Int. J. Nonlinear Anal. Appl. 8 (2017) No. 1, 81-93
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2016.504

Uncertainty in linear fractional transportation problem

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(Communicated by M. Eshaghi)

Abstract

In this paper, we study the linear fractional transportation problem with uncertain parameters. After recalling some definitions, concepts and theorems in uncertainty theory we present three approaches for solving this problem. First we consider the expected value of the objective function together with the expectation of satisfying constraints. Optimizing the expected value of the objective function with considering chance constrained method for the restrictions is our second approach. In the third approach we add the objective function to the constraints and solve again the problem by chance constrained method. A numerical example is solved by three approaches and their solutions are compared.

Keywords: Transportation Problem; Linear Fractional Programming; Uncertain Measure; Uncertain Variable; Uncertain Programming.
2010 MSC: Primary 26A25; Secondary 39B62.

1. Introduction

Linear fractional transportation problem was first proposed by Swarup [21]. He studied the problem of finding optimal ratio of two linear function subject to a set of linear constraints and non negativity conditions on the variables. Dorina Moanta has also presented a solution to a three dimensional problem with an objective function which is the ratio of two linear functions [16]. Sivri et all, proposed an optimal or near optimal initial solution and the optimality condition for linear fractional

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Received: December 2015 *Revised:* May 2016

transportation problem [19]. Guzel et all. have presented a solution to the interval linear fractional transportation problem [6].

More studies concern on the deterministic cases for which the parameters in the models are precisely known. However, in real world applications, there are cases that the parameters might be inexact and have to be estimated. Some indeterminacy factors might occur in the problems due to the lack of history data in practice. So it is not suitable to employ the classical models and algorithms in these situations.

We are frequently lack of observed data about the unknown state of nature, not only for technical difficulties, but also for economic reasons. In this case, the probability theory is no longer valid. In order to deal with this indeterminacy phenomenon, uncertainty theory was founded and redefined by Liu [13, 14]. Today, uncertain measure plays an important role in dealing with belief degrees in uncertainty theory. In this regard, uncertain variable was defined by Liu to indicate the quantities with uncertainty. Liu and Ha derived a useful formula for calculating the expected values of strictly monotone function of independent uncertain variables [15]. Up to now, uncertainty theory has become a completely mathematical system.

Initially, uncertain programming was founded by Liu [11]. Since then, it was widely applied to deal with uncertain problems by many researchers. Sheng and Yao presented a transportation model with uncertain costs and demands and an uncertain programming model for fixed charge transportation problem [18, 17]. Cui and Sheng proposed an uncertain model for solid transportation problem [3]. Guo et all, presented a transportation problem with uncertain costs and random supplies [5]. Nowadays, the uncertainty theory has become a branch of mathematics which models all uncertainties in human world.

This article concerns about linear fractional transportation problem in uncertain environment in which supplies, demands and the coefficients of objective function are supposed to be uncertain variables.

The rest of this paper proceeds as follows. In Section 2, some basic concepts and results of uncertainty theory are presented. Then, in Section 3, we construct an uncertain linear fractional transportation model. In section 4, according to expected value of an uncertain variable and inverse uncertainty distribution, we will show that the model can be transformed to its deterministic form by three approaches. Finally, in last two sections we bring a numerical example and present a brief summary.

2. Preliminaries

In this section, we recall some notations, definitions, and the concept of uncertainty theory, which will be used throughout in the paper.

