Int. J. Nonlinear Anal. Appl. 12 (2021) No. 2, 219-235
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2019.17188.1913

Dhage iteration principle for IVPs of nonlinear first order impulsive functional differential equations

Bapurao C. Dhage[a,*], Janhavi B. Dhage[a]

[a]Kasubai, Gurukul Colony, Thodga Road, Ahmedpur-413 515, Dist: Latur, Maharashtra, India

(Communicated by Madjid Eshaghi Gordji)

Abstract

In this paper we prove the existence and approximation theorems for the initial value problems of first order nonlinear impulsive functional differential equations under certain mixed partial Lipschitz and partial compactness type conditions. Our results are based on the Dhage monotone iteration principle embodied in a hybrid fixed point theorem of Dhage involving the sum of two monotone order preserving operators in a partially ordered Banach space. The novelty of the present approach lies the fact that we obtain an algorithm for the solution. Our abstract main result is also illustrated by indicating a numerical example.

Keywords: Impulsive functional differential equation; Dhage iteration method; hybrid fixed point principle; existence and approximate solution.
2010 MSC: Primary 34A12; Secondary 47H10.

1. Introduction

The nice blend of two characteristics, namely, the dependence upon back history and jumps or the sudden changes of the complex dynamic systems at some intervals leads to the consideration of mathematical models of nonlinear impulsive functional differential equations. Therefore, any universal phenomenon depending upon its past history and the jumps at finite number of points can be represented better with the help of nonlinear impulsive functional differential equations. The importance of both the aspects of delay and jumps in the study of behavior of such dynamic systems as well as exhaustive account of various topics related to this problem may be found in the research monographs of Hale [21], Samoilenko and Perestyuk [25] Bainov and Simenov [1], Lakshmikantam et

*Corresponding author
Email addresses: bcdhage@gmail.com (Bapurao C. Dhage), jbdhage@gmail.com (Janhavi B. Dhage)

Received: February 2019 *Accepted:* November 2019

al [24] and the references therein. The various topics such as existence, uniqueness and stability theory for nonlinear impulsive functional differential equations have received much attention during the last two decade, but the theory of approximation of the solutions via construction of the algorithms for such nonlinear equations is relatively rare in the literature. The existence and approximation theorems for nonlinear functional differential equations are studied in Dhage [7, 8, 9] whereas such results have been obtained for impulsive differential equations in Dhage [14, 15] via Dhage iteration method. Hence, it is desirable to extend this new iteration method to nonlinear impulsive functional differential equations. The existence theorems so far discussed in the literature for such nonlinear impulsive functional differential equations involve either the use of usual Lipschitz or compactness type condition on the nonlinearities which are considered to be very strong conditions in the subject of nonlinear analysis. Here in the present set up of new Dhage iteration method, we do not need usual Lipschitz and compactness type conditions but require only partial Lipschitz and partial compactness type conditions of the nonlinearity and the existence as well as approximation of the solutions is obtained on whole interval under certain monotonic conditions. We claim that the results of this paper are new to the literature on impulsive functional differential equations.

Let $\mathbb{R}$ be the real line and let $I_0 = [-r, 0]$ be a closed interval in $\mathbb{R}$ for some real number $r > 0$. We denote the class $C(I_0, \mathbb{R})$ of continuous real-valued functions on I_0 by $\mathcal{C}$ called the history space which is obviously a Banach space under the supremum norm $\|\cdot\|_{\mathcal{C}}$ defined by

$$\|\cdot\|_{\mathcal{C}} = \sup_{t\in I_0} |x(t)|.$$

Again, let $I = [0, T]$ be a closed and bounded interval in $\mathbb{R}$. Suppose that $t_0, \ldots t_{p+1}$ are the points in I such that $0 = t_0 < t_1 < \cdots, < t_p < t_{p+1} = T$ and let $I' = I \setminus \{t_1, \ldots, t_p\}$. Denote $I_j = (t_j, t_{j+1}) \subset I$ for $j = 1, 2, \ldots, p$. Similarly, denote $J = [-r, T] = I_0 \cup I$ and $J' = I_0 \cup I'$. For a given $t \in I'$, we define a function $x_t \in \mathcal{C}$ by $x_t(\theta) = x(t+\theta)$. By $X = C(I, \mathbb{R})$ and $L^1(I, \mathbb{R})$ we denote respectively the spaces of continuous and Lebesgue integrable real-valued functions defined on J.

Now, given the functions $h \in L^1(I, \mathbb{R}_+)$ and $\varphi \in \mathcal{C}$, we consider the initial value problem (in short IVP) of nonlinear the first order impulsive functional differential equation (in short IFDE)

$$\left.\begin{aligned} x'(t) + h(t)x(t) = f(t, x_t),\ \ t \in I \setminus \{t_1, \ldots, t_p\}, \\ x(t_j^+) - x(t_j^-) = \mathcal{I}_j(x(t_j)), \\ x_0 = \varphi, \end{aligned}\right\} \tag{1.1}$$

where, the limits $x(t_j^+)$ and $x(t_j^-)$ are respectively the right and left limit of x at $t = t_j$ such that $x(t_j) = x(t_j^-)$, $\mathcal{I}_j \in C(\mathbb{R}, \mathbb{R})$, $\mathcal{I}_j(x(t_j))$ are the impulsive effects at the points $t = t_j$, $j = 1, \ldots, p$ and $f : I \times \mathcal{C} \to \mathbb{R}$ is such that the function $t \to f(t, x)$ is continuous on $I' = I - \{t_1, ..., t_p\}$ for each $x \in \mathbb{R}$, and there exist the limits

$$\lim_{t\to t_j^-} f(t, u) = f(t_j, u) \quad \text{and} \quad \lim_{t\to t_{j+}} f(t, u),\ u \in \mathcal{C},$$

for each $j = 1, \ldots, p$.

By a *impulsive solution* of the IFDE (1.1) we mean a function $x \in PC^1(J, \mathbb{R})$ that satisfies the functional differential equation and the conditions in (1.1), where $PC^1(J, \mathbb{R})$ is the space of piecewise continuously differentiable real-valued functions defined on J.

The IFDE (1.1) has already been discussed in the literature under continuity and compactness type conditions of the function f for various aspects of the solutions. The existence and unqueness theorems for the IFDE (1.1) may be proved using the classical and hybrid fixed point theorems of Schauder, Banach and Dhage given in Dhage [2, 11] and references therein. Here in the present study, we discuss the IFDE (1.1) for existence and approximate impulsive solution under partial Lipschit and partial compactness type conditions via Dhage iteration method based on a hbrid fixed point theorems of Dhage [4, 5].

2. Auxiliary Results

Throughout this paper, unless otherwise mentioned, let $(E, \preceq, \|\cdot\|)$ denote a partially ordered normed linear space. Two elements x and y in E are said to be **comparable** if either the relation $x \preceq y$ or $y \preceq x$ holds. A non-empty subset C of E is called a **chain** or **totally ordered** if all the elements of C are comparable. It is known that E is **regular** if $\{x_n\}$ is a nondecreasing (resp. nonincreasing) sequence in E such that $x_n \to x^*$ as $n \to \infty$, then $x_n \preceq x^*$ (resp. $x_n \succeq x^*$) for all $n \in \mathbb{N}$. The conditions guaranteeing the regularity of E may be found in Heikkilä and Lakshmikantham [23] and the references therein.

