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On the Ψ-instability of a nonlinear Lyapunov matrix differential equation with integral term as right side

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Abstract

The aim of this paper is to give sufficient conditions for Ψ-instability of trivial solution of a nonlinear Lyapunov matrix differential equation with integral term as right side.

Keywords: Ψ-instability, nonlinear Lyapunov matrix differential equation, integral term as right side.
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1. Introduction

The Lyapunov matrix differential equations occur in many branches of control theory such as optimal control and stability analysis. Recent works for Ψ-stability, Ψ-asymptotic stability, Ψ-instability, Ψ-boundedness, controllability, dichotomy and conditioning for Lyapunov matrix differential equations have been given in many papers. See [4, 5, 6, 7, 8, 9, 10, 13, 14, 15, 16, 17] and the references therein.

In this paper are presented a several new sufficient conditions for Ψ-instability of the trivial solution to the nonlinear Lyapunov matrix differential equation with integral term as right side:

$$Z' = A(t)Z + ZB(t) + F(t,Z) + \int_0^t G(t,s,Z(s))ds. \tag{1.1}$$

These conditions can be expressed in the terms of a fundamental matrices of the matrix differential equations

$$X' = A(t)X \tag{1.2}$$

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$$Y' = YB(t) \tag{1.3}$$

$$Z' = A(t)Z + ZB(t) \tag{1.4}$$

and on the functions F and G. Here, Ψ is a matrix function whose introduction permits to obtaining a mixed asymptotic behavior for the components of solutions. The main tools used in this paper are the technique of the variation of constants formula and Kronecker product of matrices, which has been successfully applied in various fields of matrix theory, group theory and particle physics. See, for example, the cited papers and the references cited therein.

2. Preliminaries

In this section we present some basic notations, definitions, hypotheses and results which are useful later on. Let R^d be the Euclidean d-dimensional space. For $x = (x_1, x_2, ..., x_d)^T \in R^d$, let $\| x \| = \max\{| x_1|, | x_2|, ..., | x_d|\}$ be the norm of x (here, ^{T} denotes transpose). Let $\mathbb{M}_{d\times d}$ be the linear space of all real $d \times d$ matrices. For $A = (a_{ij}) \in \mathbb{M}_{d\times d}$, we define the norm $| A |$ by formula $| A | = \sup\limits_{\|x\|\leq 1} \| Ax \|$. It is well-known that $| A | = \max\limits_{1\leq i\leq d}\{\sum\limits_{j=1}^{d} | a_{ij}|\}$. By a solution of the equation (1.1) we mean a continuous differentiable $d \times d$ matrix function satisfying the equation (1.1) for all $t \in R_+ = [0, \infty)$. In equation (1.1), we assume that $A(t)$, $B(t)$, $F(t, Z)$ and $G(t, s, Z)$ are continuous $d \times d$ matrices for $t \in R_+$, $Z \in \mathbb{M}_{d\times d}$ and $t \geq s \geq 0$. We will admit that for all $t_0 \in R_+$ and $Z_0 \in \mathbb{M}_{d\times d}$, the equation (1) has a unique solution $Z(t)$, defined on $\mathbb{R}_+$, such that $Z(t_0) = Z_0$.

Let $\Psi_i : R_+ \longrightarrow (0, \infty)$, $i = 1, 2, ..., d$, be continuous functions and the matrix

$$\Psi = \text{diag } [\Psi_1, \Psi_2, \cdots \Psi_d].$$

Definition 2.1. *([4], [9])The trivial solution of the equation $X' = F(t, X)$ (where $X \in \mathbb{M}_{d\times d}$ and F is a continuous $d \times d$ matrix function) is said to be $\Psi-$ stable over R_+ if for each $\varepsilon > 0$ and each $t_0 \in R_+$, there is a a corresponding $\delta = \delta(\varepsilon, t_0) > 0$ such that any solution $X(t)$ of the equation which satisfies the inequality $| \Psi(t_0)X(t_0) |< \delta$, exists and satisfies the inequality $| \Psi(t)X(t) |< \varepsilon$ for all $t \geq t_0$.*
Otherwise, we say that the trivial solution is $\Psi-$ unstable over R_+.

Remark 2.2. *1. The Definition extends the definition of stability (instability) from (vector) differential equations to matrix differential equations.*
2. For $\Psi = I_d$, one obtain the notion of classical stability (instability) (see [2]).
3. It is easy to see that if Ψ and Ψ^{-1} are bounded on R_+, then the $\Psi-$ stability (instability) is equivalent with the classical stability (instability).

Definition 2.3. *([6], [7])The matrix function $M : R_+ \longrightarrow \mathbb{M}_{d\times d}$ is said to be $\Psi-$ bounded on R_+ if the matrix function $\Psi(t)M(t)$ is bounded on R_+ (i.e. there exists $m > 0$ such that $| \Psi(t)M(t) |\leq m$, for all $t \in R_+$).*
Otherwise, is said that the matrix function M is $\Psi-$ unbounded on R_+.

Definition 2.4. *([1]) Let $A = (a_{ij}) \in M_{m\times n}$ and $B = (b_{ij}) \in M_{p\times q}$. The Kronecker product of A and B, written $A \otimes B$, is defined to be the partitioned matrix*

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}.$$

Obviously, $A \otimes B \in \mathbb{M}_{mp\times nq}$.

The important rules of calculation of the Kronecker product are given in [1], [12], Chapter 2 and Lemma 1, [9].

Definition 2.5. *([12]) The application* $\mathcal{V}ec : \mathbb{M}_{m\times n} \longrightarrow R^{mn}$*, defined by*

$$\mathcal{V}ec(A) = (a_{11}, a_{21}, \cdots, a_{m1}, a_{12}, a_{22}, \cdots, a_{m2}, \cdots, a_{1n}, a_{2n}, \cdots, a_{mn})^T,$$

where $A = (a_{ij}) \in \mathbb{M}_{m\times n}$*, is called the vectorization operator.*

For important properties and rules of calculation of the $\mathcal{V}ec$ operator, see Lemmas 2, 3, 4, [9].

