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Almost α-F-contraction, fixed points and applications

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Abstract

In this manuscript, we initiate an almost α-F-contraction and an almost α-F-weak contraction in the setting of partial metric spaces and establish adequate conditions for the existence of fixed points. The obtained results generalize the classical and recent results of the literature, which are validated by suitable examples. As applications of these established results, we solve a nonlinear fractional differential equation and a boundary value problem.

Keywords: α-admissible; 0-complete partial metric space; Almost α-F-contraction; Almost α-F-weak contraction; Fixed point.
2010 MSC: 47H10; 54H25; 46J10; 46J15.

1. Introduction

Matthews [12] exhibited the idea of a partial metric space which is a productive instrument in developing some procedures that emerge normally in the fields of software engineering and is a generalization of normal metric space portrayed in 1906 by Maurice Frèchet. Inspired by the requirements of computer science, researchers generalized Banach's contraction to a partial metric space which allowed the possibility of non-zero self-distances. In fact, partial metrics are more adaptable having broader topological properties than that of metrics and create partial orders. It is essential from a realistic perspective that the fixed point exists which is possibly unique and we are capable to establish that

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fixed point.
Acknowledging the work of Berinde [3, 4], Wardowski [23] and Samet et al.[19], we familiarize with an almost α-F-contraction and an almost α-F-weak contraction in the framework of partial metric space and then establish adequate hypotheses for the existence of a single fixed point. Also, inspired by the fact that fractional nonlinear differential equations are of great significance in the numerous fields of science and engineering, we give an application of our result in establishing a solution of the fractional differential equations satisfying periodic boundary conditions. Further, prompted by the reality that concentrating solar energy to produce electricity in bulk is one of the technologies best suited to mitigate climate change in a reasonable manner as well as reducing the consumption of fossil fuels, we solve a boundary value problem emerging while transforming solar energy to electrical energy.

2. Main Results

Now we familiarize with an almost α-F-contraction and an almost α-F-weak contraction in the framework of partial metric space and then establish a fixed point of mappings satisfying these contractions. For the sake of convenience, we assume that an expression $-\infty.0$ has the value $-\infty$. Also $\tilde{p}^w(\rho,\sigma) = \tilde{p}(\rho,\sigma) - \min\{\tilde{p}(\rho,\rho), \tilde{p}(\sigma,\sigma)\}$.

Definition 2.1. *A mapping $T : X \to X$ of a partial metric space $(X, \tilde{p})$ is known as an almost α-F- contraction if there exist $\tau > 0$, $L \geq 0$ and functions $F \in \mathcal{F}$ and $\alpha : X \times X \to \{-\infty\} \cup (0, +\infty)$ so that for $\rho, \sigma \in X$ satisfying $\tilde{p}(T\rho, T\sigma) > 0$, the subsequent inequality holds*

$$\tau + \alpha(\rho,\sigma)F(\tilde{p}(T\rho,T\sigma)) \leq F(\tilde{p}(\rho,\sigma)) + L\tilde{p}^w(\sigma,T\rho).$$

Definition 2.2. *A mapping $T : X \to X$ of a partial metric space $(X, \tilde{p})$ is known as an almost α- F-weak contraction if there exist $\tau > 0$, $L \geq 0$ and functions $F \in \mathcal{F}$ and $\alpha : X \times X \to \{-\infty\} \cup (0, +\infty)$ so that for $\rho, \sigma \in X$ satisfying $\tilde{p}(T\rho, T\sigma) > 0$, the subsequent inequality holds*

$$\begin{aligned} \tau + \alpha(\rho,\sigma)F(\tilde{p}(T\rho,T\sigma)) \ &\leq \ F\left(\max\left\{ \tilde{p}(\rho,\sigma), \tilde{p}(\rho,T\rho), \tilde{p}(\sigma,T\sigma), \frac{\tilde{p}(\rho,T\sigma)+\tilde{p}(\sigma,T\rho)}{2} \right\}\right) \\ &+ \ L\tilde{p}^w(\sigma,T\rho). \end{aligned}$$

In each of the above definitions, a self map T is α-admissible ([19], [24]) and $\mathcal{F}$ is a family of the functions [23] $F : \mathbb{R}^+ \to \mathbb{R}$ so that:

(F1) $\mu < \nu \Rightarrow F(\mu) < F(\nu)$, $\mu, \nu \in \mathbb{R}^+$,

(F2) for each sequence $\{\mu_n\}$ in $\mathbb{R}^+$, $\lim_{n\to\infty} \mu_n = 0$ iff $\lim_{n\to\infty} F(\mu_n) = -\infty$,

(F3) $\lim_{\mu\to 0^+} \mu^k F(\mu) = 0$, $k \in (0, 1)$.

Example 2.3.
1. $F(\mu) = \ln\mu,\ \mu > 0$,
2. $F(\mu) = \ln\mu + \mu,\ \mu > 0$,
3. $F(\mu) = -\frac{1}{\sqrt{\mu}}, \mu > 0$,
4. $F(\mu) = \ln(\mu^2 + \mu), \mu > 0$.

