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Some drifts on posets and its application to fuzzy subalgebras

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Abstract

In this paper, given a poset $(X, \leq)$, we introduce some drifts on a groupoid $(X, *)$ with respect to $(X, \leq)$, and we obtain several properties of these drifts related to the notion of $Bin(X)$. We discuss some connections between fuzzy subalgebras and upward drifts.

Keywords: $Bin(X)$; (strong, oriented, positive, strict) upward drift; selective; BCK-algebra; fuzzy subalgebra.
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1. Introduction

The notion of a semiring was first introduced by H. S. Vandiver in 1934. Semirings have proven to be useful in some areas of applied mathematics and computer sciences. Semirings have also proved useful in studying automata and formal languages ([1]). The notion of a fuzzy subset of a set was introduced by L. A. Zadeh ([10]). His seminal paper in 1965 has opened up new insights and applications in a wide range of scientific fields. J. N. Mordeson and D. S. Malik ([7]) published a remarkable book, *Fuzzy commutative algebra*, presented a fuzzy ideal theory of commutative rings and applied the results to the solution of fuzzy intersection equations. The fuzzy semiring and the K-fuzzy semiring, where K denotes some subset of $\mathbf{R}$ closed under the operation min, +, or max. Min-max-plus computations (and suitable semirings) are used in several areas, for example, in mathematical physics in the study of several partial differential equations. It is interesting to observe

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that the fuzzy calculus, which is used for artificial intelligence purposes, indeed involves essentially "(min, max) semirings" ([3]).

The notion of the semigroup $(Bin(X), \Box)$ was introduced by H. S. Kim and J. Neggers ([5]). H. Fayoumi ([2]) introduced the notion of the center $ZBin(X)$ in the semigroup $Bin(X)$ of all binary systems on a set X, and showed that if $(X, \bullet) \in ZBin(X)$, then $x \neq y$ implies $\{x, y\} = \{x \bullet y, y \bullet x\}$. Moreover, she showed that a groupoid $(X, \bullet) \in ZBin(X)$ if and only if it is a locally-zero groupoid. J. S. Han et al. ([4]) introduced the notion of hypergroupoids $(HBin(X), \Box)$, and showed that $(HBin(X), \Box)$ is a supersemigroup of the semigroup $(Bin(X), \Box)$ via the identification $x \longleftrightarrow \{x\}$. They also proved that $(HBin^*(X), \ominus, [\emptyset])$ is a BCK-algebra.

H. S. Kim et al. ([6]) introduced several types of drifts. Using the product "$\Box$" in $(Bin(X), \Box)$, they discussed connections between order relations of $Bin(\mathbf{R})$ with drifts, fuzzy universal for upward drifts, and suitable groupoids.

In this paper, given a poset $(X, \leq)$, we introduce several upward drifts on a groupoid $(X, *)$ with respect to $(X, \leq)$, and we obtain some properties related to the notion of $Bin(X)$. We discuss relations between fuzzy subalgebras and upward drifts. We concentrate on several types of upward drifts, although other types are mentioned also. For basic notions on partially ordered sets, we refer to ([8]).

2. Preliminaries

Given a non-empty set X, we let $Bin(X)$ be the collection of all groupoids $(X, *)$, where $* : X \times X \to X$ is a map and where $*(x, y)$ is written in the usual product form. Given elements $(X, *)$ and $(X, \bullet)$ of $Bin(X)$, define a product "$\Box$" on these groupoids as follows:

$$(X, *) \,\Box\, (X, \bullet) = (X, \Box)$$

where

$$x \,\Box\, y = (x * y) \bullet (y * x)$$

for any $x, y \in X$. Using this notion, H. S. Kim and J. Neggers proved the following theorem.

Theorem 2.1. ([5]) $(Bin(X), \Box)$ is a semigroup, i.e., the operation "$\Box$" as defined in general is associative. Furthermore, the left zero semigroup is the identity for this operation.