Definition 2.1. (Liu, [10], [13]) Let Γ be a nonempty set and $\mathcal{L}$ be a σ-algebra over Γ. Each element $\Lambda \in \mathcal{L}$ is called an event. A set function $\mathscr{M}$ from $\mathcal{L}$ to [0,1] is called an uncertain measure if it satisfies the following axioms:
Axiom 1. (Normality Axiom) $\mathscr{M}\{\Lambda\} = 1$ for the universal set Γ;
Axiom 2. (Duality Axiom) $\mathscr{M}\{\Lambda\} + \mathscr{M}\{\Lambda^c\} = 1$ for any event Λ;
Axiom 3. (Subadditivity Axiom) For every countable sequence of events $\Lambda_1, \Lambda_2, \ldots$, we have

$$\mathscr{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathscr{M}\left\{\Lambda_i\right\}$$

The triplet $(\Gamma, \mathcal{L}, \mathscr{M})$ is called an uncertainty space. The product uncertain measure was defined via the following product axiom:
Axiom 4.(Product Axiom) Let $(\Gamma_k, \mathcal{L}_k, \mathscr{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots$. The product uncertain measure $\mathscr{M}$ is an uncertain measure satisfying

$$\mathscr{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathscr{M}_k\left\{\Lambda_k\right\}$$

where Λ_k are arbitrary chosen events from $\mathcal{L}_k$ for $k = 1, 2, \ldots$, respectively.

Definition 2.2. (Liu, [13]) An uncertain variable is a measurable function ξ from an uncertainry space $(\Gamma, \mathcal{L}, \mathscr{M})$ to the set of real numbers, i.e., for any Borel set B of real numbers, the set $\xi^{-1}(B) = \{\gamma \in \Gamma | \xi(\gamma) \in B\}$ is an event.

Definition 2.3. (Liu, [13]) An uncertain variable ξ on the uncertainty space $(\Gamma, \mathcal{L}, \mathscr{M})$ is said to be positive if $\mathscr{M}\{\xi \leq 0\} = 0$.

Definition 2.4. (Liu, [13]) The uncertainty distribution Φ of an uncertain variable ξ is defined by $\Phi(x) = \mathscr{M}\left\{\xi \leq x\right\}$ for any real number x.

Definition 2.5. (Liu, [13]) An uncertainty distribution $\Phi(x)$ is said to be regular if it is a continuous and strictly increasing function with respect to x at which $0 < \Phi(x) < 1$ and

$$\lim_{x \to -\infty} \Phi(x) = 0, \lim_{x \to +\infty} \Phi(x) = 1.$$

Clearly, a regular uncertainty distribution $\Phi(x)$ has an inverse function on the range of x with $0 < \Phi(x) < 1$, and the inverse function $\Phi^{-1}(\alpha)$ exists on the open interval $(0, 1)$.

Definition 2.6. (Liu, [13]) The uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$ are said to be independent if

$$\mathscr{M}\left\{\bigcap_{i=1}^{n} (\xi_i \in B_i)\right\} = \bigwedge_{i=1}^{n} \mathscr{M}\left\{\xi_i \in B_i\right\}$$

for any Borel sets $B_1, B_2, \ldots, B_n$ of real numbers.

Theorem 2.7. (Liu, [14]) Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If the function $f(x_1, x_2, \ldots, x_n)$ is strictly increasing with respect to $x_1, x_2, \ldots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \ldots, x_n$ then the uncertain variable $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ has an inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = f\left(\Phi_1^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)\right)$$

Definition 2.8. (Liu, [13]) Let ξ be an uncertain variable. Then the expected value of ξ is defined by

$$E[\xi] = \int_{0}^{+\infty} \mathscr{M}\{\xi \geq x\} dx - \int_{-\infty}^{0} \mathscr{M}\{\xi \leq x\} dx$$

provided that at least one of the two integrals is finite.

Let ξ is an uncertain variable with regular uncertainty distribution Φ. Liu [13] proved that $E[\xi] = \int_0^1 \Phi^{-1}(\alpha)d\alpha$.

Theorem 2.9. (Liu and Ha, [15]) Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If the function $f(x_1, x_2, \ldots, x_n)$ is strictly increasing with respect to $x_1, x_2, \ldots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \ldots, x_n$ then the uncertain variable $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ has an expected value

$$E[\xi] = \int_0^1 f\left(\Phi_1^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)\right)\, \mathrm{d}\alpha$$

Definition 2.10. (Liu, [13]) An uncertain variable ξ is called linear if it has a linear uncertainty distribution

$$\Phi(x) = \begin{cases} 0, & \text{if } x \le a \\ \frac{x-a}{b-a}, & \text{if } a \le x \le b \\ 1, & \text{if } x \ge b \end{cases}$$

denoted by $L(a, b)$ where a and b are real numbers with $a < b$.