Here, it is worth mentioning that on changing the elements of F , we may deduce distinct contractions existing in the literature.

Theorem 2.4. *Let $T : X \to X$ be a continuous almost α-F-weak contraction on a 0-complete partial metric space $(X, \tilde{p})$ so that $\alpha(\rho_0, T\rho_0) \geq 1$, $\rho_0 \in X$. Then T has a fixed point $\rho^* \in X$ such that the sequence $\{T^n \rho_0\}_{n\in\mathbb{N}}$ is convergent to ρ^*, $\rho_0 \in X$.*

Proof . Suppose $\rho_0 \in X$ be so that $\alpha(\rho_0, T\rho_0) \geq 1$. Define a sequence $\{\rho_n\}$ in X as $\rho_{n+1} = T\rho_n$ $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ so that $\rho_{n_0+1} = \rho_{n_0}$, then $T\rho_{n_0} = \rho_{n_0}$ and the proof is finished. Let, $\rho_{n+1} \neq \rho_n$ $n \in \mathbb{N}$. Therefore by given hypotheses, we obtain

$$\alpha(\rho_0, \rho_1) = \alpha(\rho_0, T\rho_0) \geq 1.$$

Using α-admissibility of T ,we get,

$$\alpha(T\rho_0, T\rho_1) = \alpha(\rho_1, \rho_2) \geq 1.$$

Again following the above steps, we get,

$$\alpha(T\rho_1, T\rho_2) = \alpha(\rho_2, \rho_3) \geq 1,$$

and so on. By mathematical induction, we get

$$\alpha(\rho_n, \rho_{n+1}) \geq 1, \, n \in \mathbb{N}. \tag{2.1}$$

Since T is an almost α- F- weak contraction, we attain

$$F(\tilde{p}(\rho_n, \rho_{n+1})) = F(\tilde{p}(T\rho_{n-1}, T\rho_n)) \leq \alpha(\rho_{n-1}, \rho_n)F(\tilde{p}(T\rho_{n-1}, T\rho_n)),$$

i.e.,

$$\begin{aligned}
&\tau + F(\tilde{p}(\rho_n, \rho_{n+1})) \leq \tau + \alpha(\rho_{n-1}, \rho_n)F(\tilde{p}(T\rho_{n-1}, T\rho_n)) \\
&\quad \leq F\left(\max\left\{\tilde{p}(\rho_{n-1}, \rho_n), \tilde{p}(\rho_{n-1}, T\rho_{n-1}), \tilde{p}(\rho_n, T\rho_n), \frac{\tilde{p}(\rho_{n-1}, T\rho_n) + \tilde{p}(\rho_n, T\rho_{n-1})}{2}\right\}\right) \\
&\qquad + L\tilde{p}^w(\rho_n, T\rho_{n-1}) \\
&\quad \leq F\left(\max\left\{\tilde{p}(\rho_{n-1}, \rho_n), \tilde{p}(\rho_{n-1}, \rho_n), \tilde{p}(\rho_n, \rho_{n+1}), \frac{\tilde{p}(\rho_{n-1}, \rho_{n+1}) + \tilde{p}(\rho_n, \rho_n)}{2}\right\}\right) \\
&\qquad + L\tilde{p}^w(\rho_n, \rho_n).
\end{aligned}$$

By using triangular inequality in partial metric space, we get

$$\begin{aligned}
\tilde{p}(\rho_{n-1}, \rho_{n+1}) &\leq \tilde{p}(\rho_{n-1}, \rho_n) + \tilde{p}(\rho_n, \rho_{n+1}) - \tilde{p}(\rho_n, \rho_n). \\
\text{or} \quad \frac{\tilde{p}(\rho_{n-1}, \rho_{n+1}) + \tilde{p}(\rho_n, \rho_n)}{2} &\leq \frac{\tilde{p}(\rho_{n-1}, \rho_n) + \tilde{p}(\rho_n, \rho_{n+1})}{2} \\
&\leq \max\left\{\tilde{p}(\rho_{n-1}, \rho_n), \tilde{p}(\rho_n, \rho_{n+1})\right\}.
\end{aligned}$$

Also $\tilde{p}^w$ is a metric on X, therefore

$$\tilde{p}^w(\rho_n, \rho_n) = \tilde{p}(\rho_n, \rho_n) - \min\left\{\tilde{p}(\rho_n, \rho_n), \tilde{p}(\rho_n, \rho_n)\right\} = 0.$$

Hence,

$$\tau + F(\tilde{p}(\rho_n, \rho_{n+1})) \leq F\Bigg(\max\Big\{ \tilde{p}(\rho_{n-1}, \rho_n), \tilde{p}(\rho_n, \rho_{n+1}) \Big\} \Bigg). \tag{2.2}$$

