}} f\|_2.$$

Put $J = hE(|u|^2) \circ \varphi^{-1}$. It follows that W is bounded on $L^2(\Sigma)$ if and only if $J \in L^\infty(\Sigma)$ (see [19] and also [7] for a discussion of $E(\cdot) \circ \varphi^{-1}$ when φ is not invertible). The role of conditional expectation operator is important in this note. We shall frequently use the following general properties of $E^{\mathcal{A}}$ and C_φ acting on $L^2(\mathcal{F})$. The proofs of these facts and some related discussions may be found in [18, 23, 26]

P(1) If f is an $E^{\mathcal{A}}$-measurable function, then $E^{\mathcal{A}}(fg) = fE^{\mathcal{A}}(g)$;
P(2) If $f \geq 0$ then $E^{\mathcal{A}}(f) \geq 0$; If $f > 0$ then $E^{\mathcal{A}}(f) > 0$;
P(3) $\sigma(f) \subseteq \sigma(E^{\mathcal{A}}(f))$, for each nonnegative $f \in L^2(\Sigma)$;
P(4) $E^{\mathcal{A}}(|f|^2) = |E^{\mathcal{A}}(f)|^2$ if and only if $f \in L(\mathcal{A})$;
P(5) $\varphi^{-1}(\sigma(h)) = X$, i.e., $h \circ \varphi > 0$;
P(6) $\int_{\varphi^{-1}A} gf \circ \varphi d\mu = \int_A hE^{\varphi^{-1}(\Sigma)}(g \circ \varphi^{-1}) f d\mu$, for all $g \in L^2(\Sigma), A \in \Sigma$;
P(7) $W^* f = hE^{\varphi^{-1}(\Sigma)}(uf) \circ \varphi^{-1}$;
P(8) $W^*Wf = hE^{\varphi^{-1}(\Sigma)}(u^2) \circ \varphi^{-1} f$;
P(9) $WW^*f = u(h \circ \varphi) E^{\varphi^{-1}(\Sigma)}(uf)$;
P(10) $E^{\varphi^{-1}(\mathcal{A})}(L^2(\mathcal{A})) = \overline{C_\varphi(L^2(\mathcal{A}))} = \{f \in L^2(\mathcal{A}) : f \;\; is \;\; \varphi^{-1}(\mathcal{A}) - \text{measurable}\}$.

Given a complex separable Hilbert space $\mathcal{H}$, let $\mathcal{B}(\mathcal{H})$ denotes the linear space of all bounded linear operators on $\mathcal{H}$. Recall that $T \in \mathcal{B}(\mathcal{H})$ has a generalized inverse if there exists an operator $S \in \mathcal{B}(\mathcal{H})$ for which $TST = T$. It is well known that $T \in \mathcal{B}(\mathcal{H})$ has a generalized inverse if and only if $\mathcal{R}(\mathcal{T})$ is closed (see [10]). In general, S is not unique. The generalized inverse S is called the Moore-Penrose inverse of T if $STS = S$ and the idempotents TS and ST are self-adjoint. In this case, S is unique and it is denoted by $T^\dagger$. Note that if $U|T|$ is the polar decomposition of T , then by definition, U^* is a generalized inverse of U and hence has closed range. Also, since $\mathcal{R}(U^*) = \mathcal{N}(U)^\perp$, U is isometry on $\mathcal{R}(U^*)$. Associated with $T \in \mathcal{B}(\mathcal{H})$ there is a useful related operator $\widetilde{T} = |T|^{1/2} U |T|^{1/2}$, called the Aluthge transform of T, first time it was studied in [1]. It is easy to check that $U^*|T^*|^\dagger$ and $|T^\dagger|^{\frac{1}{2}} U^* |T^\dagger|^{\frac{1}{2}}$ are the polar decomposition and Aluthge transform of T, respectively[33]. The Moore-Penrose inverse is designed as a measure for the invertibility of an operator. If $T = U|T|$ is invertible, then $T^{-1} = T^\dagger$, U is unitary and so $|T| = (T^*T)^{1/2}$ is invertible.

From now on, we assume that $CR(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$ with closed range. For other important properties of $T^\dagger$ see([2, 3, 10, 14, 15, 22]).

To avoid tedious calculations we consider only the composition case. Suppose that $C_\varphi \in B(L^2(\Sigma))$ has closed range. Then h is bounded away from zero on $\sigma(h)$. Put $S = M_{\frac{\chi_{\sigma(h)}}{h}} C_\varphi^*$. Then $S \in B(L^2(\Sigma))$. Since $\sigma(h \circ \varphi) = X$, ${C_\varphi}^* C_\varphi = M_h$ and $C_\varphi {C_\varphi}^* = M_{h\circ\varphi} E^{\varphi^{-1}(\Sigma)} E$, we have $C_\varphi S C_\varphi = C_\varphi$ and $S C_\varphi S = S$. Also it is easy to check that $C_\varphi S = E$ and $S C_\varphi = M_{\chi_{\sigma(h)}} = (S C_\varphi)^*$. Hence, S is the Moore-Penrose inverse of C_φ. Also, it is easy to check that

$$(C_\varphi^\dagger)^* = C_\varphi M_{\frac{\chi_{\sigma(h)}}{h}} = M_{\frac{1}{h\circ\varphi}} C_\varphi,$$