For "*corresponding Kronecker product system associated with (1.1)"*, see Lemma 5, [9].

The Lemmas 6 and 9, [9], play an important role in the proofs of main results of present paper.

For $\Psi-$ instability of matrix differential equations (1.2), (1.3) and (1.4), see essential details in [9].

3. Main results

In this section, we obtain sufficient conditions for $\Psi-$ instability of trivial solution of nonlinear Lyapunov matrix differential equation (1), in three cases.

Case 1. We start from $\Psi-$ instability of equation $Z' = A(t)Z$.

Theorem 3.1. *Suppose that:*
(1). There exist supplementary projections P_1 *and* $P_{2,}$ $P_{2,} \neq O_d$*, and a positive constant* K *such that the fundamental matrix* $X(t)$ *for (2) satisfies the condition*

$$\int_0^t \left|\Psi(t)X(t)P_1X^{-1}(s)\Psi^{-1}(s)\right| ds + \int_t^\infty \left|\Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s)\right| ds \le K,$$

for all $t \ge 0$*;*
(2). The matrix function $F(t, Z)$ *satisfies the inequality*

$$| \Psi(t)F(t, Z) | \le \gamma | \Psi(t)Z |,$$

for all $t \in \mathbb{R}_+$ *and* $Z \in \mathbb{M}_{d\times d}$*, where* γ *is a positive constant;*
(3). The matrix function $B(t)$ *satisfies the condition* $| B(t) | \le b$*, for all* $t \ge 0$*, where* b *is a positive constant;*
(4). The matrix function $G(t, s, Z)$ *satisfies the inequality*

$$| \Psi(t)G(t, s, Z) | \le g(t, s) | \Psi(s)Z |,$$

for $t \ge s \ge 0$ *and* $Z \in \mathbb{M}_{d\times d}$*, where* $g(t, s)$ *is a continuous nonnegative function for* $t \ge s \ge 0$ *such that*

$$\int_0^t g(t, s)ds \le M,$$

for all $t > 0$*,* M *being a positive constant;*
(5). $(b + \gamma + M) K < 1$*.*
Then, the trivial solution of (1.1) is $\Psi-$ *unstable over* R_+*.*

Proof . We may reason by reduction to absurdity. Suppose the contrary. Then, by Definition, it results that the trivial solution of the equation (1) is $\Psi-$ stable over R_+. Therefore, for each $\varepsilon > 0$ and each $t_0 \in R_+$, there is a a corresponding $\delta = \delta(\varepsilon, t_0) > 0$ such that any solution $Z(t)$ of the equation (1) which satisfies the inequality $\mid \Psi(t_0)Z(t_0) \mid < \delta$, exists and satisfies the inequality $\mid \Psi(t)Z(t) \mid < \varepsilon$ for all $t \geq t_0$.
Without loss of generality, we may assume that $X(0) = I_d$. We choose $Z_0 \in \mathbb{M}_{d\times d}$ such that $P_1 Z_0 = O_d$ and $0 < \mid \Psi(0)Z_0 \mid < \delta(\varepsilon, 0)$. Let $Z(t)$ be the solution of (1) with $Z(0) = Z_0$. Then, $\mid \Psi(t)Z(t) \mid < \varepsilon$ for all $t \geq 0$.
Let $W(t)$ be the matrix function

$$W(t) = Z(t) - \int_0^t X(t)P_1X^{-1}(s)H(s)ds + \int_t^\infty X(t)P_2X^{-1}(s)H(s)ds$$

for $t \geq 0$, where

$$H(s) = Z(s)B(s) + F(s, Z(s)) + \int_0^s G(s, u, Z(u))du, \text{ for } s \geq 0.$$

For $s \geq 0$, we have

$$\begin{aligned}
&\mid \Psi(s)H(s) \mid = \\
&= \mid \Psi(s) \left(Z(s)B(s) + F(s, Z(s)) + \textstyle\int_0^s G(s, u, Z(u))du \right) \mid \leq \\
&\leq \mid \Psi(s)Z(s) \mid\mid B(s) \mid + \mid \Psi(s)F(s, Z(s)) \mid + \textstyle\int_0^s \mid \Psi(s)G(s, u, Z(u)) \mid du \leq \\
&\leq b \mid \Psi(s)Z(s) \mid + \gamma \mid \Psi(s)Z(s) \mid + \textstyle\int_0^s g(s, u) \mid \Psi(u)Z(u) \mid du \leq \\
&\leq \varepsilon(b + \gamma) + \varepsilon \textstyle\int_0^s g(s, u)du \leq \varepsilon(b + \gamma + M),
\end{aligned}$$

from which, for $v \geq t \geq 0$, we obtain

$$\begin{aligned}
&\mid \textstyle\int_t^v X(t)P_2X^{-1}(s)H(s)ds \mid = \\
&= \mid \Psi^{-1}(t) \textstyle\int_t^v \Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s)\Psi(s)H(s)ds \mid \leq \\
&\leq \mid \Psi^{-1}(t) \mid \textstyle\int_t^v \mid \Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s) \mid\mid \Psi(s)H(s) \mid ds \leq \\
&\leq \varepsilon\,(b + \gamma + M) \mid \Psi^{-1}(t) \mid \textstyle\int_t^v \mid \Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s) \mid ds.
\end{aligned}$$