We need the following definitions (see Dhage [3, 4, 5] and the references therein) in what follows.

A mapping $\mathcal{T} : E \to E$ is called **isotone** or **monotone nondecreasing** if it preserves the order relation $\preceq$, that is, if $x \preceq y$ implies $\mathcal{T}x \preceq \mathcal{T}y$ for all $x, y \in E$. Similarly, $\mathcal{T}$ is called **monotone nonincreasing** if $x \preceq y$ implies $\mathcal{T}x \succeq \mathcal{T}y$ for all $x, y \in E$. Finally, $\mathcal{T}$ is called **monotonic** or simply **monotone** if it is either monotone nondecreasing or monotone nonincreasing on E. A mapping $\mathcal{T} : E \to E$ is called **partially continuous** at a point $a \in E$ if for given $\epsilon > 0$ there exists a $\delta > 0$ such that $\|\mathcal{T}x - \mathcal{T}a\| < \epsilon$ whenever x is comparable to a and $\|x - a\| < \delta$. $\mathcal{T}$ is called partially continuous on E if it is partially continuous at every point of it. It is clear that if $\mathcal{T}$ is partially continuous on E, then it is continuous on every chain C contained in E and vice-versa. A non-empty subset S of the partially ordered metric space E is called **partially bounded** if every chain C in S is bounded. A mapping $\mathcal{T}$ on a partially ordered metric space E into itself is called **partially bounded** if $\mathcal{T}(E)$ is a partially bounded subset of E. $\mathcal{T}$ is called **uniformly partially bounded** if all chains C in $\mathcal{T}(E)$ are bounded by a unique constant. A non-empty subset S of the partially ordered metric space E is called **partially compact** if every chain C in S is a compact subset of E. A mapping $\mathcal{T} : E \to E$ is called **partially compact** if every chain C in $\mathcal{T}(E)$ is a relatively compact subset of E. $\mathcal{T}$ is called **uniformly partially compact** if $\mathcal{T}$ is a uniformly partially bounded and partially compact operator on E. $\mathcal{T}$ is called **partially totally bounded** if for any bounded subset S of E, $\mathcal{T}(S)$ is a partially totally bounded subset of E. If $\mathcal{T}$ is partially continuous and partially totally bounded, then it is called **partially completely continuous** on E.

Remark 2.1. *Suppose that $\mathcal{T}$ is a monotone operator on E into itself. Then $\mathcal{T}$ is a partially bounded or partially compact on E if $\mathcal{T}(C)$ is a bounded or compact subset of E for each chain C in E.*

Definition 2.2 (Dhage [6, 7]). *The order relation $\preceq$ and the metric d on a non-empty set E are said to be $\mathcal{D}$-**compatible** if $\{x_n\}$ is a monotone sequence, that is, monotone nondecreasing or monotone nonincreasing sequence in E and if a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x^* implies that the original sequence $\{x_n\}$ converges to x^*. Similarly, given a partially ordered normed linear space $(E, \preceq, \|\cdot\|)$, the order relation $\preceq$ and the norm $\|\cdot\|$ are said to be $\mathcal{D}$-compatible if $\preceq$ and the metric d defined through the norm $\|\cdot\|$ are $\mathcal{D}$-compatible. A subset S of E is called **Janhavi set** if the order relation $\preceq$ and the metric d or the norm $\|\cdot\|$ are $\mathcal{D}$-compatible in it. In particular, if $S = E$, then E is called a **Janhavi metric space** or **Janhavi Banach space**.*

Clearly, the set $\mathbb{R}$ of real numbers with usual order relation $\leq$ and the norm defined by the absolute value function $|\cdot|$ has this property. Similarly, the finite dimensional Euclidean space $\mathbb{R}^n$ with usual componentwise order relation and the standard norm possesses the compatibility property and so is a **Janhavi Banach space**.

Definition 2.3. *An upper semi-continuous and monotone nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a $\mathcal{D}$-function provided $\psi(0) = 0$. A monotone operator $\mathcal{T} : E \to E$ is called nonlinear partial $\mathcal{D}$-contraction if there exists a $\mathcal{D}$-function ψ such that*

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \psi\big(\|x - y\|\big) \tag{2.1}$$

for all comparable elements $x, y \in E$, where $0 < \psi(r) < r$ for $r > 0$. In particular, if $\psi(r) = k\,r$, $k > 0$, $\mathcal{T}$ is called a partial Lipschitz operator with a Lipschitz constant k and moreover, if $0 < k < 1$, $\mathcal{T}$ is called a linear partial contraction on E with the contraction constant k.

The **Dhage monotone iteration principle** or **Dhage monotone iteration method** embodied in the following applicable hybrid fixed point theorems of Dhage [4] in a partially ordered normed linear space is used as a key tool for our work contained in this paper. The details of the Dhage monotone iteration principle or method along with some nice applications are given in Dhage [6, 7, 8], Dhage and Dhage [16, 17], Dhage *et al.* [18, 19], Dhage and Otrocol [20] and the references therein.

Theorem 2.4 (Dhage [4, 5]). *Let $(E, \preceq, \|\cdot\|)$ be a partially ordered Banach space and let $\mathcal{T} : E \to E$ be a monotone nondecreasing and nonlinear partial $\mathcal{D}$-contraction. Suppose that there exists an element $x_0 \in E$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$. If $\mathcal{T}$ is continuous or E is regular, then $\mathcal{T}$ has a unique comparable fixed point x^* and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges monotonically to x^*. Moreover, the fixed point x^* is unique if every pair of elements in E has a lower bound or an upper bound.*

Theorem 2.5 (Dhage [4, 5]). *Let $\big(E, \preceq, \|\cdot\|\big)$ be a regular partially ordered complete normed linear space and let every compact chain C in E be s Janhavi set. Suppose that $\mathcal{A}, \mathcal{B} : E \to E$ are two monotone nondecreasing operators such that*

(a) $\mathcal{A}$ is partially bounded and nonlinear partial $\mathcal{D}$-contraction,

(b) $\mathcal{B}$ is partially continuous and partially compact, and

(c) there exists an element $x_0 \in E$ such that $x_0 \preceq \mathcal{A}x_0 + \mathcal{B}x_0$ or $x_0 \succeq \mathcal{A}x_0 + \mathcal{B}x_0$.

Then the hybrid operator equation $\mathcal{A}x + \mathcal{B}x = x$ has a solution x^* in E and the sequence $\{x_n\}$ of successive iterations defined by $x_{n+1} = \mathcal{A}x_n + \mathcal{B}x_n$, n=0,1,..., converges monotonically to x^*.