$$(C_\varphi^\dagger)^* C_\varphi^\dagger = M_{\frac{1}{h\circ\varphi}} E = (M_{\frac{1}{\sqrt{h\circ\varphi}}} E)^2,$$

$$C_\varphi^\dagger (C_\varphi^\dagger)^* = M_{\frac{\chi_{\sigma(h)}}{h}}.$$

Composition operators as an extension of shift operators are a good tool for separating weak hyponormal classes. Classic seminormal (weighted) composition operators have been extensively studied by Harrington and Whitley [17], Lambert [19, 23], Singh [28], Campbell [7, 8, 9] and Stochel [13]. In [5] and [6] some weak hyponormal classes of composition operators are studied. In those works, examples were given which show that composition operators can be used to separate each partial normality class from quasinormal through w-hyponormal. In [20] some examples were presented to illustrate that composition operators lie between those classes. This note is a continuation of the work done in [20]. The plan of this note is to present some characterizations of weak p-hyponormal and weak p-paranormal classes of weighted composition operators on $L^2(\Sigma)$. We then give specific examples illustrating these classes.

The Cauchy dual of left invertible operators is introduced in [25] as a powerful tool in the model theory of left-invertible operators. To be precise, if T is left invertible, it easy to see that T^*T is invertible and the operator given by $L_T := (T^*T)^{-1}T^*$ is a canonical left inverse of T. The Cauchy dual of T is then defined as

$$\omega(T) := T(T^*T)^{-1} = L_T^*,$$

which is a right inverse of T^*. For more details on the properties of Cauchy dual see [11, 25, 29]. We introduce now the notion of Cauchy dual for Moore−Penrose inverse.

Definition 1.1. *Let $T \in CR(L^2(\Sigma))$. The Cauchy dual T is is defined as*

$$\omega(T) = T(T^*T)^\dagger.$$

This article has been organized in two sections. In section 2, we study Cauchy dual of operators with closed range. We give several basic properties such as complex symmetry of these types of operators with a special conjugation. In section 3, we provide necessary and sufficient conditions for composition operators to be weak p-hyponormal and weak p-paranormal. Finally, some specific examples is provided to illustrate the obtained results.

2. Cauchy dual and Complex symmetric

The main goal of this section is to study the Cauchy dual of composition operators. Also we investigate the complex symmetric for Cauchy dual of composition operators with a special conjugation. We start with the following results that extend the case of left invertible operators and are easy to obtain.

Proposition 2.1. *Let $C_\varphi \in CR(L^2(\Sigma))$. we have*

(a) $\omega(C_\varphi) = \frac{\chi_{\sigma(h\circ\varphi)}}{h\circ\varphi} C_\varphi = (C_\varphi^\dagger)^*$.

(b) $\omega(C_\varphi^*) = \frac{\chi_{\sigma(J)}}{J} C_\varphi^* = C_\varphi^\dagger = (\omega(C_\varphi))^*$.

(c) $\omega(C_\varphi)^*\omega(C_\varphi) = \frac{\chi_{\sigma(h)}}{h}$, $\omega(C_\varphi)\omega(C_\varphi)^* = \frac{\chi_{\sigma(h\circ\varphi)}}{h\circ\varphi} E$,

(d) $\omega(\omega(C_\varphi)) = C_\varphi$.

(e) $\omega(C_\varphi^\dagger) = \omega(C_\varphi)^\dagger$.

Proof .(a) we know that $(T^*T)^\dagger = T^\dagger T^{*\dagger}$. So, for each $f \in L^2(\Sigma)$, we have

$$\omega(C_\varphi)f = C_\varphi C_\varphi^\dagger C_\varphi^{*\dagger} f = \frac{\chi_{\sigma(h\circ\varphi)}}{h\circ\varphi} C_\varphi f = (C_\varphi^\dagger)^* f.$$

(b) We have

$$\omega(C_\varphi^*)f = C_\varphi^* C_\varphi^{*\dagger} C_\varphi^\dagger f = C_\varphi^*(\frac{\chi_{\sigma(h\circ\varphi)}}{(h\circ\varphi)} E(f))$$

$$= \frac{\chi_{\sigma(h)}}{h} C_\varphi^* f = (\omega(C_\varphi))^* f.$$

(c)

$$\omega(C_\varphi)^*\omega(C_\varphi)f = \frac{\chi_{\sigma(h)}}{h} C_\varphi^\dagger(\frac{\chi_{\sigma(h\circ\varphi)}}{h\circ\varphi} C_\varphi f) = \frac{\chi_{\sigma(h)}}{h} f$$

and

$$\omega(C_\varphi)\omega(C_\varphi)^* f = \frac{\chi_{\sigma(h\circ\varphi)}}{h\circ\varphi} C_\varphi(\frac{\chi_{\sigma(h)}}{h} C_\varphi^\dagger f) = \frac{\chi_{\sigma(h\circ\varphi)}}{h\circ\varphi} E(f).$$

(d) $\omega(\omega(C_\varphi)) = \omega((C_\varphi^\dagger)^*) = C_\varphi f$.
(e) By direct computations, we get that

$$\omega(C_\varphi^\dagger) f = C_\varphi^\dagger C_\varphi C_\varphi^* f = hE(f)\circ\varphi^{-1} = C_\varphi^* f = (\omega(C_\varphi))^\dagger f.$$