Given the real numbers $\mathbf{R}$, we shall discuss certain groupoids $(\mathbf{R}, *)$ which we will refer to as *drifts*. Let "$*$" be a binary operation on $\mathbf{R}$, i.e., $(\mathbf{R}, *)$ is a groupoid. We recognize the following types. A groupoid $(\mathbf{R}, *)$ is said to be an *upward drift* if $x * y \geq \min\{x, y\}$ for any $x, y \in \mathbf{R}$. It is said to be *strict* if $x * y > \min\{x, y\}$ for any $x \neq y \in \mathbf{R}$. Thus, if we consider $\mu(x) = x$, then although μ is not a fuzzy subset, $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$, i.e., it satisfies a similar identity. A groupoid $(\mathbf{R}, *)$ is said to be a *strong upward drift* if $x * y \geq \max\{x, y\}$ for any $x, y \in \mathbf{R}$. It is said to be *strict* if $x * y > \max\{x, y\}$ for any $x, y \in \mathbf{R}$. A groupoid $(\mathbf{R}, *)$ is said to be a *downward drift* if $x * y \leq \max\{x, y\}$ for any $x, y \in \mathbf{R}$. It is said to be *strict* if $x * y < \min\{x, y\}$ for all $x \neq y \in \mathbf{R}$. It is said to be a *strong down drift* if $x * y \leq \min\{x, y\}$ for any $x, y \in \mathbf{R}$. It is said to be *strict* if $x * y < \min\{x, y\}$ for all $x \neq y \in \mathbf{R}$. A groupoid $(\mathbf{R}, *)$ is said to be an *average drift* if $\min\{x, y\} \leq x * y \leq \max\{x, y\}$ for any $x, y \in \mathbf{R}$, i.e., it is both upward and downward as a drift. Again, it is said to be *strict* if $\min\{x, y\} < x * y < \max\{x, y\}$ for any $x \neq y \in \mathbf{R}$, i.e., it is both a strict upward drift and a strict downward drift.

Example 2.2. ([6]) (1). Let $x * y := \lambda x + (1 - \lambda)y$ for any $x, y \in \mathbf{R}$ where $0 \le \lambda \le 1$. Then $x * y \ge \min\{x, y\}$ and $x * y \le \max\{x, y\}$ for any $x, y \in \mathbf{R}$, so that $(\mathbf{R}, *)$ is an example of an average drift.
(2). Let $x * y := |x| + |y|$ for any $x, y \in \mathbf{R}$. Then $|x| \ge x$ implies $x * y \ge \max\{x, y\}$, i.e., $(\mathbf{R}, *)$ is an example of a strong upward drift.
(3). Let $x * y := -(|x| + |y|)$ for any $x, y \in \mathbf{R}$. Then $-|x| \le x$ implies $x * y \le \min\{x, y\}$, i.e., $(\mathbf{R}, *)$ is an example of a strong downward drift.
(4). Let $(\mathbf{R}, *)$ be an (strong, resp.) upward (downward, resp.) drift. If we define a new binary operation $\star$ on $\mathbf{R}$ by $x \star y := -(x * y)$ for any $x, y \in \mathbf{R}$, then $(\mathbf{R}, \star)$ is a (strong, resp.) downward (upward, resp.) drift.

Theorem 2.3. ([6]) The set of all upward drifts forms a subsemigroup of $(Bin(\mathbf{R}), \square)$.

Proposition 2.4. ([6]) Let $(\mathbf{R}, *)$ and $(\mathbf{R}, \bullet)$ be groupoids defined by $x * y := \lambda x + (1 - \lambda)y$ and $x \bullet y := \mu x + (1 - \mu)y$ for any $x, y \in \mathbf{R}$ where $0 \le \lambda, \mu \le 1$. If we define $(\mathbf{R}, \square) := (\mathbf{R}, *)\square(\mathbf{R}, \bullet)$, then $(\mathbf{R}, \square)$ is an average drift.

Remark 2.5. Let $(\mathbf{R}, *) \in Bin(\mathbf{R})$ and let $e \in \mathbf{R}$ such that $x * x = e$ for any $x \in \mathbf{R}$. Then $(\mathbf{R}, *)$ is neither an upward drift nor a downward drift. In fact, assume that $(\mathbf{R}, *)$ is an upward drift. Then $e = x * x \ge \min\{x, x\} = x$ for any $x \in \mathbf{R}$, a contradiction. Similarly, if we assume that $(\mathbf{R}, *)$ is a downward drift, then $e = x * x \le \max\{x, x\} = x$ for any $x \in \mathbf{R}$, a contradiction.

Example 2.6. ([6]) Define a binary operation "$*$" on $\mathbf{R}$ by $x * y := x + y$ for any $x, y \in \mathbf{R}$. Then $1 + 3 > \max\{1, 3\}; (-1) + (-3) < \min\{-1, -3\}$, and $\min\{1, 2\} < 1 + 2 < \max\{1, 2\}$ does not hold, i.e., $(\mathbf{R}, +)$ is not a drift of any kind.

Remark 2.7. Given any groupoid $(\mathbf{R}, *)$ we have an associated partition of the plane into an (strong) upward drift, a (strong) downward drift and an average drift region.