Remark 2.6. *The condition that every compact chain of E is Janhavi holds if every partially compact subset of E possesses the compatibility property with respect to the order relation $\preceq$ and the norm $\|\cdot\|$ in it.*

Remark 2.7. *We remark that hypothesis (a) of Theorem 2.5 implies that the operator $\mathcal{A}$ is partially continuous and consequently both the operators $\mathcal{A}$ and $\mathcal{B}$ in the theorem are partially continuous on E. The regularity of E in above Theorems 2.4 and 2.5 may be replaced with a stronger continuity condition respectively of the operators $\mathcal{T}$ and $\mathcal{A}$ and $\mathcal{B}$ on E which are the results proved in Dhage [3, 4].*

3. Existence and Approximation Theorems

Let $X_j = C(I_j, \mathbb{R})$ denote the class of continuous real-valued functions on the interval $I_j = (t_j, t_{j+1})$. Denote by $PC(J, \mathbb{R})$ the space of piecewise continuous real-valued functions on $J = I_0 \cup I$ defined by

$$PC(J,\mathbb{R}) = \Big\{x \in \mathcal{C} \cap X_j \mid x(t_j^-) \text{ and } x(t_j^+) \text{ exists for } j = 1, \dots, p; \\ \text{and } x(t_j^-) = x(t_j)\Big\}. \tag{3.1}$$

Define a supremum norm $\|\cdot\|$ in $PC(J, \mathbb{R})$ by

$$\|x\|_{PC} = \sup_{t\in J} |x(t)| \tag{3.2}$$

and define the order cone K in $PC(J, \mathbb{R})$ by

$$K = \{x \in PC(J, \mathbb{R}) \mid x(t) \geq 0 \text{ for all } t \in J\}, \tag{3.3}$$

which is obviously a normal cone in $PC(J, \mathbb{R})$. Now, define the order relation $\preceq$ in $PC(J, \mathbb{R})$ by

$$x \preceq y \iff y - x \in K \tag{3.4}$$

which is equivalent to

$$x \preceq y \iff x(t) \leq y(t) \text{ for all } t \in J.$$

Clearly, $\big(PC(J, \mathbb{R}), K\big)$ becomes a regular ordered Banach space with respect to the above norm and order relation in $PC(J, \mathbb{R})$ and every compact chain C in $PC(J, \mathbb{R})$ is a Janhavi set in view of the following lemmas proved in Dhage [7, 8].

Lemma 3.1 (Dhage [12, 13]). *Every ordered Banach space (E, K) is regular.*

Lemma 3.2 (Dhage [12, 13]). *Every partial compact subset S of an ordered Banach space (E, K) is a Janhavi set in E.*

We introduce an order relation $\preceq_{\mathcal{C}}$ in the history space $\mathcal{C}$ induced by the order relation $\preceq$ defined in $PC(J, \mathbb{R})$.Thus, for any $x, y \in \mathcal{C}, x \preceq_{\mathcal{C}} y$ implies $x(\theta) \preceq y(\theta)$ for all $\theta \in [-r, 0]$. Moreover, if $x, y \in PC(J, \mathbb{R})$ and $x \preceq y$, then $x_t \preceq_{\mathcal{C}} y_t$ for all $t \in [0, T]$ (cf. Dhage [10, 11] and references therein).

We need the following definition in what follows.

Definition 3.3. *A function $u \in PC^1(J, \mathbb{R})$ is said to be a lower impulsive solution of the IFDE (1.1) if it satisfies*

$$\left.\begin{array}{c} u'(t) + h(t)u(t) \leq f(t, u_t), \ \ t \in I \setminus \{t_1, \dots, t_p\}, \\ u(t_j^+) - u(t_j^-) \leq \mathcal{I}_j(u(t_j)), \\ u_0 \leq \varphi, \end{array}\right\}$$

for $j = 1, 2, ..., p$. Similarly, a function $v \in PC^1(J, \mathbb{R})$ is called an upper impulsive solution of the IFDE (1.1) if the above inequality is satisfied with reverse sign.

We consider the following set of assumptions in what follows:

(H$_1$) The impulsive functions $\mathcal{I}_j \in C(\mathbb{R},\mathbb{R})$ are bounded on X with bounds $M_{\mathcal{I}_j}$ for each $j = 1,\ldots,p.$,

(H$_2$) There exists a constants $L_{\mathcal{I}_j} > 0$ such that

$$0 \le \mathcal{I}_j x - \mathcal{I}_j y \le L_{\mathcal{I}_j}\big(x-y\big)$$

for all $x, y \in \mathbb{R}$, $x \ge y$, where $j = 1,\ldots,p$.

(H$_3$) The function f is bounded on $I \times \mathcal{C}$ with bound M_f.

(H$_4$) $f(t,x)$ is nondecreasing in x for each $t \in J$.

(H$_5$) There exists a constant $L_f > 0$ such that

$$0 \le f(t,x) - f(t,y) \le L_f\big(x-y\big)$$

for all $t \in J$ and $x, y \in \mathcal{C}$, $x \ge_{\mathcal{C}} y$.

(LS) The IFDE (1.1) has a lower impulsive solution $u \in PC^1(J,\mathbb{R})$.

(US) The IFDE (1.1) has an upper impulsive solution $v \in PC^1(J,\mathbb{R})$.

Below we prove some useful results in what follows.

Lemma 3.4. *Given $\sigma \in L^1(I,\mathbb{R})$, a function $x \in PC(J,\mathbb{R})$ is a impulsive solution to the IFDE*

$$\left.\begin{array}{c} x'(t) + h(t)x(t) = \sigma(t), \ \ t \in I \setminus \{t_1,\ldots,t_p\}, \\ x(t_j^+) - x(t_j^-) = \mathcal{I}_j(x(t_j)), \\ x_0 = \varphi, \end{array}\right\} \tag{3.5}$$

if and only if it is an impulsive solution of the impulsive integral equation

$$x(t) = \begin{cases} \varphi(0)\, e^{-H(t)} + \displaystyle\sum_{0<t_j<t} k(t,t_j)\mathcal{I}_j(x(t_j)) + \int_0^t k(t,s)\sigma(s)\,ds, \ \ t \in I, \\ \varphi(t), \ \ t \in I_0, \end{cases} \tag{3.6}$$

where the kernel function k is given by

$$k(t,s) = e^{-H(t)+H(s)} \quad \text{and} \quad H(t) = \int_0^t h(s)\,ds. \tag{3.7}$$

Proof . The proof of second case in the expression (3.6) is obvious, so we prove only the first case. First note that the integral in $H(t)$ is a continuous and nonnegative real-valued function on J. Therefore, we have $H(t) > 0$ on I provided h is not an identically zero function. Otherwise $H(t) \equiv 0$ on I. Moreover, we have $H(t^-) = H(t) = H(t^+)$ for all $t \in I$.

First suppose that x is an impulsive solution of the IFDE (3.5) on J. Then, we have

$$\left.\begin{array}{c} \big(e^{H(t)}x(t)\big)' = e^{H(t)}\sigma(t), \ \ t \in J \setminus \{t_1,\ldots,t_p\}, \\ x(t_j^+) - x(t_j^-) = \mathcal{I}_j(x(t_j)), \\ x_0 = \varphi, \end{array}\right\} \tag{3.8}$$

for $j = 1, 2, \dots, p$.

From the theory of integral calculus, it follows that

$$\begin{aligned} e^{H(t_1^-)}x(t_1^-) - e^{H(0)}x(0) &= \int_0^{t_1} \left(e^{H(s)}x(s)\right)' \, ds \\ e^{H(t_2^-)}x(t_2^-) - e^{H(t_1^+)}x(t_1^+) &= \int_{t_1}^{t_2} \left(e^{H(s)}x(s)\right)' \, ds \\ &\vdots \\ e^{H(t)}x(t) - e^{H(t_p^+)}x(t_p^+) &= \int_{t_p}^{t} \left(e^{H(s)}x(s)\right)' \, ds. \end{aligned}$$

Summing up the above equations,

$$e^{H(t)}x(t) - \sum_{0<t_j<t} e^{H(t_j)}\mathcal{I}_j(x(t_j)) = \varphi(0) + \int_0^t e^{H(s)}h(s)\, ds,$$

or

$$x(t) = \varphi(0)e^{-H(t)} + \sum_{0<t_j<t} k(t, t_j)\mathcal{I}_j(x(t_j)) + \int_0^t k(t, s)\sigma(s)\, ds.$$

for $t \in I$.