□

Proposition 2.2. *Let $C_\varphi^\dagger \in CR(L^2(\Sigma))$. Then $\omega(C_\varphi^\dagger) = V|\omega(C_\varphi^\dagger)|$ is the polar decomposition of $\omega(C_\varphi^\dagger)$, such that*

$$|\omega(C_\varphi^\dagger)|(f) = \frac{1}{\sqrt{h}}(f);$$
$$V(f) = \frac{\chi_{\sigma(h\circ\varphi)}}{\sqrt{h\circ\varphi}}C_\varphi^\dagger(f).$$

We know that $\widetilde{C_\varphi^\dagger} f = \sqrt[4]{\frac{h}{h\circ\varphi}} f\circ\varphi$. Now turn to the computation of $\omega(\widetilde{C_\varphi^\dagger})$ and $\widetilde{\omega(C_\varphi^\dagger)}$. By combining the previous results we obtain the following proposition.

Proposition 2.3. *Let $C_\varphi^\dagger \in CR(L^2(\Sigma))$. Then*

(a) $\widetilde{\omega(C_\varphi^\dagger)} = \frac{\chi_h}{\sqrt[4]{h(h\circ\varphi)^3}}C_\varphi$.

(b) $\omega(\widetilde{C_\varphi^\dagger}) = \frac{\chi_{\sigma(E(h))}}{h\circ\varphi E(\sqrt{h})}\sqrt[4]{h(h\circ\varphi)}C_\varphi$.

Corollary 2.4. *Let $C_\varphi^\dagger \in CR(L^2(\Sigma))$. Then $\widetilde{\omega(C_\varphi^\dagger)} = \omega(\widetilde{C_\varphi^\dagger})$ if and only if $E(\sqrt{h}) = \sqrt{h}$.*

A conjugation on a Hilbert space $\mathcal{H}$ is an anti-linear operator $S : \mathcal{H} \to \mathcal{H}$ which satisfies $\langle Sx, Sy\rangle = \langle y, x\rangle$ for all $x, y \in \mathcal{H}$ and $S^2 = I$. An operator $T \in \mathcal{H}$ is said to be complex symmetric if there exists a conjugation S on $\mathcal{H}$ such that $T = ST^*S$. The class of complex symmetric operators is unexpectedly large. We refer the reader to [16, 21] for more details, including historical comments and references. In the following, we show that the Cauchy dual of composition operators on $L^2(\Sigma)$ is complex symmetric.

Proposition 2.5. *Let $\sigma(h) = X$, $\varphi^2 = I$, the identify transformation. If $h(h\circ\varphi) = 1$, where $\mathcal{A} = \varphi^{-1}(\Sigma)$. Then $\omega(C_\varphi)$ is complex symmetric.*

Proof . Define $S(f) = \frac{\bar{f}\circ\varphi}{\sqrt{h\circ\varphi}}$. Then S is conjugate linear, $S^2 = I$ and for each $f \in L^2(\Sigma)$, we have

$$S\omega(C_\varphi)^*S(f) = S(C_\varphi^\dagger)S(f) = S(\frac{\chi_{\sigma(h)}}{h}hE(S(f))\circ\varphi^{-1}) = \frac{\sqrt{h\circ\varphi}}{h\circ\varphi}S(\bar{f})$$

$$= \frac{f\circ\varphi}{h\circ\varphi} = \omega(C_\varphi)f.$$

Also

$$\langle Sf, Sg\rangle = \int_X \frac{(\bar{f}\circ\varphi)(g\circ\varphi)}{h\circ\varphi}d\mu = \int_X \frac{h\bar{f}g}{h}d\mu = \langle g, f\rangle.$$

So, $\omega(C_\varphi)$ is complex symmetric. □

Corollary 2.6. *Let $\sigma(h) = X$, $\varphi^2 = I$, the identify transformation. If $h(h\circ\varphi) = 1$, where $\mathcal{A} = \varphi^{-1}(\Sigma)$. Then C_φ is complex symmetric.*

Example 2.7. *Suppose that $1 < a < \infty$. Let $X = [\frac{1}{a}, a]$, $d\mu = dx$ and Σ be the Lebesgue sets. Define the non-singular transformation $\varphi : X \to X$ by $\varphi(x) = \frac{1}{x}$. Put $\mathcal{A} = \varphi^{-1}(\Sigma)$. Then $h(x) = \frac{1}{x^2}$ and $E = I$. Simple computations show that $\sigma(h) = X$, $\varphi^2 = I$ and $h(h \circ \varphi) = 1$. Define $S(f) = \frac{\bar{f} \circ \varphi}{\sqrt{h \circ \varphi}}$. It is clear that S is conjugate linear. By direct computation we get that $\omega(C_\varphi) = \frac{1}{x^2} f(\frac{1}{x})$ and*

$$S\omega(C_\varphi)^* S(f) = \omega(C_\varphi) f.$$

Thus $\omega(C_\varphi)$ is complex symmetric operator on $L^2(\Sigma)$.