Example 2.8. ([6]) Suppose $(\mathbf{R}, *)$ is the groupoid with $x * y := x^2 + y^2$ for any $x, y \in \mathbf{R}$. Then partitioning the plane can be done as follows: $\mathbf{R}^2 = A \cup B$ where $A = \{(x, y) \mid x^2 + (y - \frac{1}{2})^2 \le (\frac{1}{2})^2\}$ and $B = \{(x, y) \mid (x - \frac{1}{2})^2 + y^2 \le (\frac{1}{2})^2\}$. It can be easily seen that A is an upward drift region and B is a strong downward drift region.

3. Upward drifts on a poset

Let $(X, \le)$ be a poset. A groupoid $(X, *)$ is said to be an *upward drift* if for all $x, y \in X$, we have

$$x * y \ge x \text{ or } x * y \ge y. \tag{3.1}$$

If $(\mathbf{R}, \le)$ is the real numbers with the standard order, then $\min\{x, y\} \in \{x, y\}$ and thus the condition (3.1) has the equivalent form, for all $x, y \in \mathbf{R}$,

$$x * y \ge \min\{x, y\}. \tag{3.2}$$

If $(X,*)$ is a left-zero semigroup, then $x*y=x$ implies $x*y\geq x$ and thus $(X,*)$ is an upward drift for any poset $(X,\leq)$.

An upward drift $(X,*)$ is said to be *strict* if for all $x,y\in X$

$$x*y>x \text{ or } x*y>y. \tag{3.3}$$

Given a poset $(X,\leq)$, we denote by $UD(X,\leq)$ the set $\{(X,*)\in Bin(X)\,|\,(X,*)$ is an upward drift with respect to $(X,\leq)\}$, and we denote by $SUD(X,\leq)$ the set $\{(X,*)\in Bin(X)\,|\,(X,*)$ is a strict upward drift with respect to $(X,\leq)\}$.

If $(X,*)$ is a left-zero semigroup, then $x*y=x$ for all $x,y\in X$. If (3.3) is satisfied, then $x*y=x>y$ for all $y\in X$, and thus $x*x=x>x$, an impossibility. Hence $(X,*)\notin SUD(X,\leq)$.

Theorem 3.1. *$(UD(X,\leq),\Box)$ is a subsemigroup of $(Bin(X),\Box)$ and $(SUD(X,\leq),\Box)$ is a two-sided ideal of $(UD(X,\leq),\Box)$.*

Proof . Given $(X,*),(X,\bullet)\in Bin(X)$, we let $(X,\Box):=(X,*)\Box(X,\bullet)$. Then $x\Box y=(x*y)\bullet(y*x)\geq x*y$ or $x\Box y=(x*y)\bullet(y*x)\geq y*x$, since $(X,\bullet)$ is an upward drift. Since $(X,*)$ is also an upward drift, we obtain $x\Box y\geq x$ or $x\Box y\geq y$, proving that $(UD(X,\leq),\Box)$ is a subsemigroup of $(Bin(X),\Box)$.

If $(X,*)\in SUD(X,\leq)$, then $x\Box y=(x*y)\bullet(y*x)>x*y$ or $x\Box y>y*x$, so that if $x\Box y>x*y$, then $x\Box y>x*y\geq x$ or $x\Box y>x*y\geq y$, i.e., the condition (3.3) is satisfied and $(X,\Box)\in SUD(X,\leq)$. Similarly, if $(X,*)\in SUD(X,\leq)$, then $x\Box y=(x*y)\bullet(y*x)\geq x*y$ or $x\Box y\geq y*x$. It follows from (3.3) that $x\Box y>x$ or $x\Box y>y$, and thus $(X,\Box)$ satisfies (3.3) as well, i.e., $(X,\Box)\in SUD(X,\leq)$. $\Box$

A groupoid $(X,*)$ is said to be *selective* [6, 7] if $x*y\in\{x,y\}$ for all $x,y\in X$. Since selective groupoids can be connected to digraphs, we have another natural connection between groupoids of a certain type and digraphs.