Conversely, suppose that x is an impulsive solution of the impulsive integral equation (3.6). Obviously x satisfies the initial and jump conditions given in (3.5). By definition of the kernel function k, we obtain

$$e^{H(t)}x(t) = \varphi(0) + \sum_{0<t_j<t} e^{H(t_j)}\mathcal{I}_j(x(t_j)) + \int_0^t e^{H(s)}\sigma(s)\, ds \tag{3.9}$$

for all $t \in I$. Since $\sigma \in L^1(I, \mathbb{R})$, one has $\int_0^t e^{H(s)}\sigma(s)\, ds \in AC(I, \mathbb{R})$. So, by a direct differentiation of (3.9) yields,

$$\left(e^{H(t)}x(t)\right)' = e^{H(t)}\sigma(t),$$

or

$$x'(t) + h(t)x(t) = \sigma(t),$$

for $t \in I$ satisfying $x(0) = \varphi(0)$ and (3.3). The proof of the lemma is complete. □

Remark 3.5. *We note that the kernal function $k(t, s)$ is continuous and nonnegative real-valued function on $I \times I$. Moreover, $\sup_{t>s} k(t, s) \le 1$.*

Lemma 3.6. *Given $\sigma \in L^1(I, \mathbb{R})$, if there is a function $u \in PC^1(J, \mathbb{R})$ satisfying the impulsive functional differential inequality*

$$\left.\begin{aligned} u'(t) + h(t)u(t) \le \sigma(t),\ \ t \in I \setminus \{t_1, \dots, t_p\}, \\ u(t_j^+) - u(t_j^-) \le \mathcal{I}_j(u(t_j)), \\ u_0 \preceq_C \varphi, \end{aligned}\right\} \tag{3.10}$$

then it satisfies the impulsive integral inequality

$$u(t) \le \begin{cases} \varphi(0)e^{-H(t)} + \sum\limits_{0<t_j<t} k(t,t_j)\mathcal{I}_j(u(t_j)) + \int_0^t k(t,s)\sigma(s)\,ds, & \text{if } t \in I, \\ \varphi(t), & \text{if } t \in I_0, \end{cases} \tag{3.11}$$

where the kernel function k is defined by the expression (3.7) *on $I \times I$.*

Proof . The proof of second case in the expression (3.11) is obvious, so we prove only the first case. Proceeding as in the proof of Lemma 3.4, we obtain

$$\left.\begin{array}{r} \left(e^{H(t)}u(t)\right)' \le e^{H(t)}\sigma(t), \;\; t \in J \setminus \{t_1,\dots,t_p\}, \\ u(t_j^+) - u(t_j^-) \le \mathcal{I}_j(u(t_j)), \\ u_0 \le_{\mathcal{C}} \varphi,, \end{array}\right\}$$

for $j = 1, 2, \dots, p$.

From the theory of integral calculus, it follows that

$$\begin{aligned} e^{H(t_1^-)}u(t_1^-) - e^{H(0)}u(0) &= \int_0^{t_1} \left(e^{H(s)}u(s)\right)' ds \\ e^{H(t_2^-)}u(t_2^-) - e^{H(t_1^+)}u(t_1^+) &= \int_{t_1}^{t_2} \left(e^{H(s)}u(s)\right)' ds \\ &\vdots \\ e^{H(t)}u(t) - e^{H(t_p^+)}u(t_p^+) &= \int_{t_p}^{t} \left(e^{H(s)}u(s)\right)' ds. \end{aligned}$$

Summing up the above equations,

$$e^{H(t)}u(t) - \sum_{0<t_j<t} e^{H(t_j)}\mathcal{I}_j(u(t_j)) \le \varphi(0) + \int_0^t e^{H(s)}h(s)\,ds,$$

or

$$u(t) \le \varphi(0)\,e^{-H(t)} + \sum_{0<t_j<t} k(t,t_j)\mathcal{I}_j(u(t_j)) + \int_0^t k(t,s)\sigma(s)\,ds$$

for $t \in I$ and the proof of the lemma is complete. □

Similarly, we have the following useful result concerning the impulsive functional differential inequality with reverse sign.

Lemma 3.7. *Given $\sigma \in L^1(J,\mathbb{R})$, if there is a function $v \in PC^1(J,\mathbb{R})$ satisfying the impulsive functional differential inequality*

$$\left.\begin{array}{r} v'(t) + h(t)v(t) \ge \sigma(t), \;\; t \in I \setminus \{t_1,\dots,t_p\}, \\ v(t_j^+) - v(t_j^-) \ge \mathcal{I}_j(v(t_j)), \\ v_0 \ge_{\mathcal{C}} \varphi, \end{array}\right\} \tag{3.12}$$

then it satisfies the impulsive integral inequality

$$v(t) \ge \begin{cases} \varphi(0)e^{-H(t)} + \sum\limits_{0<t_j<t} k(t,t_j)\mathcal{I}_j(v(t_j)) + \int_0^t k(t,s)\sigma(s)\,ds, & t \in I, \\ \varphi(t), & t \in I_0, \end{cases} \tag{3.13}$$

where the kernel function k is defined by the expression (3.7) *on $I \times I$.*

Theorem 3.8. *Suppose that hypotheses (H_1)-(H_4) and (LS) hold. Then the IFDE (1.1) has a impulsive solution x^* defined on J and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by*

$$x_0(t) = u(t), \ \ t \in J,$$
$$x_{n+1}(t) = \begin{cases} \varphi(0)e^{-H(t)} + \sum\limits_{0<t_j<t} k(t,t_j)\mathcal{I}_j(x_n(t_j)) + \int_0^t k(t,s)f(s,x_n(s))\,ds, \ \ t \in I, \\ \varphi(t), \ \ t \in I_0, \end{cases} \tag{3.14}$$

converges monotone nondecreasingly to x^*.

Proof . Set $E = PC(J,\mathbb{R})$. Then, by Lemma 3.2, every compact chain C in E possesses the compatibility property with respect to the norm $\|\cdot\|_{PC}$ and the order relation $\preceq$ so that every compact chain C in E is a Janhavi set.

Now, by Lemma 3.4, the IFDE (1.1) is equivalent to the nonlinear impulsive functional integral equation (in short IFIE)

$$x(t) = \begin{cases} \varphi(0)e^{-H(t)} + \sum\limits_{0<t_j<t} k(t,t_j)\mathcal{I}_j(x(t_j)) + \int_0^t k(t,s)f(s,x(s))\,ds, \ \ t \in I, \\ \varphi(t), \ \ t \in I_0. \end{cases} \tag{3.15}$$

Define two operators $\mathcal{A}$ and $\mathcal{B}$ on E by

$$\mathcal{A}x(t) = \begin{cases} \sum\limits_{0<t_j<t} k(t,t_j)\mathcal{I}_j(x(t_j)), \ t \in I, \\ 0, \ t \in I_0, \end{cases} \tag{3.16}$$

and

$$\mathcal{B}x(t) = \begin{cases} \varphi(0)\,e^{-H(t)} + \int_0^t k(t,s)f(s,x(s))\,ds, \ t \in I, \\ \varphi(t), \ t \in I_0. \end{cases} \tag{3.17}$$

From the continuity of the integral, it follows that $\mathcal{A}$ and $\mathcal{B}$ define the operators $\mathcal{A},\mathcal{B} : E \to E$ and the impulsive integral equation (3.15) is transformed into the operator equation as

$$\mathcal{A}x(t) + \mathcal{B}x(t) = x(t), \ \ t \in J. \tag{3.18}$$

Now, the problem of finding the impulsive solution of the IFDE (1.1) is just reduced to finding impulsive solution of the operator equation (3.18) on J. We show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 2.5 in a series of following steps.