It is clear that a linear uncertain variable is regular and the inverse uncertainty distribution of linear uncertain variable $L(a, b)$ is

$$\Phi^{-1}(\alpha) = (1-\alpha)a + \alpha b.$$

The linear uncertain variable $\xi = L(a, b)$ has an expected value

$$E[\xi] = \frac{a+b}{2}.$$

Definition 2.11. (Liu, [13]) An uncertain variable ξ is called zigzag if it has a zigzag uncertainty distribution

$$\Phi(x) = \begin{cases} 0, & \text{if } x \le a \\ \frac{x-a}{2(b-a)}, & \text{if } a \le x \le b \\ \frac{x+c-2b}{2(c-b)}, & \text{if } b \le x \le c \\ 1, & \text{if } x \ge c \end{cases}$$

denoted by $\mathcal{Z}(a, b, c)$ where a, b, c are real numbers with $a < b < c$.

It is clear that a zigzag uncertain variable is regular. The inverse uncertainty distribution of zigzag uncertain variable $\mathcal{Z}(a, b, c)$ is

$$\Phi^{-1}(\alpha) = \begin{cases} (1-2\alpha)a + 2ab, & \text{if } \alpha < 0.5 \\ (2-2\alpha)b + (2\alpha-1)c, & \text{if } \alpha \ge 0.5 \end{cases}$$

The zigzag uncertain variable $\xi = \mathcal{Z}(a, b, c)$ has an expected value

$$E[\xi] = \frac{a+2b+c}{4}.$$

Definition 2.12. (Liu, [13]) An uncertain variable ξ is called normal if it has a normal uncertainty distribution

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(\mu - x)}{\sqrt{3}\sigma}\right)\right)^{-1}, \quad x \in \mathbb{R}$$

denoted by $\mathcal{N}(\mu, \sigma)$ where μ and σ are real numbers with $\sigma > 0$.

It is clear that a normal uncertain variable is regular. The inverse uncertainty distribution of normal uncertain variable $\mathcal{N}(\mu, \sigma)$ is

$$\Phi^{-1}(\alpha) = \mu + \frac{\sigma\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}.$$

The normal uncertain variable $\xi = \mathcal{N}(\mu, \sigma)$ has an expected value

$$E[\xi] = \mu.$$

Definition 2.13. (Liu, [13]) An uncertain variable ξ is called lognormal if $\ln \xi$ is a normal uncertain variable $\mathcal{N}(\mu, \sigma)$. In other words, a lognormal uncertain variable has an uncertainty distribution

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(\mu - \ln x)}{\sqrt{3}\sigma}\right)\right)^{-1}, \quad x > 0$$

denoted by $\mathcal{LOGN}(\mu, \sigma)$ where μ and σ are real numbers with $\sigma > 0$.

It is clear that a lognormal uncertain variable is regular. The inverse uncertainty distribution of lognormal uncertain variable $\mathcal{LOGN}(\mu, \sigma)$ is

$$\Phi^{-1}(\alpha) = \exp\left(\mu + \frac{\sigma\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}\right).$$

The lognormal uncertain variable $\xi = \mathcal{LOGN}(\mu, \sigma)$ has an expected value

$$E[\xi] = \begin{cases} \sigma\sqrt{3}\exp(\mu)\csc(\sqrt{3}), & \text{if } \sigma < \pi/\sqrt{3} \\ +\infty, & \text{if } \sigma \geq \pi/\sqrt{3} \end{cases}.$$

3. Uncertain Linear Fractional Transportation Model

Consider the following form of linear fractional transportation problem with m sources and n destinations.