If

$$\max\Big\{ \tilde{p}(\rho_{n-1}, \rho_n), \tilde{p}(\rho_n, \rho_{n+1}) \Big\} = \tilde{p}(\rho_n, \rho_{n+1}),$$

then from equation (2.2), we have

$$F(\tilde{p}(\rho_n, \rho_{n+1}) \leq F(\tilde{p}(\rho_n, \rho_{n+1})) - \tau,$$

which is impossible. So

$$\max\Big\{ \tilde{p}(\rho_{n-1}, \rho_n), \tilde{p}(\rho_n, \rho_{n+1}) \Big\} = \tilde{p}(\rho_{n-1}, \rho_n), \ \text{n} \in \mathbb{N}.$$

Consequently, we attain

$$F(\tilde{p}(\rho_n, \rho_{n+1})) \leq F(\tilde{p}(\rho_{n-1}, \rho_n)) - \tau, \text{ n} \in \mathbb{N}$$

$$\begin{aligned} \Rightarrow F(\tilde{p}(\rho_n, \rho_{n+1})) &\leq F(\tilde{p}(\rho_{n-2}, \rho_{n-1})) - \tau - \tau \\ &\leq F(\tilde{p}(\rho_{n-2}, \rho_{n-1})) - 2\tau. \end{aligned}$$

On generalizing

$$F(\tilde{p}(\rho_n, \rho_{n+1})) \leq F(\tilde{p}(\rho_0, \rho_1)) - n\tau \ , \text{ n} \in \mathbb{N}. \tag{2.3}$$

As $n \to \infty$, we attain

$$F(\tilde{p}(\rho_n, \rho_{n+1})) = -\infty.$$

Using $(F2)$ we get,

$$\lim_{n\to\infty} \tilde{p}(\rho_n, \rho_{n+1}) = 0. \tag{2.4}$$

Using $(F3)$, we claim that there exists $k \in (0, 1)$ so that,

$$\lim_{n\to\infty} (\tilde{p}(\rho_n, \rho_{n+1}))^k F(\tilde{p}(\rho_n, \rho_{n+1})) = 0. \tag{2.5}$$

Therefore from equation (2.3), for $n \in \mathbb{N}$, we attain that

$$(\tilde{p}(\rho_n, \rho_{n+1}))^k \Bigg(F(\tilde{p}(\rho_n, \rho_{n+1})) - F(\tilde{p}(\rho_0, \rho_1)) \Bigg) \leq -(\tilde{p}(\rho_n, \rho_{n+1}))^k n\tau \leq 0. \tag{2.6}$$

Using equations (2.4) and (2.5) and taking limit as $n \to \infty$ in equation (2.6), we attain

$$\lim_{n\to\infty} n(-(\tilde{p}(\rho_n, \rho_{n+1}))^k) = 0.$$

So there exists $n_1 \in \mathbb{N}$ and $n(\tilde{p}(\rho_n, \rho_{n+1}))^k \leq 1$, $n \geq n_1$, i.e.,

$$\tilde{p}(\rho_n, \rho_{n+1}) \leq \frac{1}{n^{\frac{1}{k}}}, \, n \geq n_1. \tag{2.7}$$

For $m > n > n_1$

$$\begin{aligned}\tilde{p}(\rho_n, \rho_m) &\leq \sum_{i=n}^{m-1} \tilde{p}(\rho_i, \rho_{i+1}) - \sum_{n}^{n-2} \tilde{p}(\rho_{i+1}, \rho_{i+1}) \\ &\leq \sum_{i=n}^{m-1} \tilde{p}(\rho_i, \rho_{i+1}) < \sum_{i=n}^{\infty} \tilde{p}(\rho_i, \rho_{i+1}) \leq \sum_{i=1}^{\infty} \frac{1}{i^{\frac{1}{k}}}.\end{aligned}$$

By p-series test, $\sum_{i=1}^{\infty} \frac{1}{i^{\frac{1}{k}}}$ is convergent as $\frac{1}{k} > 1$, Hence

$$\lim_{n,m\to\infty} \tilde{p}(\rho_n, \rho_m) = 0,$$

i.e., $\{\rho_n\}$ is a 0-Cauchy sequence. Exploiting 0-completeness of partial metric space $(X, \tilde{p})$, there exists $\rho^* \in X$ so that $\lim_{n\to\infty} \rho_n = \rho^*$. But T is a continuous, therefore

$$\tilde{p}(\rho^*, T\rho^*) = \lim_{n\to+\infty} \tilde{p}(\rho_n, T\rho_n) = \lim_{n\to+\infty} \tilde{p}(\rho_n, \rho_{n+1}) = 0,$$

i.e., ρ^* is a fixed point of T. Further, $\lim_{n\to\infty} T^n\rho_0 = \lim_{n\to\infty} T^{n-1}\rho_1 = \lim_{n\to\infty} T^{n-2}\rho_2 = ... = \lim_{n\to\infty} \rho_n = \rho^*$. This finishes the proof. □