Proposition 3.2. *Let $(X,\leq)$ be an anti-chain, i.e., $x\leq y\Leftrightarrow x=y$, for all $x,y\in X$. Then $(X,*)\in UD(X,\leq)$ if and only if $(X,*)$ is selective.*

Proof . If $(X,*)\in UD(X,\leq)$, then either $x*y\geq x$ or $x*y\geq y$ for all $x,y\in X$. Since $(X,\leq)$ is an anti-chain, it follows that either $x*y=x$ or $x*y=y$ for all $x,y\in X$, which is equivalent to $x*y\in\{x,y\}$ for all $x,y\in X$, proving that $(X,*)$ is a selective groupoid. The converse is straightforward. $\Box$

Proposition 3.3. *Let $(X,*)$ be a selective groupoid. If $(X,\leq)$ is any poset, then $(X,*)\in UD(X,\leq)$.*

Proof . Assume $(X,*)\notin UD(X,\leq)$. Then there exist elements x,y $(x\neq y)$ in X such that $x*y\not\geq x$ and $x*y\not\geq y$. Since $(X,*)$ is selective, we have $x*y\in\{x,y\}$. The condition $x*y\not\geq x$ implies $x*y=y$. In fact, if $x*y=x$, then $x=x*y\not\geq x$, a contradiction. Similarly, $x*y\in\{x,y\}$ and $x*y\not\geq y$ implies that $x*y=x$. Hence we obtain $x=x*y=y$, a contradiction. $\Box$

If a selective groupoid is a strict upward drift, then $x*y>x$ or $x*y>y$. Hence, since $x*x>x$ is impossible in this case, we may restrict our definition of the strictness to a weaker form:

An upward drift $(X, *)$ is said to be *oriented* if for all $x, y \in X$ with $x \neq y$,

$$x * y > x \text{ or } x * y > y. \tag{3.4}$$

Given a poset $(X, \leq)$, we denote by $OUD(X, \leq)$ the set $\{(X, *) \in Bin(X) \,|\, (X, *) \text{ is an oriented upward drift with respect to } (X, \leq)\}$.

Example 3.4. Let $(X, \leq)$ be a poset and let $(X, *)$ be a left-zero semigroup. Assume $(X, *)$ is an oriented upward drift. Then for all $x \neq y$ in X, either $x * y > x$ or $x * y > y$. Since $(X, *)$ is a left-zero semigroup, we have $x > y$ for all $y \neq x$ in X. Similarly, the fact $y * x = y$ implies that $y > x$ for all $x \in y$ in X. It follows that, if $x \neq y$, then $x > y$, $y > x$, a contradiction, so that if $|X| \geq 2$, then $(X, *)$ is not an oriented upward drift with respect to $(X, \leq)$.

Theorem 3.5. *$(OUD(X, \leq), \Box)$ is a two-sided ideal of $(UD(X, \leq), \Box)$.*

Proof . Let $(X, *), (X, \bullet) \in UD(X, \leq)$ and let $(X, \bullet) \in OUD(X, \leq)$. If $x \neq y$ in X, then $x \Box y = (x * y) \bullet (y * x) > x * y$ or $x \Box y > y * x$. Since $(X, *) \in UD(X, \leq)$, we have either $(x * y \geq x$ or $x * y \geq y)$ or $(y * x \geq y$ or $y * x \geq x)$. It follows that $x \Box y > x$ or $x \Box y > y$, proving that $(X, \Box) \in OUD(X, \leq)$. Similarly, if $(X, *) \in OUD(X, \leq)$ and $(X, \bullet) \in UD(X, \leq)$, then $x \Box y = (x * y) \bullet (y * x) \geq x * y$ or $x \Box y \geq y * x$. Since $(X, \bullet) \in OUD(X, \leq)$, we have either $(x * y > x$ or $x * y > y)$ or $(x * y > y$ or $x * y > x)$. It follows that $x \Box y > x$ or $x \Box y > y$, proving that $(X, \Box) \in OUD(X, \leq)$. $\Box$

4. Strong upward drifts on a poset

In [6] we defined the notion of the strong upward drift on the real numbers. In this section we define it on partially ordered sets.

Let $(X, \leq)$ be a poset. A groupoid $(X, *)$ is said to be a *strong upward drift* if, for all $x, y \in X$, we have

$$x * y \geq x \text{ and } x * y \geq y. \tag{4.1}$$

If $(\mathbf{R}, \leq)$ is the real numbers with the standard order, then $\max\{x, y\} \in \{x, y\}$ and thus condition (4.1) has the equivalent form, for all $x, y \in \mathbf{R}$,

$$x * y \geq \max\{x, y\}. \tag{4.2}$$

We have *associated strict conditions*:

$$x * y > x \quad \text{and} \quad x * y \geq y \qquad (4.1)^*_L,$$

$$x * y \geq x \quad \text{and} \quad x * y > y \qquad (4.1)^*_R,$$

$$x * y > x \quad \text{and} \quad x * y > y \qquad (4.1)^*.$$

Example 4.1. (a). Let $X_1 := [0, \infty)$ and let $\lambda, \mu \geq 0$. If we define $x * y := \lambda x + \mu y$ for all $x, y \in X_1$, then $(3.3)^*_L$ does not hold. In fact, if $x * y > x, x * y \geq y$ for all $x, y \in X_1$, then $\lambda x + \mu y > x$, $\lambda x + \mu y \geq y$. It follows that $\mu y > (1 - \lambda)x$ for all $x, y \in X_1$. If we let $y := 0$, then $0 > (1 - \lambda)x$ for all $x \in X_1$ and hence $0 > (1 - \lambda)0 = 0$, a contradiction.
(b). Let $X_2 := (0, \infty)$ and let $\lambda, \mu \geq 1$. If we define $x * y := \lambda x + \mu y$ for all $x, y \in X_2$, then $(3.3)^*_L, (3.3)^*_R$ and $(3.3)^*$ hold. In fact, $x * y \geq y$ is equivalent to $\lambda x > 0 \geq (1 - \mu)y$ which is true for all $x, y \in X_2$. Moreover, $x * y > x$ is equivalent to $\mu y > 0 \geq (1 - \mu)x$ which is also true for all $x, y \in X_2$.

Example 4.2. (a). Let $X := \mathbf{R}$, the set of all real numbers. If we define $x * y := \max\{x + 1, y\}$ for all $x, y \in X$, then $x * y = \max\{x + 1, y\} \geq x + 1 > x$, and $x * y = \max\{x + 1, y\} \geq y$. Hence $(3.3)^*_L$ holds. But $(3.3)^*_R$ does not hold, since $2 * 5 = \max\{2 + 1, 5\} \not> 5$.
(b). Let $X := \mathbf{R}$, the set of all real numbers. If we define $x * y := \max\{x, y + 1\}$ for all $x, y \in X$, then it is easy to see that $(3.3)^*_R$ holds, but not $(3.3)^*_L$.
(c). Let $X := \mathbf{R}$, the set of all real numbers. If we define $x * y := \max\{x, y + 1\}$ for all $x, y \in X$, then $x * y = \max\{x + 1, y + 1\} \geq x + 1 > x$ and $x * y = \max\{x + 1, y + 1\} \geq y + 1 > y$, i.e., $(3.3)^*$ holds.

Given a poset $(X, \leq)$, we denote the set of all strong upward drifts $(X, *)$ with respect to $(X, \leq)$ by $SUD(X, \leq)$, i.e.,

$$SUD(X, \leq) = \{(X, *) \in Bin(X) \,|\, (X, *) : \text{strong upper drift w. r. t.} \, (X, \leq)\}.$$

Theorem 4.3. *$(SUD(X, \leq), \Box)$ is a subsemigroup of $(Bin(X), \Box)$.*

Proof . Given $(X, *), (X, \bullet) \in SUD(X, \leq)$, we let $(X, \Box) := (X, *)\Box\, (X, \bullet)$. Then $x \Box y = (x * y) \bullet (y * x) \geq x * y$ and $x \Box y = (x * y) \bullet (y * x) \geq y * x$, since $(X, \bullet)$ is a strong upward drift. Since $(X, *)$ is also a strong upward drift, we obtain $x \Box y \geq x$ and $x \Box y \geq y$, proving the theorem. □

Proposition 4.4. *$(SUD(X, \leq), \Box)$ is a right ideal of $(UD(X, \leq), \Box)$.*

Proof . Given $(X, *) \in SUD(X, \leq)$ and $(X, \bullet) \in UD(X, \leq)$, we have $x \Box y = (x * y) \bullet (y * x) \geq x * y$ or $x \Box y = (x * y) \bullet (y * x) \geq y * x$ for all $x, y \in X$. Since $(X, *) \in SUD(X, \leq)$, we obtain $x \Box y \geq x$ and $x \Box y \geq y$, proving that $(X, \Box) = (X, *)\Box(X, \bullet) \in SUD(X, \leq)$. □

Corollary 4.5. ([6]) *If $(\mathbf{R}, *)$ is a strong upward drift and $(\mathbf{R}, \bullet)$ is an upward drift, where $\mathbf{R}$ is the set of all real numbers, then $(\mathbf{R}, \Box) := (\mathbf{R}, *)\Box(\mathbf{R}, \bullet)$ is also a strong upward drift.*

Proposition 4.6. *Let $(X, *)$ be a strong upward drift and let $(X, \bullet)$ satisfy the condition $(3.3)^*_L$. If $(X, \Box) := (X, *)\Box(X, \bullet)$, then $(X, \Box)$ satisfies the condition $(3.3)^*$.*