$$\begin{cases} \min \quad Q(x) = \dfrac{\sum_{i=1}^{m}\sum_{j=1}^{n} p_{ij}x_{ij}}{\sum_{i=1}^{m}\sum_{j=1}^{n} d_{ij}x_{ij}} \\ \text{subject to} \\ \qquad \sum_{j=1}^{n} x_{ij} \leq a_i, \qquad i = 1, 2, ..., m, \\ \qquad \sum_{i=1}^{m} x_{ij} \geq b_j, \qquad j = 1, 2, ..., n, \\ \qquad x_{ij} \geq 0, \qquad i = 1, 2, ...m, \qquad j = 1, 2, ..., n. \end{cases} \tag{3.1}$$

A single product is to be shipped from the sources to destinations. The total supply from source i is a_i, and total demand at destiantion j is b_j. The cost and the profit of transporting a unite from source i to destination j, are p_{ij} and d_{ij} respectively. The variable x_{ij} denotes the amount transported from source i to destination j. We assume that the denominator of objective function is positive throughout the constraints and total supply is not less than total demand.

In model (3.1) the quantities a_i, b_j, p_{ij} and d_{ij} are precisely known. Due to complexity of real world, the parameters might be inexact but obtained from experience evaluation or expert knowledge. In this case, we may assume the quantities are uncertain variables. We assume that a_i, b_j, p_{ij} and d_{ij} are all uncertain variables and denote them as $\tilde{a}_i, \tilde{b}_j, \xi_{ij}$ and η_{ij}, respectively. Also suppose that, all the uncertain variables $\tilde{a}_i, \tilde{b}_j, \xi_{ij}$ and η_{ij} are independent. Thus, the uncertainty linear fractional transportation problem is formulated as follows,

$$\begin{cases} 2\min \quad \dfrac{\sum_{i=1}^{m}\sum_{j=1}^{n}\xi_{ij}x_{ij}}{\sum_{i=1}^{m}\sum_{j=1}^{n}\eta_{ij}x_{ij}} \\ \text{subject to} \\ \qquad \displaystyle\sum_{j=1}^{n} x_{ij} \leq \tilde{a}_i, \qquad i=1,2,...,m, \\ \qquad \displaystyle\sum_{i=1}^{m} x_{ij} \geq \tilde{b}_j, \qquad j=1,2,...,n, \\ \qquad x_{ij} \geq 0, \qquad i=1,2,...m, \qquad j=1,2,...,n. \end{cases} \tag{3.2}$$

4. The Crisp Equivalences of Model

In this section, by using three methods, we will convert model 3.2 to deterministic model. The expected value model is a most understandable method for modeling. Here we take expected value critrion on the numerator and denominator of objective function and on the constraint functions. Then the model (3.2) turns into the following mathematical model,

$$\begin{cases} \min \quad \dfrac{E\left[\sum_{i=1}^{m}\sum_{j=1}^{n}\xi_{ij}x_{ij}\right]}{E\left[\sum_{i=1}^{m}\sum_{j=1}^{n}\eta_{ij}x_{ij}\right]} \\ \text{subject to} \\ \qquad E\left[\displaystyle\sum_{j=1}^{n} x_{ij} - \tilde{a}_i\right] \leq 0, \qquad i=1,2,\ldots,m, \\ \qquad E\left[\displaystyle\sum_{i=1}^{m} x_{ij} - \tilde{b}_j\right] \geq 0, \qquad j=1,2,\ldots,n, \\ \qquad x_{ij} \geq 0, \qquad i=1,2,...m, \qquad j=1,2,...,n. \end{cases} \tag{4.1}$$