Example 2.5. *Let $X = \mathbb{R}$, where $(X, \tilde{p})$ is a 0-complete partial metric space with partial metric*

$$\tilde{p}(\rho, \sigma) = \begin{cases} \max\{\rho, \sigma\}, & \rho \neq \sigma \\ 0, & \rho = \sigma. \end{cases}$$

The mapping $T\rho = \frac{\rho}{2}$, for all $\rho \in X$ is continuous. Let us define the function α by

$$\alpha(\rho, \sigma) = \begin{cases} 1, & \rho \geq \sigma \\ 0, & otherwise. \end{cases}$$

Clearly, T is an almost α- F- weak contraction mapping, for all $\rho, \sigma \in X, \tau = 0.01, L = 1$ and $F(\alpha) = -\frac{1}{\alpha}$. Also, for $\rho_0 = 1$, we have

$$\alpha(1, T1) = \alpha(1, \frac{1}{2}) = 1.$$

Further,

$$\alpha(\rho, \sigma) \geq 1 \Rightarrow \rho \geq \sigma \Rightarrow T\rho \geq T\sigma \Rightarrow \alpha(T\rho, T\sigma) \geq 1.$$

Thus, T is α-admissible. Now a sequence $\{\rho_n\} = \{\frac{1}{n}\}_{n\in\mathbb{N}}$ is 0-Cauchy as $\lim_{n\to\infty} \tilde{p}(\frac{1}{n}, 0) = \tilde{p}(0, 0) = \lim_{n\to\infty} \tilde{p}(\frac{1}{n}, \frac{1}{m})$ and converges to a point in X as X is 0-complete. Consequently, all the postulates of Theorem (2.4) are verified and 0 is the fixed point of T.

Theorem 2.6. *Let $T : X \to X$ be a continuous almost α-F-contraction on a 0-complete partial metric space $(X, \tilde{p})$ so that $\alpha(\rho_0, T\rho_0) \geq 1$, $\rho_0 \in X$. Then T has a fixed point $\rho^* \in X$ so that the sequence $\{T^n\rho_0\}_{n\in\mathbb{N}}$ is convergent to ρ^*,$\rho_0 \in X$.*

Proof . The proof is identical to Theorem (2.4).□

Theorem 2.7. *Let $T : X \to X$ be an almost α-F-weak contraction on a 0-complete partial metric space $(X, \tilde{p})$ satisfying the subsequent hypotheses:*

(i) there exists $\rho_0 \in X$ so that $\alpha(\rho_0, T\rho_0) \geq 1$,

(ii) if $\{\rho_n\}$ is a sequence in X so that $\rho_n \to \rho$ as $n \to +\infty$ and $\alpha(\rho_n, \rho_{n+1}) \geq 1$, then $\alpha(\rho_n, \rho) \geq 1$, $n \in \mathbb{N}$.

Then T has a fixed point $\rho^* \in X$ and the sequence $\{T^n\rho_0\}_{n\in\mathbb{N}}$ is convergent to ρ^*, $\rho_0 \in X$.

Proof . Suppose $\rho_0 \in X$ be so that $\alpha(\rho_0, T\rho_0) \geq 1$ and let $\rho_n = T\rho_{n-1}$,for all $n \in \mathbb{N}$. On the same lines of Theorem 2.4, we obtain that $\{\rho_n\}$ is a 0-Cauchy sequence in the 0-complete partial metric space $(X, \tilde{p})$. So, there exists $\rho^* \in X$ and $\rho_n \to \rho^*$ as $n \to +\infty$. From the hypothesis (ii) and equation (2.1), we attain

$$\alpha(\rho_n, \rho^*) \geq 1, \text{ n} \in \mathbb{N}.$$

Case (I) Let there exists $i_n \in \mathbb{N}$ and $\rho_{i_n+1} = T\rho^*$ and $i_n > i_{n-1}$. So,

$$\rho^* = \lim_{n\to+\infty} \rho_{i_n+1} = \lim_{n\to+\infty} T\rho^* = T\rho^*,$$

i.e., ρ^* is a fixed point of T.

Case (II) Let that there exists $n_0 \in \mathbb{N}$ so that $\rho_{n+1} \neq T\rho^*$, $n \geq n_0$, i.e., $\tilde{p}(T\rho_n, T\rho^*) > 0$, $n \geq n_0$. Utilizing Definition (2.2) and $(F1)$, we get