Proof . Given $x, y \in X$, since $(X, \bullet)$ is strong and $(X, *)$ satisfies the condition $(3.3)^*_L$, we have

$$x \Box y = (x * y) \bullet (y * x) > x * y \geq x, y,$$

i.e., $x \Box y > x$ and $x \Box y > y$. Hence $(X, \Box)$ satisfies the condition $(3.3)^*$. $\Box$

Proposition 4.7. *Let $(X, *)$ satisfy $(3.3)^*_L$. Let $(X, \bullet)$ be a strong upward drift. If $(X, \Box) := (X, *)\Box(X, \bullet)$, then $(X, \Box)$ satisfies the condition $(3.3)^*$.*

Proof . Given $x, y \in X$, since $(X, \bullet)$ is strong, we have

$$x \Box y = (x * y) \bullet (y * x) \geq x * y, y * x.$$

Since $(X, *)$ satisfies the condition $(3.3)^*_L$, we obtain $x * y > x, x * y \geq y$ and $y * x > y, y * x \geq x$. Hence we have $x \Box y \geq x * y > x$ and $x \Box y \geq y * x > y$. This shows that $(X, \Box)$ satisfies the condition $(3.3)^*$. $\Box$

Proposition 4.8. *Let $(X, \leq)$ be a poset and let $(X, *)$ be a leftoid for $f : X \to X$ such that $x \leq f(x)$ for all $x \in X$. Then $(X, *) \in UD(X, \leq)$.*

Proof . Given $x, y \in X$, we have $x * y = f(x) \geq x$. It follows that $(X, *) \in UD(X, \leq)$. $\Box$

Proposition 4.9. *Let $(X, \leq)$ be a poset and let $(X, *)$ be a leftoid for $f : X \to X$. If $(X, *) \in SUD(X, \leq)$, then $(X, *)$ is a trivial groupoid.*

Proof . If $(X, *) \in SUD(X, \leq)$, then $f(x) = x * y \geq y$ for all $x, y \in X$. It follows that $f(x) \geq f(y)$ for all $x, y \in X$. If we exchange x with y, then we have $f(y) \geq f(x)$, proving that $f(x) = f(y)$ for all $x, y \in X$, proving that $x * y = f(x)$, a constant. This shows that $(X, *)$ is a trivial groupoid. $\Box$

Proposition 4.10. *Let $(X, *)$ be a leftoid for $f : X \to X$ such that $x \leq f(x)$ for all $x \in X$ and let $(X, \bullet)$ be a rightoid for $g : X \to X$ such that $y \leq g(y)$ for all $y \in X$. If $(X, \Box) := (X, *)\Box(X, \bullet)$, then $(X, \Box) \in UD(X, \leq)$.*

Proof . Given $x, y \in X$, we have

$$\begin{aligned} x \Box y &= (x * y) \bullet (y * x) \\ &= f(x) \bullet f(y) \\ &= g(f(y)) \\ &\geq f(y) \\ &\geq y. \end{aligned}$$

It follows that $(X, \Box) \in UD(X, \leq)$. $\Box$

5. Positive and strict upwards

Let $(X, \leq)$ be a poset. A groupoid $(X, *)$ is said to be a *positive upward drift* with respect to the poset $(X, \leq)$ if for all $x, y \in X$,

$$x * y > x \quad \text{and} \quad x * y > y. \tag{5.1}$$

For example, if $X := (0, \infty)$ and $x * y := x + y$ for all $x, y \in X$ where "+" is the usual addition on the set of all real numbers. Then $(X, *)$ is a positive upward drift with respect to the standard order relation on the set of all real numbers, since $x + y > x$ and $x + y > y$ for all $x, y \in X$.

Proposition 5.1. *Every positive upward drift is infinite.*

Proof . Let $x \in X$. Since $(X, *)$ is a positive upward drift, we have $x < x * x$. It follows that $x < x * x < (x * x) * (x * x)$. If we define $x^{2^{n+1}} := x^{2^n} * x^{2^n}$, then we obtain $x < x^2 < \cdots < x^{2^{n-1}} < x^{2^n} < x^{2^{n+1}} < \cdots$, proving that $|X| = \infty$. □

Given a poset $(X, \leq)$, we denote the set of all positive upward drifts $(X, *)$ with respect to $(X, \leq)$ by $PD(X, \leq)$, i.e.,

$$PD(X, \leq) = \{(X, *) \in Bin(X) \,|\, (X, *) : \text{positive upward w.r.t.} \,(X, \leq)\}.$$