If we take expected value criterion on the numerator and denominator of objective function and confidence level on the constraint functions, then the model (3.2) turns into the following mathematical model,

$$\begin{cases} \min & \dfrac{E\left[\sum_{i=1}^{m}\sum_{j=1}^{n}\xi_{ij}x_{ij}\right]}{E\left[\sum_{i=1}^{m}\sum_{j=1}^{n}\eta_{ij}x_{ij}\right]} \\ \text{subject to} & \\ & \mathscr{M}\left\{\sum_{j=1}^{n}x_{ij}\le\tilde{a}_i\right\}\ge\alpha_i, \qquad i=1,2,...,m, \\ & \mathscr{M}\left\{\sum_{i=1}^{m}x_{ij}\ge\tilde{b}_j\right\}\ge\beta_j, \qquad j=1,2,...,n, \\ & x_{ij}\ge 0, \qquad i=1,2,...m, \qquad j=1,2,...,n. \end{cases} \tag{4.2}$$

where α_i and β_j are some predetermined confidence levels for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

The chance constrained programming is another method to deal with optimal problem in uncertain environment. In this method the decision maker hopes to get a smallest value $\bar{f}$ such that uncertain variable $\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}\xi_{ij}x_{ij}}{\sum_{i=1}^{m}\sum_{j=1}^{n}\eta_{ij}x_{ij}}$ is less than or equal to $\bar{f}$ with confidence level $\alpha \in (0, 1)$. If the decision maker prefers treating the problem under the chance constraints, the model (3.2) can be constructed as the following model.

$$\begin{cases} \min & \bar{f} \\ \text{subject to} & \\ & \mathscr{M}\left\{\dfrac{\sum_{i=1}^{m}\sum_{j=1}^{n}\xi_{ij}x_{ij}}{\sum_{i=1}^{m}\sum_{j=1}^{n}\eta_{ij}x_{ij}}\le\bar{f}\right\}\ge\alpha \\ & \mathscr{M}\left\{\sum_{j=1}^{n}x_{ij}\le\tilde{a}_i\right\}\ge\alpha_i, \qquad i=1,2,...,m, \\ & \mathscr{M}\left\{\sum_{i=1}^{m}x_{ij}\ge\tilde{b}_j\right\}\ge\beta_j, \qquad j=1,2,...,n, \\ & x_{ij}\ge 0, \qquad i=1,2,...m, \qquad j=1,2,...,n. \end{cases} \tag{4.3}$$

The next theorems help us to convert the above models to determined models.

Theorem 4.1. *Suppose that $\tilde{a}_i, \tilde{b}_j, \xi_{ij}$ and η_{ij} are independent uncertain variables with regular uncertainty distribution $\Psi_i, \Upsilon_j, \Phi_{ij}$ and Θ_{ij}, respectively. Then* (4.1) *is equivalent to*

$$\begin{cases} \min & \dfrac{\sum_{i=1}^{m}\sum_{j=1}^{n}x_{ij}\int_0^1\Phi_{ij}^{-1}(\alpha)\,\mathrm{d}\alpha}{\sum_{i=1}^{m}\sum_{j=1}^{n}x_{ij}\int_0^1\Theta_{ij}^{-1}(\alpha)\,\mathrm{d}\alpha} \\ \text{subject to} & \\ & \sum_{j=1}^{n}x_{ij}\le\int_0^1\Psi_i^{-1}(\alpha)\,\mathrm{d}\alpha, \qquad i=1,2,...,m, \\ & \sum_{i=1}^{m}x_{ij}\ge\int_0^1\Upsilon_j^{-1}(\alpha)\,\mathrm{d}\alpha, \qquad j=1,2,...,n, \\ & x_{ij}\ge 0, \qquad i=1,2,...m, \qquad j=1,2,...,n. \end{cases} \tag{4.4}$$

Proof . Liu [13] showed that for independent uncertaint variables ξ and η with regular uncertainty distribution Φ and Θ and for any real numbers a and b we have

$$E[a\xi + b\eta] = aE[\xi] + bE[\eta] = a\int_o^1 \Phi^{-1}(\alpha)\ \mathrm{d}\alpha + b\int_0^1 \Theta^{-1}(\alpha)\ \mathrm{d}\alpha.$$