Lemma 2.8. *[20] Let $f \in L^2(\Sigma)$, and $Af := u(h \circ \varphi)E(uf)$. Then for all $p \in (0, \infty)$,*

$$A^p f = u(h^p \circ \varphi)(E(u^2))^{p-1} E(uf).$$

Lemma 2.9. *Let $C_\varphi \in CR(L^2(\Sigma))$. Then $C_\varphi^\dagger = V|C_\varphi^\dagger|$ is the polar decomposition of $C_\varphi^\dagger$. Such that for each $f \in L^2(\Sigma)$,*

$$|C_\varphi^\dagger|(f) = \frac{E(f)}{\sqrt{h \circ \varphi}}, \qquad V(f) = \sqrt{h} E(f) \circ \varphi^{-1}.$$

Proof . Let $f \in L^2(\Sigma)$. Then $(C_\varphi^\dagger)^* C_\varphi^\dagger f = M_{\frac{1}{h \circ \varphi}} E(f)$. Now $|C_\varphi^\dagger|$ follows from Lemma 2.8. Moreover, by a direct calculation we get that

$$V(f) = \sqrt{h} E(f) \circ \varphi^{-1}.$$

Moreover, it is easy to check that $V|C_\varphi^\dagger| = C_\varphi^\dagger$, $VV^*V = V$ and $\mathcal{N}(V) = \mathcal{N}(C_\varphi^{\dagger *}) = \mathcal{N}(C_\varphi^\dagger)$. This completes the proof. □

In the following, we will obtain the Aluthge transformation of C_φ and $C_\varphi^\dagger$. Recall that the Aluthge transformation of C_φ is defined by $\widetilde{C_\varphi} = |C_\varphi|^{\frac{1}{2}} U |C_\varphi|^{\frac{1}{2}}$. Let $U|C_\varphi|$ and $C_\varphi^\dagger = V|C_\varphi^\dagger|$ be the polar decompositions of C_φ and $C_\varphi^\dagger$ respectively. Since $C_\varphi^*(f) = hE(f) \circ \varphi^{-1}$, we obtain $|C_\varphi|(f) = \sqrt{h} f$ and $U(f) = \frac{f \circ \varphi}{\sqrt{h \circ \varphi}}$. It follows that

$$\widetilde{C_\varphi} f = (\frac{h}{h \circ \varphi})^{\frac{1}{4}} C_\varphi f, \qquad f \in L^2(\Sigma).$$

We now turn to the computation of $(\widetilde{C_\varphi})^\dagger$ and $\widetilde{C_\varphi^\dagger}$. Now, let $C_\varphi \in CR(L^2(\Sigma))$. Put

$$Pf = \sqrt[4]{h} E(\frac{f}{\sqrt[4]{h}}) \circ \varphi^{-1}, \qquad f \in L^2(\Sigma).$$

Then it is easy to check that P satisfy all conditions of the Moore-Penrose inverse. Thus

$$P = (\widetilde{C_\varphi})^\dagger.$$

Now, by using Lemma 2.8 and 2.9, we obtain

$$\widetilde{C_\varphi^\dagger} f = |C_\varphi^\dagger|^{\frac{1}{2}} V |C_\varphi^\dagger|^{\frac{1}{2}} = \frac{1}{\sqrt[4]{h \circ \varphi}} E(\sqrt[4]{h} E(f) \circ \varphi^{-1}).$$

These observations establish the following proposition.

Proposition 2.10. *Let $C_\varphi \in CR(L^2(\Sigma))$. Then the following assertions hold.*

(i) $\widetilde{C_\varphi} f = (\frac{h}{h\circ\varphi})^{\frac{1}{4}} C_\varphi f$.

(ii) Let $U_\varphi|\widetilde{C_\varphi}|$ *be the polar decomposition of* $\widetilde{C_\varphi}$. *Then*

$$|\widetilde{C_\varphi}|\ (f) = \sqrt{hE(\frac{h}{h\circ\varphi})^{\frac{1}{2}} \circ \varphi^{-1} f};$$

$$U_\varphi(f) = \frac{\sqrt[4]{h}}{\sqrt{h\circ\varphi E(\sqrt{h})}} f\circ\varphi.$$

(iii) If $\widetilde{C_\varphi} \in CR(L^2(\Sigma))$, *then* $(\widetilde{C_\varphi})^\dagger f = \sqrt[4]{h} E(\frac{f}{\sqrt[4]{h}}) \circ \varphi^{-1}$.

(iv) $\widetilde{C^\dagger_\varphi} f = \frac{1}{\sqrt[4]{h\circ\varphi}} E(\sqrt[4]{h} E(f) \circ \varphi^{-1})$.

3. On Some Characterization of $C^\dagger_\varphi$

In [20] Jabbarzadeh and Azimi determined when composition operators were p-quasihyponormal, p-paranormal and absolute p-paranormal. But in some cases composition operators can not be separated some of these classes. In the following, we will show that Moore-Penrose inverse of composition operators, $C^\dagger_\varphi$, can be used to separate each partial normality class from quasinormal through w-hyponormal. An operator $T \in B(H)$ is p-quasihyponormal if for each $p \in (0,\infty)$, $T^*(T^*T)^pT \geq T^*(TT^*)^pT$. For all $x \in \mathcal{H}$, if $\||T|^pU|T|^px\| \geq \||T|^px\|^2$, then T is called a p-paranormal operator and if $\||T|^pTx\| \geq \|Tx\|^{p+1}$, then T is called a absolute p-paranormal operator, and T is p-$*$-paranormal operator if $\||T|^pU^*|T|^px\|\|x\| \geq \||T^*|^px\|^2$,$T$ is called a absolute p-$*$-paranormal operator if $\||T|^pTx\| \geq \|T^*x\|^{p+1}$ [4, 12, 27].