$$\begin{aligned}
\tau + F(\tilde{p}(\rho_{n+1}, T\rho^*)) &= \tau + F(\tilde{p}(T\rho_n, T\rho^*)) \\
&\leq \tau + \alpha(\rho_n, \rho^*)F(\tilde{p}(T\rho_n, T\rho^*)) \\
&\leq F\Bigg(\max\Bigg\{\tilde{p}(\rho_n, \rho^*), \tilde{p}(\rho_n, T\rho_n), \tilde{p}(\rho^*, T\rho^*), \frac{\tilde{p}(\rho_n, T\rho^*) + \tilde{p}(\rho^*, T\rho_n)}{2}\Bigg\}\Bigg) \\
&\quad + L\tilde{p}^w(\rho^*, T\rho_n) \\
&\leq F\Bigg(\max\Bigg\{\tilde{p}(\rho_n, \rho^*), \tilde{p}(\rho_n, \rho_{n+1}), \tilde{p}(\rho^*, T\rho^*), \frac{\tilde{p}(\rho_n, T\rho^*) + \tilde{p}(\rho^*, \rho_{n+1})}{2}\Bigg\}\Bigg) \\
&\quad + L\tilde{p}^w(\rho^*, \rho_{n+1}).
\end{aligned}$$

Now, $\tilde{p}(\rho_n, T\rho^*) \leq \tilde{p}(\rho_n, \rho^*) + \tilde{p}(\rho^*, T\rho^*) - \tilde{p}(\rho^*, \rho^*)$, therefore

$$\begin{aligned}
\tau + F(\tilde{p}(\rho_{n+1}, T\rho^*)) &\leq F\Bigg(\max\Bigg\{\tilde{p}(\rho_n, \rho^*), \tilde{p}(\rho_n, \rho_{n+1}), \tilde{p}(\rho^*, T\rho^*), \qquad (2.8) \\
&\quad \frac{\tilde{p}(\rho_n, \rho*) + \tilde{p}(\rho^*, T\rho^*) - \tilde{p}(\rho^*, \rho^*) + \tilde{p}(\rho^*, \rho_{n+1})}{2}\Bigg\}\Bigg) \\
&\quad + L\Bigg\{\tilde{p}(\rho^*, \rho_{n+1}) - \min\{\tilde{p}(\rho^*, \rho^*), \tilde{p}(\rho_{n+1}, \rho_{n+1})\}\Bigg\}. \qquad (2.9)
\end{aligned}$$

If $\tilde{p}(\rho^*, T\rho^*) > 0$ and $\lim_{n\to\infty} \tilde{p}(\rho_n, \rho^*) = \lim_{n\to\infty} \tilde{p}(\rho_{n+1}, \rho^*) = \tilde{p}(\rho^*, \rho^*)$.
Taking the limit as $n \to \infty$ in equation (2.8), we get

$$\tau + F(\tilde{p}(\rho^*, T\rho^*))$$

$$
\begin{aligned}
&\leq F\Bigg(\max\Bigg\{\tilde{p}(\rho^*,\rho^*),\tilde{p}(\rho^*,\rho^*),\tilde{p}(\rho^*,T\rho^*),\frac{\tilde{p}(\rho^*,\rho^*)+\tilde{p}(\rho^*,T\rho^*)-\tilde{p}(\rho^*,\rho^*)+\tilde{p}(\rho^*,\rho^*)}{2}\Bigg\}\Bigg)\\
&+ L\Big\{\tilde{p}(\rho^*,\rho^*)-\min\{\tilde{p}(\rho^*,\rho^*),\tilde{p}(\rho^*,\rho^*)\}\Big\}\\
&\leq F\Bigg(\max\Bigg\{\tilde{p}(\rho^*,\rho^*),\tilde{p}(\rho^*,\rho^*),\tilde{p}(\rho^*,T\rho^*),\frac{\tilde{p}(\rho^*,T\rho^*)+\tilde{p}(\rho^*,\rho^*)}{2}\Bigg\}\Bigg)\\
&\leq F\Big(\max\Big\{\tilde{p}(\rho^*,\rho^*),\tilde{p}(\rho^*,T\rho^*)\Big\}\Big)\\
&\leq F(\tilde{p}(\rho^*,T\rho^*)).
\end{aligned}
$$

It is not possible as $\tau > 0$, therefore our supposition is wrong and hence $\tilde{p}(\rho,T\rho^*) = 0$,i.e., ρ^* is a fixed point of T. Further $\lim_{n\to\infty} T^n\rho_0 = \lim_{n\to\infty} T^{n-1}\rho_1 = \lim_{n\to\infty} T^{n-2}\rho_2 = \ldots = \lim_{n\to\infty}\rho_n = \rho^*$. This finishes the proof. □

Theorem 2.8. *Let $T : X \to X$ be an almost α-F-contraction on a 0-complete partial metric space $(X,\tilde{p})$ satisfying the subsequent hypotheses:*

(i) there exists $\rho_0 \in X$ so that $\alpha(\rho_0, T\rho_0) \geq 1$,

(ii) if $\{\rho_n\}$ is a sequence in X so that $\rho_n \to \rho$ as $n \to +\infty$ and $\alpha(\rho_n,\rho_{n+1}) \geq 1$, then $\alpha(\rho_n,\rho) \geq 1$, $n \in \mathbb{N}$.