Thus we have

$$E\left[\sum_{i=1}^{m}\sum_{j=1}^{n}\xi_{ij}x_{ij}\right] = \sum_{i=1}^{m}\sum_{j=1}^{n}E[\xi_{ij}]x_{ij} = \sum_{i=1}^{m}\sum_{j=1}^{n}x_{ij}\int_0^1 \Phi_{ij}^{-1}(\alpha)\ \mathrm{d}\alpha$$
$$E\left[\sum_{i=1}^{m}\sum_{j=1}^{n}\eta_{ij}x_{ij}\right] = \sum_{i=1}^{m}\sum_{j=1}^{n}E[\eta_{ij}]x_{ij} = \sum_{i=1}^{m}\sum_{j=1}^{n}x_{ij}\int_0^1 \Theta_{ij}^{-1}(\alpha)\ \mathrm{d}\alpha.$$

For every $i \in \{1, 2, \ldots, m\}$ we have

$$\begin{aligned} E\left[\sum_{j=1}^{n}x_{ij} - \tilde{a}_i\right] \le 0 \quad &\Longleftrightarrow \quad \sum_{j=1}^{n}x_{ij} - E\left[\tilde{a}_i\right] \le 0 \\ &\Longleftrightarrow \quad \sum_{j=1}^{n}x_{ij} \le E\left[\tilde{a}_i\right] \\ &\Longleftrightarrow \quad \sum_{j=1}^{n}x_{ij} \le \int_0^1 \Psi_i^{-1}(\alpha)\ \mathrm{d}\alpha. \end{aligned}$$

Similarly

$$E\left[\sum_{i=1}^{m}x_{ij} - \tilde{b}_j\right] \ge 0, \qquad j = 1, 2, \ldots, n$$

is equivalent to

$$\sum_{i=1}^{m}x_{ij} \ge \int_0^1 \Upsilon_j^{-1}(\alpha)\ \mathrm{d}\alpha, \qquad j = 1, 2, \ldots, n.$$

The theorem is proved. □

Theorem 4.2. *Suppose that $\tilde{a}_i, \tilde{b}_j, \xi_{ij}$ and η_{ij} are independent uncertain variables with regular uncertainty distribution $\Psi_i, \Upsilon_j, \Phi_{ij}$ and Θ_{ij}, respectively. Then the model (4.2) is equivalent to the following model.*

$$\begin{cases} \min \quad \dfrac{\sum_{i=1}^{m}\sum_{j=1}^{n}x_{ij}\int_0^1 \Phi_{ij}^{-1}(\alpha)\ \mathrm{d}\alpha}{\sum_{i=1}^{m}\sum_{j=1}^{n}x_{ij}\int_0^1 \Theta_{ij}^{-1}(\alpha)\ \mathrm{d}\alpha} \\ \text{subject to} \\ \qquad \displaystyle\sum_{j=1}^{n}x_{ij} \le \Psi_i^{-1}(1-\alpha_i), \qquad i = 1, 2, ..., m, \\ \qquad \displaystyle\sum_{i=1}^{m}x_{ij} \ge \Upsilon_j^{-1}(\beta_j), \qquad j = 1, 2, ..., n, \\ \qquad x_{ij} \ge 0, \qquad i = 1, 2, ...m, \qquad j = 1, 2, ..., n. \end{cases} \tag{4.5}$$

Proof . According to the previous theorem the objective function of model (4.5) is equivalent to the objective function of model (4.2). Since $\tilde{a}_i$ and $\tilde{b}_j$ have uncertainty distributions Ψ_i and Υ_j,we have

$$\begin{aligned}
\mathscr{M}\left\{\sum_{j=1}^{n} x_{ij} \leq \tilde{a}_i\right\} \geq \alpha_i \quad &\Longleftrightarrow \quad 1-\mathscr{M}\left\{\sum_{j=1}^{n} x_{ij} > \tilde{a}_i\right\} \geq \alpha_i \\
&\Longleftrightarrow \quad 1-\Psi_i\left(\sum_{j=1}^{n} x_{ij}\right) \geq \alpha_i \\
&\Longleftrightarrow \quad \Psi_i\left(\sum_{j=1}^{n} x_{ij}\right) \leq 1-\alpha_i \\
&\Longleftrightarrow \quad \sum_{j=1}^{n} x_{ij} \leq \Psi_i^{-1}(1-\alpha_i)
\end{aligned}$$