Lemma 3.1. *[17] The following are equivalent:*
(i) $\Sigma_{\sigma(h)} \subseteq \varphi^{-1}(\Sigma)$,

(ii) $ker(C^*_\varphi) \subseteq ker(C_\varphi)$.

Theorem 3.2. *[17] The adjoint* C^*_φ *is quasinormal if and only if* $\Sigma_{\sigma(h)} \subseteq \varphi^{-1}(\Sigma)$ *and* $h = h\circ\varphi$ *a.e., on* $\sigma(h)$.

Theorem 3.3. *Let* $C_\varphi \in CR(L^2(\Sigma))$. *Then* $C^\dagger_\varphi$ *is quasinormal if and only if* $\Sigma_{\sigma(h)} \subseteq \varphi^{-1}(\Sigma)$ *and* $h = h\circ\varphi$ *a.e., on* $\sigma(h)$.

Proof . By definition, $C^\dagger_\varphi$ is quasinormal if and only if $C^\dagger_\varphi(C^\dagger_\varphi)^*C^\dagger_\varphi(f) = (C^\dagger_\varphi)^*C^\dagger_\varphi C^\dagger_\varphi(f)$, for all f. By a direct calculation we have

$$\frac{1}{h}(C^\dagger_\varphi f) = \frac{1}{(h\circ\varphi)} E(C^\dagger_\varphi f) \quad \text{on } \sigma(h)$$

$$\Rightarrow \frac{1}{h^2} C^*_\varphi f = \frac{1}{(h\circ\varphi)} E(\frac{1}{h} C^*_\varphi f) \quad \text{on } \sigma(h).$$

Since $ker(E) = \mathcal{R}(C_\varphi)^\perp = ker(C_\varphi^*)$, this becomes

$$\frac{1}{h^2}C_\varphi^* E(f) = \frac{1}{(h\circ\varphi)}E(\frac{1}{h}C_\varphi^* E(f)) \quad \text{on } \sigma(h). \tag{3.1}$$

Now suppose that $\Sigma_{\sigma(h)} \subseteq \varphi^{-1}(\Sigma)$ and $h = h\circ\varphi$ *a.e.*, on $\sigma(h)$, then by Lemma 3.1, for $g \in \mathcal{R}(C_\varphi)^\perp$, $hg = C_\varphi^* C_\varphi f = 0$, and (3.1) reduce to

$$\frac{1}{h^2}E(C_\varphi^* E(f)) = M_{\frac{1}{(h\circ\varphi)}}E(\frac{1}{h}C_\varphi^* E(f)) \quad \text{on } \sigma(h). \tag{3.2}$$

It follow that $C_\varphi^\dagger$ is quasinormal.
Conversely, suppose that $C_\varphi^\dagger$ is quasinormal. Then $ker(C_\varphi^*) \subseteq ker(C_\varphi)$, then $\Sigma_{\sigma(h)} \subseteq \varphi^{-1}(\Sigma)$ follows from Lemma 3.1, consequently, h is $\varphi^{-1}(\Sigma)$ measurable, write $\sigma(h) = \cup A_n$ with $A_n \in \varphi^{-1}(\Sigma)$ and $0 < \mu(A_n) < \infty$. Set $f := C_\varphi \chi_{A_n}$ in (3.1), we obtain $\frac{1}{h}\chi_{A_n} = \frac{1}{(h\circ\varphi)}\chi_{A_n}$. Then $h = h\circ\varphi$ on $\sigma(h)$. □

Corollary 3.4. *Let $C_\varphi \in CR(L^2(\Sigma))$. Then $C_\varphi^\dagger$ is quasinormal iff C_φ^* is quasinormal.*

Lemma 3.5. *Let $T \in \mathcal{B}(\mathcal{H})$ and let $U|T|$ be its polar decomposition. Suppose p is positive real number. Then the following hold:*

(i) [31] T is p-paranormal if and only if for each $p \in (0,\infty)$,

$$|T|^p U^* |T|^{2p} U |T|^p - 2\lambda |T|^{2p} + \lambda^2 \geq 0, \quad for\ all\ \lambda \geq 0.$$

(ii) [31] T is absolute p-paranormal if and only if for each $p \in (0,\infty)$,

$$T^* |T|^{2p} T - (p+1)\lambda^p |T|^2 + p\lambda^{p+1} \geq 0, \quad for\ all\ \lambda \geq 0$$

(iii) [32] T is p-∗-paranormal if and only if for each $p \in (0,\infty)$,

$$|T|^p U^* |T|^{2p} U |T|^p + 2\lambda |T^*|^{2p} + \lambda^2 \geq 0, \quad for\ all\ real\ \lambda$$

(iv) [32] T is absolute p-∗-paranormal if and only if for each $p \in (0,\infty)$,

$$T^* |T|^{2p} T - (p+1)\lambda^p |T^*|^2 + p\lambda^{p+1} \geq 0, \quad for\ all\ \lambda > 0$$

Theorem 3.6. *[20] Let $C_\varphi \in L^2(\mathcal{F})$. Then the following statements are equivalent.*

(i) C_φ is p-quasihyponormal.

(ii) C_φ is p-paranormal.

(ii) C_φ is p-∗-paranormal.

iii) C_φ is absolute p-paranormal.