Step I: *$\mathcal{A}$ and $\mathcal{B}$ are nondecreasing on E.*

Let $x, y \in E$ be such that $x \succeq y$. Then, by hypothesis (H_2), we get

$$\begin{aligned} \mathcal{A}x(t) &= \begin{cases} \sum\limits_{0<t_j<t} k(t,t_j)\mathcal{I}_j(x(t_j)), \ t \in I, \\ 0, \ t \in I_0, \end{cases} \\ &\geq \begin{cases} \sum\limits_{0<t_j<t} k(t,t_j)\mathcal{I}_j(y(t_j)), \ t \in I, \\ 0, \ t \in I_0, \end{cases} \end{aligned}$$

$$= \mathcal{A}y(t),$$

for all $t \in J$. By definition of the order relation in E, we obtain $\mathcal{A}x \succeq \mathcal{A}y$ and a fortiori, $\mathcal{A}$ is a nondecreasing operator on E. Similarly, using the hypothesis (H$_4$),

$$\begin{aligned}
\mathcal{B}x(t) &= \begin{cases} \varphi(0)\, e^{-H(t)} + \displaystyle\int_0^t k(t,s) f(s,x(s))\, ds, \ t \in I, \\ \varphi(t), \ t \in I_0, \end{cases} \\
&\geq \begin{cases} \varphi(0)\, e^{-H(t)} + \displaystyle\int_0^t k(t,s) f(s,x(s))\, ds, \ t \in I, \\ \varphi(t), \ t \in I_0, \end{cases} \\
&= \mathcal{B}y(t),
\end{aligned}$$

for all $t \in I$. Therefore, the operator $\mathcal{B}$ is also nondecreasing on E into itself.

Step II: *$\mathcal{A}$ is partially bounded and partially contraction on E.*

Let $x \in E$ be arbitrary. Then by (H_1) we have

$$|\mathcal{A}x(t)| \leq \left| \sum_{0<t_j<t} k(t,t_j) \mathcal{I}_j(x(t_j)) \right| \leq \sum_{0<t_j<t} \big|k(t,t_j)\big| \, \big|\mathcal{I}_j(y(t_j))\big| \leq \sum_{j=1}^{p} M_{\mathcal{I}_j}$$

for all $t \in J$. Taking the supremum over t, we obtain $\|\mathcal{A}x\| \leq \sum_{j=1}^{p} M_{\mathcal{I}_j}$ for all $x \in E$, so $\mathcal{A}$ is a bounded operator on E. This further implies that $\mathcal{A}$ is partially bounded on E.

Next, let $x, y \in E$ be such that $x \succeq y$. Then by (H_2), we have

$$\begin{aligned}
|\mathcal{A}x(t) - \mathcal{A}y(t)| &\leq \left| \sum_{0<t_j<t} k(t,t_j) \mathcal{I}_j(x(t_j)) - \sum_{0<t_j<t} k(t,t_j) \mathcal{I}_j(y(t_j)) \right| \\
&\leq \left| \sum_{0<t_j<t} k(t,t_j) \big[\mathcal{I}_j(x(t_j)) - \mathcal{I}_j(x(t_j))\big] \right| \\
&\leq \sum_{0<t_j<t} k(t,t_j)\, L_{\mathcal{I}_j} \big[x(t_j) - x(t_j)\big] \\
&\leq L\, \|x - y\|_{PC},
\end{aligned}$$

for all $t \in J$, where $L = \sum_{j=1}^{p} L_{\mathcal{I}_j} < 1$. Taking the supremum over t, we obtain

$$\|\mathcal{A}x - \mathcal{A}y\|_{PC} \leq L\, \|x - y\|_{PC}$$

for all $x, y \in E$ with $xx \succeq y$. Hence $\mathcal{A}$ is a partially contraction on E which also implies that $\mathcal{A}$ is partially continuous on E.

Step III: *$\mathcal{B}$ is a partially continuous on E.*

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of the points of a chain C in the partially ordered Banach space E such that $x_n \to x$ as $n \to \infty$. Then, we have

$$\lim_{n\to\infty} \mathcal{B}x_n(t) = \begin{cases} \lim\limits_{n\to\infty} \Big[\varphi(0)\, e^{-H(t)} + \displaystyle\int_0^t k(t,s) f(s,x_n(s))\, ds\Big], \ t \in I, \\ \varphi(t), \ t \in I_0, \end{cases}$$

$$
\begin{aligned}
&= \begin{cases} \varphi(0)\, e^{-H(t)} + \displaystyle\int_0^t k(t,s)\left[\lim_{n\to\infty} f(s,x_n(s))\right] ds, \ t \in I, \\ \varphi(t), \ t \in I_0, \end{cases} \\
&= \begin{cases} \varphi(0)\, e^{-H(t)} + \displaystyle\int_0^t k(t,s) f(s,x(s))\, ds, \ t \in I, \\ \varphi(t), \ t \in I_0, \end{cases} \\
8 &= \mathcal{B}x(t),
\end{aligned}
$$

for all $t \in J$. This shows that $\mathcal{B}x_n$ converges to $\mathcal{B}x$ pointwise on J. To show that the conergence is uniform, we show that $\{\mathcal{B}x_n\}_{n\in\mathbb{N}}$ is a quasi-equicontinuous sequence of functions in E. We consider the following three cases:

Case I: Let $\tau_1, \tau_2 \in (t_j, t_{j+1}] \cap I$, $j = 1, \dots, p$. Then, we have that

$$
\begin{aligned}
\Big|\mathcal{B}x_n(\tau_2) - \mathcal{B}x_n(\tau_1)\Big| &= \Big| x_0 e^{-H(\tau_1)} + \int_0^{\tau_1} k(\tau_1,s) f(s,x_n(s))\, ds \\
&\qquad - x_0 e^{-H(\tau_2)} - \int_0^{\tau_2} k(\tau_2,s) f(s,x_n(s))\, ds \Big| \\
&\le \left| \int_0^{\tau_1} k(\tau_1,s) f(s,x_n(s))\, ds - \int_0^{\tau_2} k(\tau_2,s) f(s,x_n(s))\, ds \right| \\
&\quad + \left| x_0 e^{-H(\tau_1)} - x_0 e^{-H(\tau_2)} \right| \\
&\le \left| \int_0^{\tau_1} k(\tau_1,s) f(s,x_n(s))\, ds - \int_0^{\tau_1} k(\tau_2,s) f(s,x_n(s))\, ds \right| \\
&\quad + \left| \int_0^{\tau_1} k(\tau_2,s) f(s,x_n(s))\, ds - \int_0^{\tau_2} k(\tau_2,s) f(s,x_n(s))\, ds \right| \\
&\quad + \left| x_0 e^{-H(\tau_1)} - x_0 e^{-H(\tau_2)} \right| \\
&\le |x_0| \left| e^{-H(\tau_1)} - e^{-H(\tau_2)} \right| \\
&\quad + \int_0^T \big| k(\tau_1,s) - k(\tau_2,s) \big| \big| f(s,x_n(s)) \big|\, ds \\
&\quad + \left| \int_{\tau_2}^{\tau_1} |k(\tau_2,s)|\, |f(s,x_n(s))|\, ds \right| \\
&\le |x_0| \left| e^{-H(\tau_1)} - e^{-H(\tau_2)} \right| + M_f \int_0^T \big| k(\tau_1,s) - k(\tau_2,s) \big|\, ds \\
&\quad + M_f |\tau_1 - \tau_2| \\
&\to 0 \quad \text{as} \quad \tau_2 \to \tau_1,
\end{aligned}
$$

uniformly for all $n \in \mathbb{N}$.