and

$$\begin{aligned}
\mathscr{M}\left\{\sum_{i=1}^{m} x_{ij} \geq \tilde{b}_j\right\} \geq \beta_j \quad &\Longleftrightarrow \quad \Upsilon_j\left(\sum_{i=1}^{m} x_{ij}\right) \geq \beta_j \\
&\Longleftrightarrow \quad \sum_{i=1}^{m} x_{ij} \geq \Upsilon_j^{-1}(\beta_j).
\end{aligned}$$

The proof is completed. □

Theorem 4.3. *Suppose that $\tilde{a}_i, \tilde{b}_j, \xi_{ij}$ and η_{ij} are independent uncertain variables with regular uncertainty distribution $\Psi_i, \Upsilon_j, \Phi_{ij}$ and Θ_{ij}, respectively. If ξ_{ij} and η_{ij} are positive uncertain variables, then the model* (4.3) *is equivalent to the following model.*

$$\begin{cases}
\min \quad \dfrac{\sum_{i=1}^{m}\sum_{j=1}^{n}\Phi_{ij}^{-1}(\alpha)x_{ij}}{\sum_{i=1}^{m}\sum_{j=1}^{n}\Theta_{ij}^{-1}(1-\alpha)x_{ij}} \\
\text{subject to} \\
\quad \sum_{j=1}^{n} x_{ij} \leq \Psi_i^{-1}(1-\alpha_i), \qquad i=1,2,...,m, \\
\quad \sum_{i=1}^{m} x_{ij} \geq \Upsilon_j^{-1}(\beta_j), \qquad j=1,2,...,n, \\
\quad x_{ij} \geq 0, \qquad i=1,2,...m, \qquad j=1,2,...,n.
\end{cases} \tag{4.6}$$

Proof . Suppose that uncertainty variable $\xi = \frac{\sum_{i=1}^{m}\sum_{j=1}^{n}\xi_{ij}x_{ij}}{\sum_{i=1}^{m}\sum_{j=1}^{n}\eta_{ij}x_{ij}}$ has uncertainty distribution Φ. Let

$$f(y_{11},\ldots,y_{1n},\ldots,y_{mn},t_{11},\ldots,t_{1n},\ldots,t_{mn}) = \frac{\sum_{i=1}^{m}\sum_{j=1}^{n}x_{ij}y_{ij}}{\sum_{i=1}^{m}\sum_{j=1}^{n}x_{ij}t_{ij}}.$$

It is obvious that this function is strictly increasing with respect to $y_{11},\ldots,y_{1n},\ldots,y_{mn}$ and strictly decreasing with respect to $t_{11},\ldots,t_{1n},\ldots,t_{mn}$. By Theorem 2.7 the uncertain variable ξ has an inverse uncertainty distribution

$$\Phi^{-1}(\alpha) = \frac{\sum_{i=1}^{m}\sum_{j=1}^{n}\Phi_{ij}^{-1}(\alpha)x_{ij}}{\sum_{i=1}^{m}\sum_{j=1}^{n}\Theta_{ij}^{-1}(1-\alpha)x_{ij}}.$$

So we have

$$\mathscr{M}\left\{\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}\xi_{ij}x_{ij}}{\sum_{i=1}^{m}\sum_{j=1}^{n}\eta_{ij}x_{ij}}\leq \bar{f}\right\}\geq\alpha \iff \Phi(\bar{f})\geq\alpha$$
$$\iff \Phi^{-1}(\alpha)\leq\bar{f}$$
$$\iff \frac{\sum_{i=1}^{m}\sum_{j=1}^{n}\Phi_{ij}^{-1}(\alpha)x_{ij}}{\sum_{i=1}^{m}\sum_{j=1}^{n}\Theta_{ij}^{-1}(1-\alpha)x_{ij}}\leq\bar{f}.$$