Then T has a fixed point $\rho^* \in X$ and the sequence $\{T^n\rho_0\}_{n\in\mathbb{N}}$ is convergent to ρ^*, $\rho_0 \in X$.

Proof . The proof follows on similar lines as in Theorem (2.7). □

3. Applications

Now we exhibit a few utilizations of obtained results to fractional calculus and differential calculus.

*3.1. **Fractional Calculus***

Fractional differential equations emerge in numerous engineering and scientific areas and serve as an exceptional tool for the explanation of inherited properties of innumerable materials and processes. However, these nonlinear fractional differential equations are difficult to solve and consequently efficient technique for solving such equations is required.

Now, we introduce an utilization of Theorem 2.8 to solve a nonlinear fractional differential equation:

$$
{}^cD^\beta(\rho(u)) + T(u,\rho(u)) = 0, (0 \leq u \leq 1, \beta < 1) \tag{3.1}
$$

by means of boundary conditions $\rho(0) = 0 = \rho(1)$, $T : [0,1]\times\mathbb{R} \to \mathbb{R}$. The Caputo derivative of fractional order β is

$$
{}^cD^\beta(S(u)) = \frac{1}{\Gamma(n-\beta)}\int_0^u (u-v)^{n-\beta-1}S^n(v)dv, (n = [\beta]+1), n \in \mathbb{N},
$$

where $S : [0,\infty) \to R$ is a continuous function, Γ is a gamma function and $[\beta]$ is the integer part of the real number β . The Green function related to (3.1) is

$$
G(u,v) = \begin{cases} (u(1-v))^{\xi-1} - (u-v)^{\xi-1}, & 0 \leq v \leq u \leq 1 \\ \frac{(u(1-v))^{\xi-1}}{\Gamma(\xi)}, & 0 \leq u \leq v \leq 1. \end{cases}
$$

Let $X = C([0,1],\mathbb{R})$ be the set of all continuous real functions from $[0,1]$ into $\mathbb{R}$. Then X is a Banach space and $\|\rho\|_\infty = \sup_{u\in[0,1]} |\rho(u)|$, for all $\rho \in X$.

Theorem 3.1. *Consider $\alpha_1 : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and the nonlinear fractional differential equation (3.1). Let, for all $u \in [0,1]$:*

1. $|T(u,a) - T(u,b)| \le e^{-\tau}|a-b|, (\tau > 0)$ *, for $a, b \in \mathbb{R}$.*
2. $\rho_0 \in C([0,1],\mathbb{R})$ *so that* $\alpha_1(\rho_0(u), T\rho_0(u)) \ge 0$, *where*
$$T\rho(u) = \int_0^1 G(u,v)T(v,\rho(v))dv.$$
3. $\alpha_1(\rho(u),\sigma(u)) > 0$ *implies* $\alpha_1(T\rho(u), T\sigma(u)) > 0$, *for* $\rho,\sigma \in C([0,1],\mathbb{R})$.
4. $\{\rho_n\}$ *is a sequence in* $C([0,1],\mathbb{R})$ *so that* $\rho_n \to \rho$ *in* $C([0,1],\mathbb{R})$ *and* $\alpha_1(\rho_n(u), \rho_{n+1}(u)) > 0$, $n \in \mathbb{N}$, *then* $\alpha_1(\rho_n(u), \rho(u)) > 0$, $n \in \mathbb{N}$.

Then the fractional differential equation (3.1) has at least one solution.

Proof . Clearly, finding the solution of equation (3.1) is analogous to establishing a fixed point $\rho \in X$. Now

$$\begin{aligned}
|T\rho(u) - T\sigma(u)| &= \left|\int_0^1 G(u,v)[T(v,\rho(v)) - T(v,\sigma(v))]dv\right| \\
&\le \int_0^1 G(u,v)|T(v,\rho(v)) - T(v,\sigma(v))|dv \\
&\le \int_0^1 G(u,v)e^{-\tau}|\rho(v) - \sigma(v)|dv \\
&\le e^{-\tau}\|\rho-\sigma\|_\infty \sup_{u\in I}\int_0^1 G(u,v)dv \\
&\le e^{-\tau}\|\rho-\sigma\|_\infty.
\end{aligned}$$

i.e.,

$$\|T\rho(u) - T\sigma(u)\|_\infty \le e^{-\tau}\|\rho-\sigma\|_\infty$$

or

$$\tilde{p}(T\rho, T\sigma) \le e^{-\tau}\tilde{p}(\rho,\sigma).$$

Taking logarithm, we get

$$log(\tilde{p}(T\rho, T\sigma)) \le log(e^{-\tau}\tilde{p}(\rho,\sigma))$$

or

$$\tau + log(\tilde{p}(T\rho, T\sigma)) \le log(\tilde{p}(\rho,\sigma)).$$

Let the function $F : \mathbb{R}^+ \to \mathbb{R}$ be defined as $F(\rho) = \ln\rho$, then $F \in \mathcal{F}$.
Let $\alpha : X \times X \to \{-\infty\} \cup (0,\infty)$ be defined by

$$\alpha(\rho,\sigma) = \begin{cases} 1, & \text{if } \alpha_1(\rho(u),\sigma(u)) > 0, u \in [0,1], \\ -\infty, & otherwise. \end{cases}$$