It follows that

$$\int_t^\infty X(t)P_2X^{-1}(s)H(s)ds, \text{ for } t \geq 0,$$

is an absolutely convergent integral on R_+. It is easy to see that the function $W(t)$ exists on R_+ and is a continuously differentiable function on R_+. For $t \in R_+$, we have

$$W'(t) = Z'(t) - \int_0^t X'(t)P_1X^{-1}(s)H(s)ds - X(t)P_1X^{-1}(t)H(t)+$$

$$+\int_t^\infty X'(t)P_2X^{-1}(s)H(s)ds - X(t)P_2X^{-1}(t)H(t) =$$

$$= A(t)Z(t) + Z(t)B(t) + F(t, Z(t)) + \int_0^t G(t, s, Z(s))ds-$$

$$-\int_0^t A(t)X(t)P_1X^{-1}(s)H(s)ds - X(t)P_1X^{-1}(t)H(t)+$$

$$+\int_t^\infty A(t)X(t)P_2X^{-1}(s)H(s)ds - X(t)P_2X^{-1}(t)H(t) =$$

$$= A(t)Z(t) + Z(t)B(t) + F(t, Z(t)) + \int_0^t G(t, s, Z(s))ds-$$

$$-A(t)\left(\int_0^t X(t)P_1X^{-1}(s)H(s)ds - \int_t^\infty X(t)P_2X^{-1}(s)H(s)ds\right) -$$

$$-X(t)\left(P_1 + P_2\right)X^{-1}(t)H(t) =$$

$$= A(t)Z(t) + H(t) - A(t)\left(Z(t) - W(t)\right) - H(t) = A(t)W(t).$$

Thus, $W(t)$ is a solution on R_+ of the linear equation (2). For $t \in R_+$, we have

$$\begin{aligned}&\mid \Psi(t)W(t) \mid\leq\mid \Psi(t)Z(t) \mid + \int_0^t \mid \Psi(t)X(t)P_1X^{-1}(s)\Psi^{-1}(s) \mid\mid \Psi(s)H(s) \mid ds+\\&+\int_t^\infty \mid \Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s) \mid\mid \Psi(s)H(s) \mid ds\\&\leq \varepsilon + \varepsilon\left(b + \gamma + M\right)K.\end{aligned}$$

This shows that the solution $W(t)$ is $\Psi-$ bounded on R_+.
On the other hand,

$$W(t) = X(t)X^{-1}(0)W(0) = X(t)\left(P_1 + P_2\right)W(0) =$$

$$= X(t)P_1\left(Z(0) + \int_0^\infty X(0)P_2X^{-1}(s)H(s)ds\right) +$$

$$+X(t)P_2W(0) = X(t)P_2W(0).$$

If $P_2W(0) \neq O_d$, from Lemma 11, [9], it follows that $\limsup\limits_{t\to\infty} \mid \Psi(t)W(t) \mid = +\infty$, which contradicts the $\Psi-$ boundedness of $W(t)$ on R_+
Thus, $P_2W(0) = O_d$ and then, $W(t) = O_d$ on R_+.
Therefore, for $t \in R_+$, we have

$$Z(t) = \int_0^t X(t)P_1X^{-1}(s)H(s)ds - \int_t^\infty X(t)P_2X^{-1}(s)H(s)ds.$$

From this,

$$\mid \Psi(t)Z(t) \mid\leq \int_0^t \mid \Psi(t)X(t)P_1X^{-1}(s)\Psi^{-1}(s) \mid\mid \Psi(s)H(s) \mid ds+$$

$$+\int_t^\infty \mid \Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s) \mid\mid \Psi(s)H(s) \mid ds \leq$$

$$\leq K\left(b + \gamma + M\right)\sup_{s\geq 0} \mid \Psi(s)Z(s) \mid .$$

Therefore,

$$\sup_{s\geq 0} \mid \Psi(s)Z(s) \mid \leq K\left(b+\gamma+M\right)\sup_{s\geq 0} \mid \Psi(s)Z(s) \mid,$$

which contradicts the hypothesis (5) of Theorem (because $\mid \Psi(0)Z(0) \mid > 0$).
This contradiction shows that the trivial solution of the equation (1) is $\Psi-$ unstable over R_+. □

Remark 3.2. *1. In particular case $B = O_d$ and $F = O_d$, one obtain variant for differential matrix equation of Theorem 5, [3].*
Indeed, in this case, for

$$Z = \begin{pmatrix} z_1 & z_1 & \cdots & z_1 \\ z_2 & z_2 & \cdots & z_2 \\ \vdots & \vdots & \vdots & \vdots \\ z_d & z_d & \cdots & z_d \end{pmatrix} \text{ and } G = \begin{pmatrix} g_1(t,s,z) & g_1(t,s,z) & \cdots & g_1(t,s,z) \\ g_2(t,s,z) & g_2(t,s,z) & \cdots & g_2(t,s,z) \\ \vdots & \vdots & \vdots & \vdots \\ g_d(t,s,z) & g_d(t,s,z) & \cdots & g_d(t,s,z) \end{pmatrix},$$

the equation (1) becomes

$$z' = A(t)z + \int_0^t G(t,s,Z(s))ds,$$

where $z = (z_1, z_2, ..., z_d)^T$, i.e. equation (1) from [3].
Now, the solution $Z(t)$ is $\Psi-$ unstable over R_+ if and only if the solution $z(t)$ is $\Psi-$ unstable over R_+.
Thus, the Theorem generalizes the result from [3].
2. For $F = O_d$, one obtain a new result in connection with $\Psi-$ instability of trivial solution of nonlinear Lyapunov matrix differential equation with integral term as right side

$$Z' = A(t)Z + ZB(t) + \int_0^t G(t,s,Z(s))ds,$$

in which the equation $Z' = A(t)Z$ is $\Psi-$ unstable over R_+.

Case 2. We start from $\Psi-$ instability of equation $Z' = A(t)Z + ZB(t)$.