Theorem 5.2. *Let $(X, \leq)$ be a poset and let $(X, *) \in PD(X, \leq)$. If $(X, \square) := (X, *)\square(X, \bullet)$ and $(X, \bullet) \in UD(X, \leq)$, then $(X, \square) \in PD(X, \leq)$.*

Proof . Given $x, y \in X$, since $(X, \bullet)$ is an upward drift, we have either $x \square y = (x * y) \bullet (y * x) \geq x * y$ or $x \square y = (x * y) \bullet (y * x) \geq y * x$. Since $(X, *)$ is a positive upward drift, $x * y > x, y$ and $y * x > y, x$. It follows that $x \square y > x$ and $x \square y > y$, proving that $(X, \square)$ is a positive upward drift. □

Note that Theorem 5.2 shows that $(PD(X, \leq), \square)$ is a right ideal of $(UD(X, \leq), \square)$.

Example 5.3. Let $X := (0, \infty)$. Define a binary operation $*$ on X by $x * y := \frac{x+y}{2} + \alpha(x, y)$ for all $x, y \in X$, where

$$\alpha(x, y) := \begin{cases} 0 & \text{if } x \neq y, \\ 1 & \text{if } x = y. \end{cases}$$

If $x < y$, then $x * y = \frac{x+y}{2} + 0 > x$, but $x * y = \frac{x+y}{2} < y$. Hence $(X, *)$ is a strict upward drift, but not a positive upward drift with respect to the natural order $(X, \leq)$.

Proposition 5.4. *Let $(X, \leq)$ be a poset and let $(X, *)$ be a positive upward drift with respect to $(X, \leq)$. If $(X, \square) := (X, \bullet)\square(X, *)$ and $(X, \bullet)$ is an upward drift, then $(X, \square)$ is a positive upward drift.*

Proof . The proof is similar to the proof of Theorem 5.2, and we omit it. $\square$

Given a poset $(X, \leq)$, we denote the set of all strict upward drifts $(X, *)$ with respect to $(X, \leq)$ by $SD(X, \leq)$, i.e.,

$$SD(X, \leq) = \{(X, *) \in Bin(X) \,|\, (X, *) : \text{strict upward w.r.t.}\, (X, \leq)\}.$$

It follows that $PD(X, \leq) \subseteq SD(X, \leq) \subseteq UD(X, \leq)$. The following can be easily proved.

Proposition 5.5. The set of all strict upward drifts $(SD(X, \leq), \square)$ is a two-sided ideal of $(UD(X, \leq), \square)$.

6. Fuzzy subalgebras and upward drifts

A map $\mu : (X, *) \to [0, 1]$ is said to be a *fuzzy subalgebra* ([9]) of $(X, *)$ if, for all $x, y \in X$, $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$.

Proposition 6.1. *Let $(X, *) \in UD(X, \leq)$. If $\mu : X \to [0, 1]$ is order-preserving, then it is a fuzzy subalgebra of $(X, *)$.*

Proof . Given $x, y \in X$, since $(X, *) \in UD(X, \leq)$, we obtain $x * y \geq x$ or $x * y \geq y$. Since μ is order-preserving, we have $\mu(x * y) \geq \mu(x)$ or $\mu(x * y) \geq \mu(y)$. It follows that $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$, proving the proposition. $\square$

Let X be a non-empty set and let $\mu : X \to [0, 1]$ be a fuzzy subset of X. Define an order relation $<_\mu$ on X by

(i) if $x, y \in \mu^{-1}(\alpha), \alpha \in [0, 1]$, then $x \,||\, y$ (incomparable);

(ii) if $x \in \mu^{-1}(\alpha), y \in \mu^{-1}(\beta)$, $\alpha < \beta$ in $[0, 1]$, then $x <_\mu y$.

Define a relation $x \leq_\mu y$ on X by $x = y$ or $x <_\mu y$. Then it is easy to show that $(X, \leq_\mu)$ is a partially ordered set induced by the fuzzy subset μ. Note that the poset $(X, \leq_\mu)$ is an ordinal sum of maximal anti-chains. It is known that it is equivalent to the condition that $(X, \leq_\mu)$ is N-free.

Remark 6.2. The mapping $\mu : (X, \leq_\mu) \to [0, 1]$ is order-preserving. In fact, if $x \leq_\mu y$, then there exist $\alpha, \beta \in [0, 1]$ such that $x \in \mu^{-1}(\alpha), y \in \mu^{-1}(\beta)$. It follows that $\mu(x) = \alpha < \beta = \mu(y)$.