Case II: Let $\tau_1, \tau_2 \in I_0$. Then, by uniform continuity of the function φ, we obtain

$$
\big|\mathcal{B}x_n(\tau_1) - \big|\mathcal{B}x_n(\tau_1) = |\varphi(\tau_1) - \varphi(\tau_2)| \to 0 \ \text{as} \ \tau_{\to}\tau_2,
$$

uniformly for all $n \in \mathbb{N}$.

Case III: Let $\tau_1 \in I_0$ and $\tau_2 \in I'$. Then, by above two cases, we have,

$$\begin{aligned}\Big|\mathcal{B}x_n(\tau_2) - \mathcal{B}x_n(\tau_1)\Big| &\leq |\mathcal{B}x_n(\tau_1) - \lfloor x_n(0)| \\ &\leq \big|\mathcal{B}x_n(\tau_2) - \mathcal{B}x_n(0)\big| + \mathcal{B}x_n(0) - \mathcal{B}x_n(\tau_1)\Big| \\ &\to 0 \quad \text{as} \quad \tau_1 \to \tau_2,\end{aligned}$$

uniformly for all $n \in \mathbb{N}$.

Thus, from above three case it follows that if $\tau_1, \tau_2 \in I_0 \cup (t_j, t_{j+1}]$ for all $j = 1, \ldots, p$, then we have

$$\big|\mathcal{B}x_n(\tau_1) - \big|\mathcal{B}x_n(\tau_1)\big| \to 0 \quad \text{as} \quad \tau_1 \to \tau_2,$$

uniformyly for all $n \in \mathbb{N}$. This shows that the sequence $\{\mathcal{B}x_n\}$ of functions is quasi- equicontinuous and consequently the convergence $\mathcal{B}x_n \to \mathcal{B}x$ is uniform in view of the arguments given in Bainov [1]. Hence $\mathcal{B}$ is partially continuous operator on E into itself.

Step IV: *$\mathcal{B}$ is partially compact operator on E.*

Let C be an arbitrary chain in of the partially ordered Banach space E. We show that $\mathcal{B}(C)$ is a uniformly bounded and quasi-equicontinuous set in E. First we show that $\mathcal{B}(C)$ is uniformly bounded. Let $y \in \mathcal{B}(C)$ be any element. Then there is an element $x \in C$ such that $y = \mathcal{B}x$. By hypothesis (H_3),

$$\begin{aligned}|y(t)| &= |\mathcal{B}x(t)| \\ &= \begin{cases}\Big|\varphi(0)e^{-H(t)} + \displaystyle\int_0^t k(t,s)f(s,x(s))\,ds\Big|, \ t \in I, \\ |\varphi(t)|, \ \ t \in I_0,\end{cases} \\ &\leq \begin{cases}\big|\varphi(0)e^{-H(t)}\big| + \displaystyle\int_0^T |k(t,s)|\,|f(s,x(s))|\,ds, \ t \in I, \\ |\varphi(t)|, \ \ t \in I_0,\end{cases} \\ &\leq \begin{cases}\big|\varphi(0)e^{-H(t)}\big| + M_f\displaystyle\int_0^T k(t,s)\,ds, \ t \in I, \\ |\varphi(t)|, \ \ t \in I_0,\end{cases} \\ &\leq \begin{cases}|\varphi(0)| + M_f T, \ t \in I, \\ |\varphi(t)|, \ \ t \in I_0,\end{cases} \\ &\leq \|\varphi\| + M_f T \\ &= M_{\mathcal{B}},\end{aligned}$$

for all $t \in J$. Taking the supremum over t we obtain $\|y\|_{PC} \leq \|\mathcal{B}x\|_{PC} \leq M_{\mathcal{B}}$, for all $y \in \mathcal{B}(C)$. Hence $\mathcal{B}(C)$ is uniformly bounded subset of functions E. Next, proceeding with the arguments as in Step III‘, it can be shown that $\mathcal{B}(C)$ is an quasi-equicontinuous subset of functions in E. So $\mathcal{B}(C)$ is a uniformly bounded and quasi-equicontinuous set of functions in E and hence it is compact in view of Arzelá-Ascoli theorem (see Bainov and Simeonov [1]). Consequently $\mathcal{B} : E \to E$ is a partially compact operator of E into itself.

Step V: *The lower impulsive solution u satisfies the operator inequality $u \preceq \mathcal{A}u + \mathcal{B}u$.*

By hypothesis (LS), the IFDE (1.1) has a lower impulsive solution u defined on J. Then, we have

$$\left.\begin{array}{c} u'(t)+h(t)u(t) \le f(t,u(t)), \ \ t\in I\setminus\{t_1,\dots,t_p\}, \\ u(t_j^+)-u(t_j^-) \le \mathcal{I}_j(u(t_j)), \\ u_0 \preceq_{\mathcal{C}} \varphi. \end{array}\right\} \tag{3.19}$$

Now, by a direct application of the impulsive functional differential inequality established in Lemma 3.6 yields that

$$u(t) \le \begin{cases} \varphi(0)\ e^{-H(t)} + \sum\limits_{0<t_j<t} k(t,t_j)\mathcal{I}_j(u(t_j)) + \int_0^t k(t,s)f(s,u(s))\,ds, & t\in I, \\ \varphi(t), \ \ t\in I_0. \end{cases} \tag{3.20}$$

Furthermore, from definitions of the operators $\mathcal{A}$ and $\mathcal{B}$ it follows that $u(t) \le \mathcal{A}u(t)+\mathcal{B}u(t)$ for all $t\in J$. Hence $u \preceq \mathcal{A}u+\mathcal{B}u$. Thus the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 2.5 and so the operator equation $\mathcal{A}x+\mathcal{B}x = x$ has a impulsive solution x^* in E. Consequently the integral equation and a fortiori, the IFDE (1.1) has a impulsive solution x^* defined on J. As a result, the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.14) converges monotone nondecreasingly to x^*. This completes the proof. □

Next, we prove the uniqueness theorem for the IFDE on the interval J.

Theorem 3.9. *Suppose that hypotheses (H_1)-(H_2) and (H_5)-(LS) hold. Then the IFDE (1.1) has a unique impulsive solution solution x^* defined on J and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by* (3.14) *converges monotone nondecreasingly to x^*.*

Proof . Set $E = PC(J,\mathbb{R})$. Then, every pair of elements in $PC(J,\mathbb{R})$ has a lower bound as well as an upper bound so it is a lattice with respect to the binary operations "meet($\wedge$)" and "join($\vee$)" defined by $\wedge\{x,y\} = \min\{x\,,\,y\}$ and $\vee\{x,y\} = \max\{x\,,\,y\}$.