Theorem 3.3. *Suppose that:*
(1). There exist supplementary projections P_1 and P_2, $P_2 \neq O_d$, and a positive constant K such that the fundamental matrices $X(t)$ and $Y(t)$ for (2) and (3) respectively satisfy for all $t \geq 0$ the condition

$$\int_0^t \left|\left(Y^T(t)\left(Y^T\right)^{-1}(s)\right) \otimes \left(\Psi(t)X(t)P_1X^{-1}(s)\Psi^{-1}(s)\right)\right| ds +$$

$$+\int_t^\infty \left|\left(Y^T(t)\left(Y^T\right)^{-1}(s)\right) \otimes \left(\Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s)\right)\right| ds \leq K,$$

(2). The matrix function $F(t, Z)$ satisfies the inequality

$$\mid \Psi(t)F(t,Z) \mid \leq \gamma \mid \Psi(t)Z \mid,$$

for all $t \in \mathbb{R}_+$ and $Z \in \mathbb{M}_{d\times d}$, where γ is a positive constant;
(3). The matrix function $G(t, s, Z)$ satisfies the inequality

$$\mid \Psi(t)G(t,s,Z) \mid \leq g(t,s) \mid \Psi(s)Z \mid,$$

for $t \geq s \geq 0$ and $Z \in \mathbb{M}_{d\times d}$, where $g(t,s)$ is a continuous nonnegative function for $t \geq s \geq 0$ such that

$$\int_0^t g(t,s)ds \leq M,$$

for all $t > 0$, M being a positive constant;
(4). $(\gamma + M)\, K < 1$.
Then, the trivial solution of (1.1) is $\Psi-$ unstable over R_+.

Proof . We may reason by reduction to absurdity. Suppose the contrary. Then by Definition, it results that the trivial solution of the equation (1) is $\Psi-$ stable over R_+. Therefore, for each $\varepsilon > 0$ and each $t_0 \in R_+$, there is a a corresponding $\delta = \delta(\varepsilon, t_0) > 0$ such that any solution $Z(t)$ of the equation (1) which satisfies the inequality $\mid \Psi(t_0)Z(t_0) \mid < \delta$, exists and satisfies the inequality $\mid \Psi(t)Z(t) \mid < \varepsilon$ for all $t \geq t_0$.
Let $z(t) = \mathcal{V}ec(Z(t))$, $t \geq 0$. From Lemma 5, [9], the function z(t) is a solution of the corresponding Kronecker product system associated with (1.1), i.e. of the differential system

$$z' = \left(I_d \otimes A(t) + B^T(t) \otimes I_d\right) z + f(t,z) + \int_0^t g(t,s,z(s))ds, \tag{3.1}$$

where $f(t,z) = \mathcal{V}ec(F(t,Z))$ and $g(t,s,z) = \mathcal{V}ec(G(t,s,Z))$, for $t \geq s \geq 0$ and $Z \in \mathbb{M}_{d\times d}$.
From Lemmas 6 and 7, [9], the trivial solution of (1) is $\Psi-$ stable over R_+ if and only if the trivial solution of (5) is $I_d \otimes \Psi-$ stable over R_+.
Without loss generality, we may assume that $X(0) = Y(0) = I_d$. We choose $z_0 \in R^{d^2}$, $z_0 \neq \theta$, such that $(I_d \otimes P_1)\, z_0 = \theta$ and $0 < \parallel (I_d \otimes \Psi(0))\, z_0 \parallel_{R^{d^2}} < \frac{\delta(\varepsilon,0)}{d}$.
Let $z(t)$ the solution of (3.1) such that $z(0) = z_0$. From the above results and Lemma 6, [9], we have $\parallel (I_d \otimes \Psi(t))\, z(t) \parallel_{R^{d^2}} < \varepsilon$, for $t \geq 0$ (and $\mid \Psi(t)Z(t) \mid < \varepsilon$ for all $t \geq 0$, where $Z(t) = \mathcal{V}ec^{-1}(z(t))$ is the corresponding solution of (1)).
Let $w(t)$ the vector function

$$w(t) = z(t) - \int_0^t \left(Y^T(t) \otimes X(t)\right) (I_d \otimes P_1) \left(\left(Y^T\right)^{-1}(s) \otimes X^{-1}(s)\right) H(s)ds +$$

$$+ \int_t^\infty \left(Y^T(t) \otimes X(t)\right) (I_d \otimes P_2) \left(\left(Y^T\right)^{-1}(s) \otimes X^{-1}(s)\right) H(s)ds,\ t \geq 0,$$

where

$$H(s) = f(s, z(s)) + \int_0^s g(s,u,z(u))du,\ s \geq 0,$$

or, in other form (see Lemma 1, [9]),

$$w(t) = z(t) - \int_0^t \left[\left(Y^T(t) \left(Y^T\right)^{-1}(s)\right) \otimes (X(t)P_1X^{-1}(s))\right] H(s)ds +$$

$$+ \int_t^\infty \left[\left(Y^T(t) \left(Y^T\right)^{-1}(s)\right) \otimes (X(t)P_2X^{-1}(s))\right] H(s)ds,\ t \geq 0.$$

For $v \geq t \geq 0$,

$$\| \int_t^v \left[\left(Y^T(t) \left(Y^T \right)^{-1}(s) \right) \otimes \left(X(t) P_2 X^{-1}(s) \right) \right] H(s) ds \|_{R^{d^2}} =$$

$$= \| \left(I_d \otimes \Psi^{-1}(t) \right) \int_t^v \Phi_2(t,s) (I_d \otimes \Psi(s)) H(s) ds \|_{R^{d^2}} \leq$$

$$\leq | \Psi^{-1}(t) | \int_t^v | \Phi_2(t,s) | \| (I_d \otimes \Psi(s)) H(s) \|_{R^{d^2}} ds \leq$$

$$\leq \varepsilon (\gamma + M) | \Psi^{-1}(t) | \int_t^v | \Phi_2(t,s) | ds,$$

(where $\Phi_i(t,s) = \left(Y^T(t) \left(Y^T \right)^{-1}(s) \right) \otimes \left(\Psi(t) X(t) P_i X^{-1}(s) \Psi^{-1}(s) \right),\ i = 1, 2$)
because, for $t \geq 0$ and the solution $Z(t) = \mathcal{V}ec^{-1}(z(t))$,

$$\| \left(I_d \otimes \Psi(t) \right) H(t) \|_{R^{d^2}} = \| \left(I_d \otimes \Psi(t) \right) \mathcal{V}ec \left(F(t, Z(t)) + \int_0^t G(t,s,Z(s)) ds \right) \|_{R^{d^2}}$$
$$\leq | \Psi(t) F(t, Z(t)) | + \int_0^t | \Psi(t) G(t,s,Z(s)) | ds$$

$$\leq \gamma | \Psi(t) Z(t) | + \int_0^t g(t,s) | \Psi(s) Z(s) | ds \leq \varepsilon (\gamma + M).$$