Theorem 6.3. *Let $(X, *) \in Bin(X)$ and let $\mu : X \to [0, 1]$ be a fuzzy subset. If either $\mu(x * y) > \min\{\mu(x), \mu(y)\}$ or $x * y \in \{x, y\}$ for all $x, y \in X$, then $(X, *) \in UD(X, \leq_\mu)$.*

Proof . Assume that $(X, *) \notin UD(X, \leq_\mu)$. Then there exist $x, y \in X$ such that $x * y \ngeq_\mu x$ and $x * y \ngeq_\mu$. It follows that $x * y \notin \{x, y\}$. In fact, $x * y \in \{x, y\}$ implies that either $x * y = x$ or $x * y = y$, i.e., $x * y \geq_\mu x$ or $x * y \geq_\mu y$, a contradiction. By the assumption, we obtain $\mu(x * y) > \min\{\mu(x), \mu(y)\}$. Suppose $\mu(x * y) > \mu(x)$. If we take $\alpha := \mu(x), \beta := \mu(x * y)$, then $x \in \mu^{-1}(\alpha), x * y \in \mu^{-1}(\beta), \alpha < \beta$. It follows that $x <_\mu x * y$, i.e., $x \leq_\mu x * y$, a contradiction. If we assume that $\mu(x * y) > \mu(y)$, then we obtain $y \leq_\mu x * y$, a contraction. $\square$

Example 6.4. Let $X := \{a, b, c, d\}$ be a set with the following table:

$*$	a	b	c	d
a	a	a	c	d
b	b	b	c	b
c	c	c	c	c
d	d	d	d	d

Define a map $\mu : X \to [0, 1]$ by $\mu(a) = \mu(b) = 0$, $\mu(c) = \mu(d) = 1$. By routine calculations, we obtain $(X, *) \in UD(X, \leq_\mu)$ where $\leq_\mu = \{(a, a), (b, b), (c, c), (d, d), (a, c), (a, d), (b, c), (b, d)\}$.

Example 6.5. Let $(X, *) \in Bin(X)$ such that $x * x = 0$ for all $x \in X$ with $|X| \geq 2$. Define a map $\nu : X \to [0, 1]$ by

$$\nu(x) := \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \neq 0. \end{cases}$$

Then it is not a fuzzy subalgebra of $(X, *)$, since $\nu(x * x) = \nu(0) = 0 < 1 = \min\{\nu(x), \nu(x)\}$ for all $x \neq 0$ in X. Moreover, $x * x = 0 \notin \{x, x\}$ for all $x \neq 0$. Let $x \neq y$ in $X - \{0\}$. Then $x, y \in \nu^{-1}(1)$, and hence $x \,||\, y$. Since $\nu(0) < \nu(x)$ for all $x \neq 0$, we obtain the poset $(X, \leq_\nu)$ where $\leq_\nu = \{(x, x) \,|\, x \in X\} \cup \{(0, x) \,|\, x \in X\}$. This shows that $(X, *) \notin UD(X, \leq_\nu)$.

Example 6.6. Let $X := \{a, b, c, d, e\}$ be a set with the following table:

$*$	a	b	c	d	e
a	a	a	c	d	e
b	b	b	d	e	e
c	c	d	c	d	e
d	d	e	d	d	e
e	e	b	e	d	e

Then $(X, *) \notin UD(X, \leq)$, since $d * b = e \notin \{a, d\}$, where $\leq := \{(a, a), (b, b), (c, c), (d, d), (e, e), (a, b), (a, c), (b, d), (c, d), (c, e)\}$. Define a map $\mu : X \to [0, 1]$ by $\mu(a) = 0, \mu(b) = \mu(c) = 1/2, \mu(d) = 3/4, \mu(e) = 1$. Then it is easy to show that μ is order-preserving. By Proposition 6.1, μ is a fuzzy subalgebra of $(X, *)$. By applying Theorem 6.3, we obtain $(X, *) \in UD(X, \leq_\mu)$ where $\leq_\mu = \{(a, a), (b, b), (c, c), (d, d), (e, e), (a, b), (a, c), (b, d), (c, d), (d, e)\}$.

7. Conclusion

In this paper we have emphasized a variety of upward drifts conditions. Certainly the theory of downward drifts is expected to be similar. The theory of average drifts probably contains results of a novel nature, given that it should have qualities different from either that of the theory of upward drifts and downward drifts. From the examples included it is also clear that much more can be discovered by considering other special types of groupoids and other constructions, involving directed products, homomorphism, etc.. All this will have to be considered in the future in the interest of keeping this paper from growing too large for an introduction of the ideas presented here.

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