(iv) $E(h^p) \geq h^p \circ \varphi$.

Theorem 3.7. *Let $C_\varphi \in CR(L^2(\mathcal{F}))$. Then the following assertions on $\sigma(h)$ hold.*

(i) $C_\varphi^\dagger$ is p-quasihyponormal if and only if $h \geq h \circ \varphi$,

(ii) $C_\varphi^\dagger$ is p-paranormal if and only if $\sqrt{h}(\frac{h\circ\varphi}{\sqrt{h^3}})^p \leq E(\frac{1}{h^{\frac{p-1}{2}}})$,

(iii) $C_\varphi^\dagger$ is p-∗-paranormal if and only if $\sqrt{h}(\frac{\sqrt{h}(h\circ\varphi)}{h^2\circ\varphi^{-1}})^p \leq E(\frac{1}{h^{\frac{p-1}{2}}})$,

(iv) $C_\varphi^\dagger$ is absolute p-paranorma if and only if $h \geq h \circ \varphi$,

(v) $C_\varphi^\dagger$ is absolute p-∗-paranormal if and only if $h^{p+1} \geq (h\circ\varphi)(h^p \circ \varphi^2)$.

Proof . (i) It is well known that, for each $f \in L^2(\Sigma)$,

$$(C_\varphi^\dagger)^* C_\varphi^\dagger f = M_{\frac{1}{h\circ\varphi}} E(f), \quad C_\varphi^\dagger (C_\varphi^\dagger)^* f = M_{\frac{\chi_{\sigma(h)}}{h}} f.$$

Then by Lemma 2.8 we have, $((C_\varphi^\dagger)^* C_\varphi^\dagger)^p f = M_{\frac{1}{(h\circ\varphi)^p}} E(f)$, therefore

$$(C_\varphi^\dagger)^*((C_\varphi^\dagger)^* C_\varphi^\dagger)^p C_\varphi^\dagger f = M_{\frac{1}{(h\circ\varphi)(h^p\circ\varphi^2)}} E(E(f)\circ\varphi^{-1}),$$
$$(C_\varphi^\dagger)^*(C_\varphi^\dagger (C_\varphi^\dagger)^*)^p C_\varphi^\dagger f = \frac{1}{(h\circ\varphi)^{p+1}} E(f).$$

Thus $C_\varphi^\dagger$ is p-quasihyponormal if and only if

$$\langle\{(C_\varphi^\dagger)^*((C_\varphi^\dagger)^* C_\varphi^\dagger)^p C_\varphi^\dagger - (C_\varphi^\dagger)^*(C_\varphi^\dagger (C_\varphi^\dagger)^*)^p C_\varphi^\dagger\} f, f\rangle \geq 0.$$

Since (X, A, μ) is a σ-finite measure space, let$f := \chi_{\varphi^{-1}B}$ with $\mu(\varphi^{-1}B) < \infty$. Hence, The above inner product is non-negative if and only if

$$\int_{\varphi^{-1}B} \{\frac{1}{(h\circ\varphi)(h^p\circ\varphi^2)} E(E(\chi_{\varphi^{-1}B})\circ\varphi^{-1}) - \frac{1}{(h\circ\varphi)^{p+1}} E(\chi_{\varphi^{-1}B})\} d\mu \geq 0.$$

Since $E(\chi_{\varphi^{-1}B})\circ\varphi^{-1} = E(\chi_B\circ\varphi)\circ\varphi^{-1} = \chi_B$ on $\sigma(h)$, by change of variable theorem the previous integral is non-negative if and only if

$$\int_{\varphi^{-1}B} \{\frac{1}{h(h^p\circ\varphi)} E(E(\chi_{\varphi^{-1}B})\circ\varphi^{-1}) - \frac{1}{h^{p+1}} \chi_B\} h d\mu \geq 0.$$

But this is equivalent to $h \geq h \circ \varphi$ on $\sigma(h)$.
(ii) Let $f \in L^2(\mathcal{F})$. Therefore by direct calculations,

$$|C_\varphi^\dagger|^p V^* |C_\varphi^\dagger|^{2p} V |C_\varphi^\dagger|^p f = \frac{E(E(\frac{E(f)\circ\varphi^{-1}}{h^{\frac{p-1}{2}}})\circ\varphi)}{(h\circ\varphi)^{\frac{p+1}{2}}(h\circ\varphi^2)^p}.$$

Now by Lemma 2.10, $C_\varphi^\dagger$ is p-paranormal if and only if

$$\langle \frac{E(E(\frac{E(f)\circ\varphi^{-1}}{h^{\frac{p-1}{2}}})\circ\varphi)}{(h\circ\varphi)^{\frac{p+1}{2}}(h\circ\varphi^2)^p} - \frac{2\lambda E(f)}{(h\circ\varphi)^p} + \lambda^2, f\rangle \geq 0.$$