It follows that

$$\int_t^\infty \left[\left(Y^T(t) \left(Y^T \right)^{-1}(s) \right) \otimes \left(X(t) P_2 X^{-1}(s) \right) \right] H(s) ds,\ t \geq 0,$$

is an absolutely convergent integral. It is easy to see that $w(t)$ exists on R_+ and is a continuously differentiable function on R_+. For $t \in R_+$, with the notation $C(t) = I_d \otimes A(t) + B^T(t) \otimes I_d$ and with Lemma 9, [9],

$$w'(t) = z'(t) - \int_0^t \left(Y^T(t) \otimes X(t) \right)' (I_d \otimes P_1) \left(\left(Y^T \right)^{-1}(s) \otimes X^{-1}(s) \right) H(s) ds -$$

$$- \left(Y^T(t) \otimes X(t) \right) (I_d \otimes P_1) \left(\left(Y^T \right)^{-1}(t) \otimes X^{-1}(t) \right) H(t) +$$

$$+ \int_t^\infty \left(Y^T(t) \otimes X(t) \right)' (I_d \otimes P_2) \left(\left(Y^T \right)^{-1}(s) \otimes X^{-1}(s) \right) H(s) ds -$$

$$- \left(Y^T(t) \otimes X(t) \right) (I_d \otimes P_2) \left(\left(Y^T \right)^{-1}(t) \otimes X^{-1}(t) \right) H(t) =$$

$$= C(t) z(t) + f(t, z(t)) + \int_0^t g(t,s,z(s)) ds -$$

$$- C(t) \int_0^t \left(Y^T(t) \otimes X(t) \right) (I_d \otimes P_1) \left(\left(Y^T \right)^{-1}(s) \otimes X^{-1}(s) \right) H(s) ds +$$

$$+ C(t) \int_t^\infty \left(Y^T(t) \otimes X(t) \right) (I_d \otimes P_2) \left(\left(Y^T \right)^{-1}(s) \otimes X^{-1}(s) \right) H(s) ds -$$

$$- \left(Y^T(t) \otimes X(t) \right) \left[I_d \otimes (P_1 + P_2) \right] \left(\left(Y^T \right)^{-1}(t) \otimes X^{-1}(t) \right) H(t) =$$

$$= C(t) z(t) + f(t, z(t)) + \int_0^t g(t,s,z(s)) ds + C(t) \left(w(t) - z(t) \right) - H(t) = C(t) w(t).$$

Thus, $w(t)$ is a solution on R_+ of the linear equation $u' = C(t)u$.
On the other hand, from Lemma 6, [9], for $t \geq 0$,

$$\| (I_d \otimes \Psi(t))\, w(t) \|_{R^{d^2}} \leq \| (I_d \otimes \Psi(t))\, z(t) \|_{R^{d^2}} +$$

$$+ \int_0^t | \Phi_1(t,s) | \| (I_d \otimes \Psi(s)) H(s) \|_{R^{d^2}} \, ds +$$

$$+ \int_t^\infty | \Phi_2(t,s) | \| (I_d \otimes \Psi(s)) H(s) \|_{R^{d^2}} \, ds \leq$$

$$\leq \varepsilon + \varepsilon\, (\gamma + M)\, K.$$

This shows that the solution $w(t)$ is $I_d \otimes \Psi(t)-$ bounded on R_+.
From Lemma 9, [9],

$$w(t) = \left(Y^T(t) \otimes X(t)\right) \left(\left(Y^T\right)^{-1}(0) \otimes X^{-1}(0)\right) w(0) =$$

$$= \left(Y^T(t) \otimes X(t)\right) [I_d \otimes (P_1 + P_2)]\, w(0) =$$

$$= \left(Y^T(t) \otimes X(t)\right) (I_d \otimes P_2)\, w(0).$$

If $(I_d \otimes P_2)\, w(0) \neq \theta$, from hypothesis (1) and Lemma 11, [9], it follows that

$$\limsup_{t \to \infty} \| (I_d \otimes \Psi(t))\, w(t) \|_{R^{d^2}} = +\infty.$$

This contradicts the $I_d \otimes \Psi(t)-$ boundedness of $w(t)$ on R_+.
Thus, $(I_d \otimes P_2)\, w(0) = \theta$ and then $w(t) = \theta$ on R_+.
Therefore, for $t \geq 0$,

$$z(t) = \int_0^t \left[\left(Y^T(t) \left(Y^T\right)^{-1}(s)\right) \otimes \left(X(t) P_1 X^{-1}(s)\right)\right] H(s) ds -$$

$$- \int_t^\infty \left[\left(Y^T(t) \left(Y^T\right)^{-1}(s)\right) \otimes \left(X(t) P_2 X^{-1}(s)\right)\right] H(s) ds, \; t \geq 0.$$

From this, for $t \geq 0$,

$$\| (I_d \otimes \Psi(t))\, z(t) \|_{R^{d^2}} \leq$$

$$\leq \int_0^t | \Phi_1(t,s) | \| (I_d \otimes \Psi(s)) H(s) \|_{R^{d^2}} \, ds +$$

$$+ \int_t^\infty | \Phi_2(t,s) | \| (I_d \otimes \Psi(s)) H(s) \|_{R^{d^2}} \, ds \leq$$

$$\leq (\gamma + M)\, K \sup_{t \geq 0} \| (I_d \otimes \Psi(t))\, z(t) \|_{R^{d^2}}$$

and then

$$\sup_{t \geq 0} \| (I_d \otimes \Psi(t))\, z(t) \|_{R^{d^2}} \leq (\gamma + M)\, K \sup_{t \geq 0} \| (I_d \otimes \Psi(t))\, z(t) \|_{R^{d^2}},$$

which contradicts the hypothesis (4) (because $\sup_{t \geq 0} \| (I_d \otimes \Psi(t))\, z(t) \|_{R^{d^2}} \neq 0$).
This contradiction shows that the trivial solution of the equation (1) is $\Psi-$ unstable over R_+. $\square$