Now, by Lemma 3.4, the IFDE (1.1) is equivalent to the nonlinear impulsive integral equation (3.15). Define two operators $\mathcal{A}$ and $\mathcal{B}$ on E by (3.16) and (3.17). Now, consider the mapping $\mathcal{T}: E\to E$ defined by

$$\mathcal{T}x(t) = \mathcal{A}x(t)+\mathcal{B}x(t), \ t\in J. \tag{3.21}$$

Then the impulsive integral equation (3.6) is reduced to the operator equation as

$$\mathcal{T}x(t) = x(t), \ t\in J. \tag{3.22}$$

Now, proceeding with the arguments as in the proof of Theorem 3.8 it can shown that the operator $\mathcal{A}$ is a partial Lipschitzian with Lipschitz constant $L_{\mathcal{A}} = \sum_{j=1}^{p} L_{\mathcal{I}_j}$. Similarly, we show that $\mathcal{B}$ is also a Lipschitzian on E into itself. Let $x,y\in E$ be such that $x \succeq y$. Then, by hypothesis (H_5), one has

$$\begin{aligned} \big|\mathcal{B}x(t)-\mathcal{B}y(t)\big| &= \left|\int_0^t k(t,s)f(s,x(s))\,ds - \int_0^t k(t,s)f(s,y(s))\,ds\right| \\ &\le \int_0^t |k(t,s)|\,\big|f(s,x(s))-f(s,y(s))\big|\,ds \\ &\le L_f\int_0^t \big|x(t)-y(t)\big|\,ds \\ &\le L_f\,T\,\|x-y\|_{PC} \end{aligned}$$

for all $t \in J$ and $x, y \in E$. Taking the supremum over t in the above inequality, we obtain

$$\|\mathcal{B}x - \mathcal{B}y\|_{PC} \le L_{\mathcal{B}} \|x - y\|_{PC}$$

for all $x, y \in E$, $x \succeq y$, where $L_{\mathcal{B}} = L_f T$. This shows that $\mathcal{B}$ is again a partial Lipschitzian operator on E into itself with a Lipschitz constant $L_{\mathcal{B}}$. Next, by definition of the operator $\mathcal{T}$, one has

$$\|\mathcal{T}x - \mathcal{T}y\|_{PC} \le \|\mathcal{A}x - \mathcal{A}y\|_{PC} + \|\mathcal{B}x - \mathcal{B}y\|_{PC} \le (L_{\mathcal{A}} + L_{\mathcal{B}}) \|x - y\|_{PC}$$

for all $x, y \in E$, $x \succeq y$, where $L_{\mathcal{A}} + L_{\mathcal{B}} = \sum_{j=1}^{p} L_{\mathcal{I}_j} + L_f T < 1$. Hence $\mathcal{T}$ is a partial contraction operator on E into itself. Since the hypothesis (H_6) holds, it is proved as in the step V of the proof of Theorem 3.8 that the operator equation (3.22) has a lower solution u in E. Then, by an application of Theorem 2.4, we obtain that the operator equation (3.22) and consequently the IFDE (1.1) has a unique impulsive solution x^* defined on J and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.15) converges monotone nondecreasingly to x^*. This competes the proof. □

Remark 3.10. *The conclusion of Theorems 3.8 and 3.9 also remains true if we replace the hypothesis (LS) with (US). The proof of Theorems 3.8 and 3.9 under this new hypothesis is similar and can be obtained by closely observing the same arguments with appropriate modifications. In this case the sequence* $\{x_n\}_{n=0}^{\infty}$ *defined by* (3.3) *with* $x_0(t) = v(t),\ \ t \in [-r, T]$, *converges monotone nonincrasingly to the solution* x^* *of he IFDE* (1.1) *on* J. *In this case we invoke the use of Lemma 3.7 in the proofs of these existence results on the whole interval* J.

4. The Example and Concluding Remarks

Example 4.1. *Given the closed intervals* $I_0 = \left[-\frac{\pi}{2}, 0\right]$ *and* $I = [0, 1]$ *of the real line* $\mathbb{R}$ *and given the points* $t_1 = \frac{1}{5}$, $t_2 = \frac{2}{5}$, $t_3 = \frac{3}{5}$, $t_4 = \frac{4}{5}$ *in* $[0, 1]$ *and the function* $\varphi : \left[-\frac{\pi}{2}, 0\right] \to \mathbb{R}$ *defined by* $\varphi(t) = \sin t$, *consider the initial value problem (in short IVP) of first order impulsive functional functional differential equation (in short IFDE)*

$$\left.\begin{aligned} x'(t) + x(t) &= \tanh x(t), \ \ t \in [0, 1] \setminus \{t_1, t_2, t_3, t_4\}, \\ x(t_j^+) - x(t_j^-) &= \mathcal{I}_j(x(t_j)), \\ x_0 &= \varphi, \end{aligned}\right\} \tag{4.1}$$

for $t_j \in \{\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\}$; *where* $x(t_j^-)$ *and* $x(t_j^-)$ *are respectively, the right and left limit of* x *at* $t = t_j$ *such that* $x(t_j) = x(t_j^-)$ *and* $\mathcal{I}_j(x(t_j))$ *are the impulsive effects at the points* $t = t_j$, $j = 1, \ldots, 4$ *given by*

$$\mathcal{I}_j(x) = \begin{cases} \dfrac{1}{2^j} \cdot \dfrac{x}{1+x} + 2, & \text{if } x > 0, \\ 2, & \text{if } x \le 0, \end{cases}$$

for all $t \in [0, 1]$. *Here* $f(t, x) = \tanh x$, *so it is continuous and bounded on* $[0, 1] \times \mathbb{R}$ *with bound* $M_f = 2$. *Again, the map* $x \mapsto f(t, x)$ *is nondecreasing for each* $t \in [0, 1]$. *Next, the impulsive function* $\mathcal{I}_j$ *are continuous and bounded on* $\mathbb{R}$ *with bound* $M_{\mathcal{I}_j} = 3$ *for each* $j = 1, \ldots, 4$. *It is easy to* *verify that the impulsive operators $\mathcal{I}_j$ satisfy the hypothesis (H$_2$) with Lipschitz constants* $L_{\mathcal{I}_j} = \dfrac{1}{2^j}$ *for* $j = 1, \ldots, 4$. *Moreover,* $\sum_{j=1}^{4} L_{\mathcal{I}_j} = \sum_{j=1}^{4} \dfrac{1}{2^j} < 1$. *Finally, the functions*

$$u(t) = \begin{cases} 7e^{-t} - 1, & t \in\in [0,1], \\ \sin t, & t \in \left[-\frac{\pi}{2}, 0\right] \end{cases}$$

and

$$v(t) = \begin{cases} 15t + 12, & t \in [0,1], \\ \sin t, & t \in \left[-\frac{\pi}{2}, 0\right] \end{cases}$$

are respectively the lower and upper impulsive solutions of the IFDE (1.1) *defined on* $[0,1]$. *Thus, all the conditions of Theorem 3.8 are satisfied and so the IFDE* (4.1) *has a impulsive solution* ξ^* *and the sequence* $\{x_n\}_{n=0}^{\infty}$ *of successive approximations defined by*

$$x_0(t) = \begin{cases} 7e^{-t} - 1, & t \in\in [0,1], \\ \sin t, & t \in \left[-\frac{\pi}{2}, 0\right], \end{cases}$$