Therefore,

$$\tau + \alpha(\rho, \sigma)F(\tilde{p}(T\rho, T\sigma)) \le F(\tilde{p}(\rho, \sigma)) \le F(\tilde{p}(\rho, \sigma)) + L\tilde{p}^w(\sigma, T\rho),$$

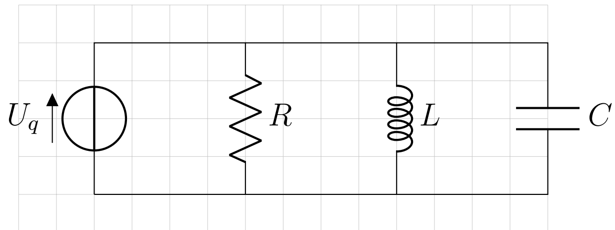
for all $\rho, \sigma \in X$ with $\tilde{p}(T\rho, T\sigma) > 0$ and $L \ge 0$. Thus, T is an almost α- F- weak contraction. From (2) there exists $\rho_0 \in X$ so that $\alpha(\rho_0, T\rho_0) \ge 1$.
Next, by using (3), we get, for all $\rho, \sigma \in X$

$$\alpha(\rho, \sigma) \ge 1 \Rightarrow \alpha_1(\rho(u), \sigma(u)) > 0 \Rightarrow \alpha_1(T\rho(u), T\sigma(u)) > 0 \Rightarrow \alpha(T\rho, T\sigma) \ge 1,$$

for all $u \in [0, 1]$. Hence T is α-admissible. Finally, using (4) all the postulates of Theorem 2.8 are verified and consequently, $\rho = T\rho$ and so ρ is a solution of (3.1). □

3.2. ***Differential Calculus.***

Nowadays solar panels are manufactured and encouraged to lessen the enslavement human race on the fewer merciful fossil fuels. The amount of energy to meet the demand needs an area of about 99858 square km. Consequently, it is essential that the region within Sun Belt should instantly begin establishing the essential technological requirements for efficient management of energy. Although , we are aware of the idea of concentrating solar radiations to create high temperatures and transform it into electricity for more than a century we did not utilize it .
The sunlight which reached the surface of the earth in one hour is sufficient to power the world for a period of nearly 12 months. Motivated by this fact, we utilize the results to solve a boundary value problem emerging in real-life model which converts solar energy to electrical energy. The reformation of solar energy to electric energy is modelled as a Cauchy problem.



Differential equation of such a problem can be given as:

$$\begin{cases} \frac{d^2I}{du^2} + \frac{R}{L}\frac{dI}{du} = K(u, I(u)); \\ I(0) = 0, I'(0) = 0, \end{cases} \tag{3.2}$$

where $K : [0, 1] \times \mathbb{R}^+ \to \mathbb{R}$ is a function. The Green function associated to (3.2) is

$$G(u, v) = \begin{cases} (u - v)e^{\tau(u-v)}, & 0 \le v \le u \le 1 \\ 0, & 0 \le u \le v \le 1, \end{cases} \tag{3.3}$$

where , $\tau > 0$ is determined in terms of R and L. Let $X = C([0, 1], R^+)$. For an arbitrary $\rho \in X$, we define

$$\|\rho\|_\tau = \sup_{u \in [0,1]} \{|\rho(u)|e^{-\tau u}\} \tag{3.4}$$

and $\tilde{p} : X \times X \to \mathbb{R}^+$ by

$$\tilde{p}(\rho, \sigma) = \|\rho - \sigma\|_\tau = \sup_{u \in [0,1]} \{|\rho(u) - \sigma(u)| e^{-\tau u}\}. \tag{3.5}$$

Clearly, $(X, \tilde{p})$ is a partial metric space [8]:

Theorem 3.2. *Let $T : C([0,1]) \to C([0,1])$ be a self mapping of a 0-complete partial metric space $(C([0,1]), \tilde{p})$ and $\alpha_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that:*