Remark 3.4. *1. In particular case $G = O_d$, one obtain Theorem 4, [9].*
2. For $F = O_d$, one obtain a new result in connection with $\Psi-$ instability of trivial solution of nonlinear Lyapunov matrix differential equation with integral term as right side

$$Z' = A(t)Z + ZB(t) + \int_0^t G(t,s,Z(s))ds,$$

in which the equation $Z' = A(t)Z + ZB(t)$ is $\Psi-$ unstable over R_+.
3. One know that the condition (1) of Theorem is a sufficient condition for $\Psi-$ instability of the equation (4) – see Theorems 2 and 3, [9].
If the linear Lyapunov matrix differential equation (4) is only $\Psi-$ unstable over R_+ (see Theorem 1, [9]), then equation (1) can not be $\Psi-$ unstable over R_+. This is shown by the next example, adapted from an example due to O. Perron, [18], and Example 3, [3].

Example 3.5. *Consider equation (1) with*

$$A(t) = \begin{pmatrix} \sin\ln(t+1) + \cos\ln(t+1) & be^{-\frac{1}{2}(t+1)} \\ 0 & \frac{1}{2} \end{pmatrix},\ F(t,Z) = \begin{pmatrix} 0 & -be^{-\frac{1}{2}(t+1)} \\ 0 & \frac{1}{2} \end{pmatrix}$$

$$B(t) = -I_2,\ G(t,s,Z) = O_2,$$

where $b \in R$, $b \neq 0$.
Consider

$$\Psi(t) = \begin{pmatrix} \frac{1}{2} & \\ 0 & e^{\frac{1}{2}(t+1)} \end{pmatrix}.$$

The equation (4) becomes

$$Z' = \begin{pmatrix} \sin\ln(t+1) + \cos\ln(t+1) - 1 & be^{-\frac{1}{2}(t+1)} \\ 0 & -\frac{1}{2} \end{pmatrix} Z.$$

From Example 3, [3], this equation is $\Psi-$ unstable over R_+.
On the other hand, the functions F and G satisfy the hypotheses of Theorem:

$$\mid \Psi(t)F(t,Z) \mid = \mid \Psi(t)F(t,Z)\Psi^{-1}(t)\Psi(t)Z \mid =$$

$$= \mid \begin{pmatrix} 0 & \frac{b}{2}e^{-(t+1)} \\ 0 & 0 \end{pmatrix} \Psi(t)Z \mid \leq$$

$$\leq \tfrac{b}{2}e^{-(t+1)} \mid \Psi(t)Z \mid \leq \tfrac{b}{2e} \mid \Psi(t)Z \mid,\ \text{for } t \geq 0.$$

Now, the equation (1.1) becomes

$$Z' = \begin{pmatrix} \sin\ln(t+1) + \cos\ln(t+1) - 1 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} Z. \tag{3.2}$$

A fundamental matrix for this equation is

$$U(t) = \begin{pmatrix} e^{(t+1)(\sin\ln(t+1)-1)} & 0 \\ 0 & e^{-\frac{1}{2}(t+1)} \end{pmatrix}.$$

It is easy to see that

$$\mid \Psi(t)U(t) \mid = \mid \begin{pmatrix} \frac{1}{2}e^{(t+1)(\sin \ln(t+1)-1)} & 0 \\ 0 & 1 \end{pmatrix} \mid \leq 1, \text{ for all } t \geq 0.$$

From Theorem 1, [9], the equation (3.2) is not $\Psi-$ *unstable over* R_+.

Case 3. We start from $\Psi-$ instability of equation $Z' = ZB(t)$.

Theorem 3.6. *Suppose that:*
(1). There exist supplementary projections P_1 *and* $P_2,$ $P_2, \neq O_d$*, and a positive constant* K *such that the fundamental matrices* $Y(t)$ *for (3) satisfies for all* $t \geq 0$ *the condition*

$$\int_0^t \left| \left(Y^T(t) \left(Y^T \right)^{-1}(s) \right) \otimes \left(\Psi(t) P_1 \Psi^{-1}(s) \right) \right| ds +$$

$$+ \int_t^\infty \left| \left(Y^T(t) \left(Y^T \right)^{-1}(s) \right) \otimes \left(\Psi(t) P_2 \Psi^{-1}(s) \right) \right| ds \leq K,$$

(2). The matrix function $F(t, Z)$ *satisfies the inequality*

$$\mid \Psi(t)F(t, Z) \mid \leq \gamma \mid \Psi(t)Z \mid,$$

for all $t \in \mathbb{R}_+$ *and* $Z \in \mathbb{M}_{d \times d}$*, where* γ *is a positive constant;*
(3). The matrix function $A(t)$ *satisfies the inequality*

$$\mid \Psi(t)A(t)\Psi^{-1}(t) \mid \leq a$$

for all $t \in \mathbb{R}_+$*,* a *being a positive constant;*
(4). The matrix function $G(t, s, Z)$ *satisfies the inequality*

$$\mid \Psi(t)G(t, s, Z) \mid \leq g(t, s) \mid \Psi(s)Z \mid,$$

for $t \geq s \geq 0$ *and* $Z \in \mathbb{M}_{d \times d}$*, where* $g(t, s)$ *is a continuous nonnegative function for* $t \geq s \geq 0$ *such that*

$$\int_0^t g(t, s)ds \leq M,$$

for all $t > 0$*,* M *being a positive constant;*
(5). $(a + \gamma + M) K < 1$.
Then, the trivial solution of (1.1) is $\Psi-$ *unstable over* R_+.