Put $f := \chi_{\varphi^{-1}B}$ with $\mu(\varphi^{-1}B) < \infty$. Hence, the above inner product is non-negative if and only if

$$\int_{\varphi^{-1}B}\{\frac{E(E(\frac{E(\chi_B\circ\varphi)\circ\varphi^{-1}}{h^{\frac{p-1}{2}}})\circ\varphi)}{(h\circ\varphi)^{\frac{p+1}{2}}(h\circ\varphi^2)^p} - \frac{2\lambda E(\chi_B\circ\varphi)}{(h\circ\varphi)^p} + \lambda^2\}d\mu =$$

$$\int_B\{\frac{E(\frac{\chi_B}{h^{\frac{p-1}{2}}})}{h^{\frac{p+1}{2}}(h\circ\varphi)^p} - \frac{2\lambda\chi_B}{h^p} + \lambda^2\}hd\mu \geq 0.$$

But this is possible if and only if $\frac{1}{h^{2p}} - \frac{E(\frac{1}{h^{\frac{p-1}{2}}})}{h^{\frac{p+1}{2}}(h\circ\varphi)^p} \leq 0$, since h is a non-negative function in $L^2(\mathcal{F})$ and B is an arbitrary element of σ-finite algebra $\mathcal{F}$. So the proof is complete.
(iii) The proof is similar to part (ii).
(iv) Let $f \in L^2(\Sigma)$. Direct computations show that

$$(C_\varphi^\dagger)^*|C_\varphi^\dagger|^{2p}C_\varphi^\dagger f = \frac{E(E(f)\circ\varphi^{-1})\circ\varphi}{(h\circ\varphi)(h\circ\varphi^2)^p}.$$

By Lemma 3.5, $C_\varphi^\dagger$ is absolute p-paranormal if and only if

$$\langle\frac{E(E(f)\circ\varphi^{-1})\circ\varphi}{(h\circ\varphi)(h\circ\varphi^2)^p} - (p+1)\frac{E(f)}{h\circ\varphi}\lambda^p + p\lambda^{p+1}, f\rangle \geq 0$$

Put $f := \chi_{\varphi^{-1}B}$ with $\mu(\varphi^{-1}B) < \infty$. Hence, the above inner product is non-negative if and only if

$$\int_{\varphi^{-1}B}\{\frac{E(E(f)\circ\varphi^{-1})\circ\varphi}{(h\circ\varphi)(h\circ\varphi^2)^p} - (p+1)\frac{E(f)}{h\circ\varphi}\lambda^p + p\lambda^{p+1}\}d\mu =$$

$$\int_B\{\frac{1}{h(h\circ\varphi)^p} - (p+1)\frac{\chi_B}{h}\lambda^p + p\lambda^{p+1}\}hd\mu \geq 0.$$

But this is possible if and only if

$$A(\lambda) := a - (p+1)b\lambda^p + p\lambda^{p+1} \geq 0, \qquad \lambda \in [0,\infty),$$

where $a = \frac{1}{h(h\circ\varphi)^p}$, $b = \frac{1}{h}$, since h is a non-negative function in $L^2(\mathcal{F})$ and B is an arbitrary element of σ-finite algebra $\mathcal{F}$.
Since this function takes its minimum value at $\lambda = b$, then

$$\begin{aligned} A(\lambda) \geq 0 &\Longleftrightarrow a \geq b^{p+1} \\ &\Longleftrightarrow \frac{1}{h(h\circ\varphi)^p} \geq \frac{1}{h^{p+1}} \\ &\Longleftrightarrow (h\circ\varphi)^p \leq h^p, \quad \text{on } \sigma(h) \\ &\Longleftrightarrow (h\circ\varphi) \leq h, \quad \text{on } \sigma(h). \end{aligned}$$

So the proof is complete.
(v) The proof is similar to part (iv). □

4. Examples

Example 4.1. *Let $X = (1, \infty)$ equipped with the Lebesgue measure $d\mu$ on the Lebesgue measurable subsets. The transformation φ is given by $\varphi(x) = \sqrt{x}$. Then $h(x) = 2x$, $E = I$, $h \circ \varphi(x) = 2\sqrt{x}$. Now, by these computations we obtain,*

$$C_\varphi^\dagger = 1, \quad C_\varphi = f(\sqrt{x}), \quad C_\varphi^* = 2x,$$
$$\widetilde{C_\varphi} = \sqrt[8]{x} f(\sqrt{x}), \quad (\widetilde{C_\varphi})^\dagger = \sqrt[4]{2x}, \quad \widetilde{C_\varphi^\dagger} = \sqrt[8]{x} f(\sqrt{x}),$$
$$V(f) = \sqrt{2x} f(2x), \quad |C_\varphi^\dagger|(f) = \frac{f(x)}{\sqrt{2\sqrt{x}}}.$$

Accoroding to above relations, $(\widetilde{C_\varphi})^\dagger \neq \widetilde{C_\varphi^\dagger}$ on $X = (1, \infty)$.
Also Theorems 3.2, 3.3, C_φ^* and $C_\varphi^\dagger$ are not quasinormal. However by Theorem 3.7, $C_\varphi^\dagger$ not only is p-quasihyponormal but it is also absolute p-paranormal. Also by simple calculations, p-paranormality, p-$*$-paranormality and absolute p-paranormality of $C_\varphi^\dagger$ are equivalent to $x^{p-1} \geq 2$, $x^p \geq 1$, $x^{3p+2} \geq 1$, respectively. Therefor $C_\varphi^\dagger$ is p-$*$-paranormal and absolute p-paranormal, but $C_\varphi^\dagger$ is not p-paranormal. However, if we change only the underlying space to $X = (0, 1)$, then by Theorem 3.7 for each $p > 0$, $C_\varphi^\dagger$ is p-quasihyponormal and p-paranormal, but $C_\varphi^\dagger$ can not be p-$*$-paranormal and absolute p-paranormal.