1. *a function $K : [0,1] \times \mathbb{R}^+ \to \mathbb{R}$ satisfying $|K(v,\rho) - K(v,\sigma)| \leq \tau^2 e^{-\tau} \tilde{p}(\rho,\sigma)$, for $\tau \in \mathbb{R}^+$, $v \in [0,1]$ and ρ, $\sigma \in \mathbb{R}^+$;*
2. *$\rho_0 \in C([0,1], \tilde{p})$ such that $\alpha_1(\rho_0(u), T\rho_0(u)) \geq 0$ and $T : C([0,1], \tilde{p}) \to C([0,1], \tilde{p})$ is defined by*
$$T\rho(u) = \int_0^u G(u,v) K(v, \rho(v)) dv;$$
3. *$\alpha_1(\rho(u), \sigma(u)) > 0$ implies $\alpha_1(T\rho(u), T\sigma(u)) > 0$, for $\rho, \sigma \in C([0,1], \tilde{p})$;*
4. *$\{\rho_n\}$ is a sequence in $C([0,1], \tilde{p})$ so that $\rho_n \to \rho$ in $C([0,1], \tilde{p})$ and $\alpha_1(\rho_n(u), \rho_{n+1}(u)) > 0$, $n \in \mathbb{N}$, then $\alpha_1(\rho_n(u), \rho(u)) > 0$, $n \in \mathbb{N}$.*

Then equation (3.2) *has a solution.*

Proof . Cauchy problem (3.2) is comparable to

$$\rho(u) = \int G(u,v) K(v, \rho(v)) dv, v \in [0,1].$$

Thus, ρ is a solution of (3.2), iff ρ is a fixed point of T. Here

$$\begin{aligned} |T\rho(u) - T\sigma(u)| &\leq \int_0^u G(u,v) |K(v,\rho(v)) - K(v,\sigma(v))| dv \\ &\leq \int_0^u G(u,v) \tau^2 e^{-\tau} \tilde{p}(\rho,\sigma) dv \\ &\leq \int_0^u \tau^2 e^{-\tau} e^{2\tau v} e^{-2\tau v} \tilde{p}(\rho,\sigma) G(u,v) dv \\ &\leq \tau^2 e^{-\tau} ||\tilde{p}(\rho,\sigma)||_\tau \times \int_0^u e^{2\tau v} G(u,v) dv \\ &\leq \tau^2 e^{-\tau} ||\tilde{p}(\rho,\sigma)||_\tau \times \left(\frac{e^{\tau u}}{\tau^2} (e^{\tau u} - u\tau - 1) \right) \end{aligned}$$

$$i.e., |T\rho(u) - T\sigma(u)| e^{-\tau u} \leq e^{-\tau} \|\tilde{p}(\rho,\sigma)\|_\tau \times (e^{\tau u} - u\tau - 1).$$

Since $(e^{\tau u} - u\tau - 1) \leq 1$,

$$\|T\rho(u) - T\sigma(u)\|_\tau \leq e^{-\tau} \|\tilde{p}(\rho,\sigma)\|_\tau.$$

Hence,

$$\tilde{p}(T\rho, T\sigma) \leq e^{-\tau}\|\tilde{p}(\rho,\sigma)\|_{\tau}.$$

Taking logarithm,

$$ln(\tilde{p}(T\rho, T\sigma)) \leq ln[e^{-\tau}(\|\tilde{p}(\rho,\sigma)\|_{\tau})]$$

or

$$\tau + ln(\tilde{p}(T\rho, T\sigma)) \leq (ln\|p\|_{\tau}).$$

Let the function $F : \mathbb{R}^{+} \to \mathbb{R}$ be defined as $F(\rho) = ln\rho$, $\rho \in X$, then $F \in \mathcal{F}$, and $\alpha : X \times X \to \{-\infty\} \cup (0, \infty)$ as

$$\alpha(\rho,\sigma) = \begin{cases} 1, & \text{if} \quad \alpha_1(\rho(u), \sigma(u)) > 0, u \in [0,1], \\ -\infty, & otherwise. \end{cases}$$

Therefore,

$$\tau + \alpha(\rho,\sigma)F(\tilde{p}(T\rho, T\sigma)) \leq F(\tilde{p}(\rho,\sigma)) \leq F(\tilde{p}(\rho,\sigma)) + L\tilde{p}^{w}(\sigma, T\rho),$$

for all $\rho, \sigma \in X$ with $\tilde{p}(T\rho, T\sigma) > 0$ and $L \geq 0$.
$\Rightarrow T$ is an almost α- F- weak contraction. From (2) there exists $\rho \in X$ so that $\alpha(\rho, T\rho) \geq 1$. Using (3), we get for all $\rho, \sigma \in X$,

$$\alpha(\rho,\sigma) \geq 1 \Rightarrow \alpha_1(\rho(u), \sigma(u)) > 0 \Rightarrow \alpha_1(T\rho(u), T\sigma(u)) > 0 \Rightarrow \alpha(T\rho, T\sigma) \geq 1,$$

for all $u \in [0, 1]$. Hence T is α-admissible. Finally, using (4) all the postulates of Theorem (2.8) are verified. Hence, $\rho = T\rho$ and so ρ is a solution of the problem (3.2). $\square$

4. Conclusion

In this manuscript, we familiarized an almost α-F-contraction and an almost α-F- weak contraction to establish a fixed point in a 0-complete partial metric space. These novel ideas prompt further examinations and applications. Our study is encouraged by the possible applications of partial metric space. These applications perform a central role in some nonlinear processes originating in biology, economics and numerical physics.

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