Proof . We may reason by reduction to absurdity. Suppose the contrary. Then by Definition, it results that the trivial solution of the equation (1) is $\Psi-$ stable over R_+. Therefore, for each $\varepsilon > 0$ and each $t_0 \in R_+$, there is a a corresponding $\delta = \delta(\varepsilon, t_0) > 0$ such that any solution $Z(t)$ of the equation (1) which satisfies the inequality $\mid \Psi(t_0)Z(t_0) \mid < \delta$, exists and satisfies the inequality $\mid \Psi(t)Z(t) \mid < \varepsilon$ for all $t \geq t_0$.
Let $z(t) = \mathcal{V}ec(Z(t))$, $t \geq 0$. From Lemma 5, [9], the function z(t) is a solution of the differential system

$$z' = \left(B^T(t) \otimes I_d \right) z + \left[\left(I_d \otimes A(t) \right) z + f(t, z) + \int_0^t g(t, s, z(s))ds \right], \tag{3.3}$$

where $f(t,z) = \mathcal{V}ec(F(t,Z))$ and $g(t,s,z) = \mathcal{V}ec(G(t,s,Z))$, for $t \geq s \geq 0$ and $Z \in \mathbb{M}_{d\times d}$.
From Lemmas 6 and 7, [9], the trivial solution of (1) is $\Psi-$ stable over R_+ if and only if the trivial solution of (7) is $I_d \otimes \Psi-$ stable over R_+.
Without loss generality, we may assume that $Y(0) = I_d$. We choose $z_0 \in R^{d^2}$, $z_0 \neq \theta$, such that $(I_d \otimes P_1)\, z_0 = \theta$ and $0 < \parallel (I_d \otimes \Psi(0))\, z_0 \parallel_{R^{d^2}} < \frac{\delta(\varepsilon,0)}{d}$.
Let $z(t)$ the solution of (3.3) such that $z(0) = z_0$. From the above results and Lemma 6, [9], we have $\parallel (I_d \otimes \Psi(t))\, z(t) \parallel_{R^{d^2}} < \varepsilon$, for $t \geq 0$ (and $\mid \Psi(t)Z(t) \mid < \varepsilon$ for all $t \geq 0$, where $Z(t) = \mathcal{V}ec^{-1}(z(t))$ is the corresponding solution of (1)).
Let $w(t)$ the vector function

$$w(t) = z(t) - \int_0^t \left(Y^T(t) \otimes I_d\right)(I_d \otimes P_1)\left(\left(Y^T\right)^{-1}(s) \otimes I_d\right) H(s)ds +$$

$$+ \int_t^\infty \left(Y^T(t) \otimes I_d\right)(I_d \otimes P_2)\left(\left(Y^T\right)^{-1}(s) \otimes I_d\right) H(s)ds,\ t \geq 0,$$

where

$$H(s) = (I_d \otimes A(s))\, z(s) + f(s, z(s)) + \int_0^s g(s,u,z(u))du,\ s \geq 0,$$

or, in other form (see Lemma 1, [9]),

$$w(t) = z(t) - \int_0^t \left[\left(Y^T(t)\left(Y^T\right)^{-1}(s)\right) \otimes P_1\right] H(s)ds +$$

$$+ \int_t^\infty \left[\left(Y^T(t)\left(Y^T\right)^{-1}(s)\right) \otimes P_2\right] H(s)ds,\ t \geq 0.$$

For $v \geq t \geq 0$,

$$\begin{aligned}
&\parallel \int_t^v \left[\left(Y^T(t)\left(Y^T\right)^{-1}(s)\right) \otimes P_2\right] H(s)ds \parallel_{R^{d^2}} = \\
&= \parallel (I_d \otimes \Psi^{-1}(t)) \int_t^v [(Y^T(t)\left(Y^T\right)^{-1}(s)) \otimes (\Psi(t)P_2\Psi^{-1}(s))](I_d \otimes \Psi(s))H(s)ds \parallel \leq \\
&\leq \mid \Psi^{-1}(t) \mid \int_t^v \mid (Y^T(t)\left(Y^T\right)^{-1}(s)) \otimes (\Psi(t)P_2\Psi^{-1}(s)) \mid \parallel (I_d \otimes \Psi(s))H(s) \parallel ds \leq \\
&\leq \varepsilon\,(a + \gamma + M) \mid \Psi^{-1}(t) \mid \int_t^v \mid (Y^T(t)\left(Y^T\right)^{-1}(s)) \otimes \Psi(t)P_2\Psi^{-1}(s) \mid ds,
\end{aligned} \tag{3.4}$$

because, for $s \geq 0$ and the solution $z(t) = \mathcal{V}ec(Z(t))$ of (3.3),

$$\begin{aligned}
&\parallel (I_d \otimes \Psi(s))\, H(s) \parallel_{R^{d^2}} = \\
&= \parallel (I_d \otimes \Psi(s))\, \mathcal{V}ec\left(A(s)Z(s) + F(s,Z(s)) + \int_0^s G(s,u,Z(u))du\right) \parallel_{R^{d^2}} \leq \\
&\leq \mid \Psi(s)A(s)\Psi^{-1}(s) \mid\mid \Psi(s)Z(s) \mid + \gamma \mid \Psi(s)Z(s) \mid + \\
&\leq \int_0^s g(s,u) \mid \Psi(u)Z(u) \mid du \leq \varepsilon\,(a + \gamma + M)
\end{aligned}$$