Example 4.2. *Let $X = [0, 1]$ equipped with the Lebesgue measure μ on the Lebesgue measurable subsets and let $\varphi : X \to X$ is defined by*

$$\varphi(x) = \begin{cases} 2x & 0 \leq x \leq \frac{1}{2}, \\ 2 - 2x & \frac{1}{2} \leq x \leq 1. \end{cases}$$

Then

$$E(f)(x) = \frac{f(x) + f(1-x)}{2},$$
$$\varphi^2(x) = \begin{cases} 4x & 0 \leq x \leq \frac{1}{4}; \\ 2 - 4x & \frac{1}{4} \leq x \leq \frac{1}{2}; \\ -2 + 4x & \frac{1}{2} \leq x \leq \frac{3}{4}; \\ 4 - 4x & \frac{3}{4} \leq x \leq 1, \end{cases}$$

and so $h(x) = 1$ and for each $f \in L^2(\Sigma)$

$$(E(f) \circ \varphi^{-1})(x) = \frac{f(\frac{x}{2}) + f(1 - \frac{x}{2})}{2}.$$

Thus by Theorem 3.7, $C_\varphi^\dagger$ is in all of the above classes. Also by proposition 2.10 and Lemma 2.9 we have,

$$C_\varphi^\dagger(f) = C_\varphi^*(f) = (\widetilde{C_\varphi})^\dagger(f) = \frac{f(\frac{x}{2}) + f(1 - \frac{x}{2})}{2};$$
$$\widetilde{C_\varphi^\dagger}(f) = E(E(f) \circ \varphi^{-1});$$
$$V(f) = \frac{f(\frac{x}{2}) + f(1 - \frac{x}{2})}{2};$$
$$|C_\varphi^\dagger|(f) = \frac{f(x) + f(1-x)}{2}.$$

Example 4.3. *Let $X = \mathbb{N}$, $\Sigma = 2^{\mathbb{N}}$ and let μ be the counting measure. Define*

$$\varphi_1(n) = \begin{cases} 1 & n = 1, 2, \\ n-1 & n \geq 3, \end{cases}$$

and

$$\varphi_2(n) = \begin{cases} 1 & n = 1, \\ n-1 & n \geq 2. \end{cases}$$

Then

$$h_1(n) = \mu(\varphi_1^{-1}(n)) = \begin{cases} 2 & n = 1, \\ 1 & n \geq 2, \end{cases} \qquad h_2(n) = \mu(\varphi_2^{-1}(n)) = 1.$$

Since φ_2 is the identity function, then by Corollary 2.3, C_{φ_2} and $C^{\dagger}_{\varphi_2}$ are complex symmetric operators but C_{φ_1} and $C^{\dagger}_{\varphi_1}$ are not complex symmetric operators.

Example 4.4. *Let $\{m_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers. Consider the space $\ell^2(m) = L^2(\mathbb{N}, 2^{\mathbb{N}}, m)$, where $2^{\mathbb{N}}$ is the power set of natural numbers and m is a measure on $2^{\mathbb{N}}$ defined by $m(\{n\}) = m_n$. Let $u = \{u_n\}_{n=1}^{\infty}$ be a sequence of non-negative real numbers. Let $\varphi : \mathbb{N} \to \mathbb{N}$ be a non-singular measurable transformation; i.e. $\mu \circ \varphi^{-1} \ll \mu$. Direct computation shows that (see [23])*

$$h(k) = \frac{1}{m_k} \sum_{j \in \varphi^{-1}(k)} m_j; \qquad E(f)(k) = \frac{\sum_{j \in \varphi^{-1}(\varphi(k))} f_j m_j}{\sum_{j \in \varphi^{-1}(\varphi(k))} m_j};$$

and

$$h \circ \varphi(k) = \frac{1}{m_{\varphi(k)}} \sum_{j \in \varphi^{-1}(\varphi(k))} m_j; \qquad (E(f) \circ \varphi^{-1})(k) = \frac{\sum_{j \in \varphi^{-1}(k)} f_j m_j}{\sum_{j \in \varphi^{-1}(k)} m_j};$$

for all non-negative sequence $f = \{f_n\}_{n=1}^{\infty} \in \ell^2(m)$ and $k \in \mathbb{N}$.
By theorem 3.7, $C^{\dagger}_{\varphi}$ is p-quasihyponormal and absolute p-paranorma if and only if

$$\frac{1}{m_k} \sum_{j \in \varphi^{-1}(k)} m_j \geq \frac{1}{m_{\varphi(k)}} \sum_{j \in \varphi^{-1}(\varphi(k))} m_j.$$

Also, $C^{\dagger}_{\varphi}$ is absolute p-$*$-paranormal if and only if

$$(\frac{1}{m_k} \sum_{j \in \varphi^{-1}(k)} m_j)^{p+1} \geq (\frac{1}{m_{\varphi(k)}} \sum_{j \in \varphi^{-1}(\varphi(k))} m_j)(\frac{1}{m_{\varphi^2(k)}} \sum_{j \in \varphi^{-1}(\varphi^2(k))} m_j)^p.$$

Acknowledgements

The author would like to thank the referee for very helpful comments and valuable suggestions.

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