$$x_{n+1}(t) = \begin{cases} \displaystyle\sum_{0<t_j<t} k(t,t_j)\mathcal{I}_j(x_n(t_j)) + \int_0^t k(t,s)\tanh x_n(s))\,ds, & t \in [0,1], \\ \sin t, & t \in \left[-\frac{\pi}{2}, 0\right], \end{cases}$$

converges monotone nondecreasingly to x^*. *Similarly, the sequence* $\{y_n\}_{n=0}^{\infty}$ *of successive approximations defined by*

$$y_0(t) = \begin{cases} t + 12, & t \in\in [0,1], \\ \sin t, & t \in \left[-\frac{\pi}{2}, 0\right], \end{cases}$$

$$y_{n+1}(t) = \begin{cases} \displaystyle\sum_{0<t_j<t} k(t,t_j)\mathcal{I}_j(y_n(t_j)) + \int_0^t k(t,s)\tanh y_n(s))\,ds, & t \in [0,1], \\ \sin t, & t \in \left[-\frac{\pi}{2}, 0\right], \end{cases}$$

also converges monotone nonincreasingly to the impulsive solution y^* *of the IFDE* (4.1) *in view of Remark 3.10.*

Remark 4.2. *We note that if the IFDE* (1.1) *has a lower impulsive solution* u *as well as an upper impulsive solution* v *such that* $u \preceq v$, *then under the given conditions of Theorem 3.8 it has corresponding impulsive solutions* x_* *and* y^* *and these impulsive solutions satisfy the inequality*

$$u = x_0 \preceq x_1 \preceq \cdots \preceq x_n \preceq x_* \preceq y^* \preceq y_n \preceq \cdots \preceq y_1 \preceq y_0 = v.$$

Hence x_* *and* y^* *are respectively the minimal and maximal impulsive solutions of the IFDE* (1.1) *in the vector segment* $[u,v]$ *of the Banach space* $E = PC(J,\mathbb{R})$, *where the vector segment* $[u,v]$ *is a set of elements in* $PC(J,\mathbb{R})$ *defined by*

$$[u,v] = \{x \in PC(J,\mathbb{R}) \mid u \preceq x \preceq v\}.$$

This is because of the order cone K defined by (3.3) is a closed set in $PC(J,\mathbb{R})$. A few details concerning the order relation by the order cones and the Janhavi sets in an ordered Banach space are given in Dhage [10, 11].

Remark 4.3. *In this paper we considered a very simple nonlinear first order impulsive functional differential equation for discussing the existence and approximation theorem via Dhage iteration principle or method, however the same method may be extended to other complex nonlinear impulsive functional differential equations of different orders and type with appropriate modifications for obtaining the algorithms and proving the existence and approximation of solution.*

References

[1] D.D. Bainov, P.S. Simeonov, *Impulsive Differential Equations: Periodic Solutions and Applications*, Longman Scientific and Technical, Harlow, 1993.

[2] B.C. Dhage, *Local fixed point theory for sum of two operators in Banach spaces*, Fixed Point Theory, 4(1) (2003) 49–60.

[3] B.C. Dhage, *Hybrid fixed point theory in partially ordered normed linear spaces and applications to fractional integral equations*, Diff. Eq. Appl. 5 (2013) 155–184.

[4] B.C. Dhage, *Partially condensing mappings in ordered normed linear spaces and applications to functional integral equations*, Tamkang J. Math. 45 (2014) 397–426.

[5] B.C. Dhage, *Nonlinear $\mathcal{D}$-set-contraction mappings in partially ordered normed linear spaces and applications to functional hybrid integral equations*, Malaya J. Mat. 3 (2015) 62–86.

[6] B.C. Dhage, *Some generalizations of a hybrid fixed point theorem in a partially ordered metric space and nonlinear functional integral equations*, Differ. Equ. Appl. 8 (2016) 77–97.

[7] B.C. Dhage, *Dhage iteration method in the theory of ordinary nonlinear PBVPs of first order functional differential equations*, Comm. Opt. Theory 2017 (2017), Article ID 32, pp. 22.

[8] B.C. Dhage, *Dhage iteration method for approximating solutions of a IVP of nonlinear first order functional differential equations*, Malaya J. Mat. **5**(4) (1917), 680-689.

[9] B.C. Dhage, *The Dhage iteration method for initial value problems of first order hybrid functional integrodifferential equations*, Recent Advances in Fixed point Theory and Applications, Nova Science Publishers Inc., 2017

[10] B.C. Dhage, *A coupled hybrid fixed point theorem involving the sum of two coupled operators in a partially ordered Banach space with applications*, Differ. Equ. Appl. **9**(4) (2017), 453-477.

[11] B.C. Dhage, *Some variants of two basic hybrid fixed point theorem of Krasnoselskii and Dhage with applications*, Nonlinear Studies **25** (3) (2018), 559-573.

[12] B.C. Dhage, *A coupled hybrid fixed point theorem for sum of two mixed monotone coupled operators in a partially ordered Banach space with applications*, Tamkang J. Math. 50(1) (2019) 1–36.

[13] B.C. Dhage, *Coupled and mixed coupled hybrid fixed point principles in a partially ordered Banach algebra and PBVPs of nonlinear coupled quadratic differential equations*, Diff. Eq. Appl. 11(1) (2019) 1–85.

[14] B.C. Dhage, *Dhage monotone iteration principle for IVPs of nonlinear first order impulsive differential equations*, Jñānābha, 50(1) (2020) 265–278. +.

[15] B.C. Dhage, *Dhage monotone iteration method for for PBVPs of nonlinear first order impulsive differential equations*, Electronic J. Math. Anal. Appl. 9(1) (2021) 37–51.

[16] B.C. Dhage, S.B. Dhage, *Approximating solutions of nonlinear first order ordinary differential equations*, GJMS Special Issue for Recent Advances in Mathematical Sciences and Applications-13, GJMS 2(2) (2014) 25–35.

[17] S.B. Dhage, B.C. Dhage, *Dhage iteration method for Approximating positive solutions of PBVPs of nonlinear hybrid differential equations with maxima*, Int. Jour. Anal. Appl. 10(2) (2016) 101–111.

[18] S. B. Dhage, B. C. Dhage and J. B. Graef, *Dhage iteration method for initial value problem for nonlinear first order integrodifferential equations*, J. Fixed Point Theory Appl. 18 (2016) 309–325.

[19] S.B. Dhage, B.C. Dhage and J.B. Graef, *Dhage iteration method for approximating the positive solutions of IVPs for nonlinear first order quadratic neutral functional differential equations with delay and maxima*, Intern. J. Appl. Math. 31(6) (2018) 1–21.

[20] B. C. Dhage and D. Otrocol, *Dhage iteration method for approximating solutions of nonlinear differential equations with maxima*, Fixed Point Theo. 19(2) (2018) 545–556.

[21] J.K. Hale, *Theory of Functional Differential Equations*, Springer-Verlag, New York-Berlin, 1977.

[22] D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, New York, London 1988.

[23] S. Heikkilä, V. Lakshmikantham, *Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations*, Marcel Dekker inc., New York 1994.

[24] V. Lakshmikantham, D.D. Bainov, and P.S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific Pub. Co., Singapore, 1989.

[25] A.N. Samoilenko and N.A. Perestyuk, *Impulsive Differential Equations*, World Scientific, Singapore, 1995.