(see Lemma 4, [9]).
It follows that

$$\int_t^\infty \left[\left(Y^T(t)\left(Y^T\right)^{-1}(s)\right) \otimes P_2\right] H(s)ds,\ t \geq 0,$$

is an absolutely convergent integral.
It is easy to see that $w(t)$ exists on R_+ and is a continuously differentiable function on R_+.
For $t \in R_+$ and with the help of Lemma 9, [9],

$$w'(t) = z'(t) - \int_0^t \left(Y^T(t) \otimes I_d\right)' (I_d \otimes P_1) \left(\left(Y^T\right)^{-1}(s) \otimes I_d\right) H(s)ds-$$

$$-\left(Y^T(t) \otimes I_d\right) (I_d \otimes P_1) \left(\left(Y^T\right)^{-1}(t) \otimes I_d\right) H(t)+$$

$$+\int_t^\infty \left(Y^T(t) \otimes I_d\right)' (I_d \otimes P_2) \left(\left(Y^T\right)^{-1}(s) \otimes I_d\right) H(s)ds-$$

$$-\left(Y^T(t) \otimes I_d\right) (I_d \otimes P_2) \left(\left(Y^T\right)^{-1}(t) \otimes I_d\right) H(t) =$$

$$= \left(B^T(t) \otimes I_d\right) z(t) + (I_d \otimes A(t))\, z(t) + f(t, z(t)) + \int_0^t g(t, s, z(s))ds-$$

$$-\left(B^T(t) \otimes I_d\right) \int_0^t \left(Y^T(t) \otimes I_d\right) (I_d \otimes P_1) \left(\left(Y^T\right)^{-1}(s) \otimes I_d\right) H(s)ds+$$

$$+\left(B^T(t) \otimes I_d\right) \int_t^\infty \left(Y^T(t) \otimes I_d\right) (I_d \otimes P_2) \left(\left(Y^T\right)^{-1}(s) \otimes I_d\right) H(s)ds-$$

$$-\left(Y^T(t) \otimes I_d\right) [I_d \otimes (P_1 + P_2)] \left(\left(Y^T\right)^{-1}(t) \otimes I_d\right) H(t) =$$

$$= \left(B^T(t) \otimes I_d\right) z(t) + \left(B^T(t) \otimes I_d\right) (w(t) - z(t)) = \left(B^T(t) \otimes I_d\right) w(t).$$

Thus, $w(t)$ is a solution on R_+ of the linear equation $u' = \left(B^T(t) \otimes I_d\right) u$.
On the other hand, from Lemma 6, [9], we have, for $t \geq 0$,

$$\| (I_d \otimes \Psi(t))\, w(t) \|_{R^{d^2}} \leq \| (I_d \otimes \Psi(t))\, z(t) \|_{R^{d^2}} +$$

$$+\int_0^t | (Y^T(t) \left(Y^T\right)^{-1}(s)) \otimes (\Psi(t) P_1 \Psi^{-1}(s)) | \| (I_d \otimes \Psi(s)) H(s) \| ds+$$

$$+\int_t^\infty | (Y^T(t) \left(Y^T\right)^{-1}(s)) \otimes (\Psi(t) P_2 \Psi^{-1}(s)) | \| (I_d \otimes \Psi(s)) H(s) \| ds \leq$$

$$\leq \varepsilon + \varepsilon\,(a + \gamma + M)\, K.$$

This shows that the solution $w(t)$ is $I_d \otimes \Psi(t)-$ bounded on R_+.
From Lemma 9, [9],

$$w(t) = \left(Y^T(t) \otimes I_d\right) \left(\left(Y^T\right)^{-1}(0) \otimes I_d\right) w(0) =$$

$$= \left(Y^T(t) \otimes I_d\right) [I_d \otimes (P_1 + P_2)]\, w(0) =$$

$$= \left(Y^T(t) \otimes I_d\right) (I_d \otimes P_2)\, w(0).$$

If $(I_d \otimes P_2)\, w(0) \neq 0$, from hypothesis (1) and Lemma 11, [9], it follows that

$$\limsup_{t\to\infty} \| (I_d \otimes \Psi(t))\, w(t) \|_{R^{d^2}} = +\infty.$$

This contradicts the $I_d \otimes \Psi(t)$-boundedness of $w(t)$ on R_+. Thus, $(I_d \otimes P_2)\, w(0) = 0$ and then $w(t) = 0$ on R_+. Therefore, for $t \geq 0$,

$$z(t) = \int_0^t \left[\left(Y^T(t) \left(Y^T\right)^{-1}(s) \right) \otimes P_1 \right] H(s)ds -$$

$$- \int_t^\infty \left[\left(Y^T(t) \left(Y^T\right)^{-1}(s) \right) \otimes P_2 \right] H(s)ds, \ t \geq 0.$$

From this, for $t \geq 0$,

$$\| \left(I_d \otimes \Psi(t)\right) z(t) \|_{R^{d^2}} \leq$$

$$\leq \int_0^t | \left(Y^T(t) \left(Y^T\right)^{-1}(s)\right) \otimes \left(\Psi(t) P_1 \Psi^{-1}(s)\right) | \| \left(I_d \otimes \Psi(s)\right) H(s) \|_{R^{d^2}} \ ds +$$

$$+ \int_t^\infty | \left(Y^T(t) \left(Y^T\right)^{-1}(s)\right) \otimes \left(\Psi(t) P_2 \Psi^{-1}(s)\right) | \| \left(I_d \otimes \Psi(s)\right) H(s) \|_{R^{d^2}} \ ds \leq$$

$$\leq (a + \gamma + M)\, K \sup_{t \geq 0} \| \left(I_d \otimes \Psi(t)\right) z(t) \|_{R^{d^2}}$$

and then

$$\sup_{t \geq 0} \| \left(I_d \otimes \Psi(t)\right) z(t) \|_{R^{d^2}} \leq (a + \gamma + M)\, K \sup_{t \geq 0} \| \left(I_d \otimes \Psi(t)\right) z(t) \|_{R^{d^2}},$$

which contradicts the hypothesis (5) (because $\sup_{t \geq 0} \| \left(I_d \otimes \Psi(t)\right) z(t) \|_{R^{d^2}} \neq 0$). This contradiction shows that the trivial solution of the equation (1) is Ψ-unstable over R_+. $\square$

Remark 3.7. *For $F = O_d$, one obtain a new result in connection with $\Psi-$ instability of trivial solution of the Lyapunov nonlinear matrix differential equation with integral term as right side*

$$Z' = A(t)Z + ZB(t) + \int_0^t G(t, s, Z(s))ds,$$

in which the equation $Z' = ZB(t)$ is $\Psi-$ unstable over R_+.

Remark 3.8. *The above Theorems have very useful corollaries in the particular cases when $g(t, s) = h(t)g(s)$ or $g(t, s) = k(t - s)$.*

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