According to previous theorem the constraints of model (4.3) is equivalent to the constraints of models (4.6). So model (4.3) is equivalent to the following model

$$\begin{cases} \min \quad \bar{f} \\ \text{subject to} \\ \qquad \dfrac{\sum_{i=1}^{m}\sum_{j=1}^{n}\Phi_{ij}^{-1}(\alpha)x_{ij}}{\sum_{i=1}^{m}\sum_{j=1}^{n}\Theta_{ij}^{-1}(1-\alpha)x_{ij}}\leq\bar{f} \\ \qquad \sum_{j=1}^{n}x_{ij}\leq\Psi_i^{-1}(1-\alpha_i), \qquad i=1,2,...,m, \\ \qquad \sum_{i=1}^{m}x_{ij}\geq\Upsilon_j^{-1}(\beta_j), \qquad j=1,2,...,n, \\ \qquad x_{ij}\geq 0, \qquad i=1,2,...m, \qquad j=1,2,...,n. \end{cases} \tag{4.7}$$

Model (4.7) is equivalent to (4.6). The theorem is proved. □

The models (4.4),(4.5) and (4.6) are deterministic linear fractional programming. Hence, we may find easily their solutions.

5. Numerical Example

In this section we consider an example to illustrate the model. Suppose that there are three coal mines to supply the coal for five cities. The task for the decision-maker is to make the transportation plan for the next month. At the beginning of this task, the decision maker needs to obtain the basic data, such as supply capacity, demand, transportation profit and cost of unit product, and so on. In real world, we generally cannot get these data exactly. According to personal experience, we assume that the profits ξ_{ij}, the costs η_{ij}, the supply $\tilde{a}_i$ of mine i and the demand $\tilde{b}_j$ of city j follow normal uncertainty distribution.

$$\begin{aligned} &\xi_{ij}\sim\mathcal{N}(\mu_{ij},\sigma_{ij}), && i=1,2,3, && j=1,2,3,4,5, \\ &\eta_{ij}\sim\mathcal{N}(\mu'_{ij},\sigma'_{ij}), && i=1,2,3, && j=1,2,3,4,5, \\ &\tilde{a}_i\sim\mathcal{N}(\mu_i,\sigma_i), && i=1,2,3, \\ &\tilde{b}_j\sim\mathcal{N}(\mu'_j,\sigma'_j), && j=1,2,3,4,5 \end{aligned}$$

The corresponding uncertain data are listed as follows:

Table 1: The parameters of uncertainty normal distribution $\mathcal{N}(\mu_{ij}, \sigma_{ij})$ of unit costs

(μ_{ij}, σ_{ij})	1	2	3	4	5
1	(18,1)	(17,1.5)	(18,1.5)	(18,1)	(20,2.5)
2	(10,2)	(10,1.5)	(12,1.5)	(9,1)	(10,2)
3	(20,1.5)	(18,1)	(20,2.5)	(22,1.5)	(18,1.5)

Table 2: The parameters of uncertainty normal distribution $\mathcal{N}(\mu'_{ij}, \sigma'_{ij})$ of unit profits

$(\mu'_{ij}, \sigma'_{ij})$	1	2	3	4	5
1	(30,1.5)	(32,2)	(34,1.5)	(30,2)	(32,1.5)
2	(20,2)	(18,1.5)	(22,1.5)	(20,2.5)	(16,1)
3	(40,2)	(32,1.5)	(32,2)	(30,2)	(36,2.5)

Table 3: The parameters of normal distribution $\mathcal{N}(\mu_i, \sigma_i)$ of supplies

i	1	2	3
(μ_i, σ_i)	(24,2)	(32,1.5)	(30,2)

Table 4: The parameters of normal distribution $\mathcal{N}(\mu'_j, \sigma'_j)$ of demands

j	1	2	3	4	5
(μ'_j, σ'_j)	(12,1.5)	(10,1)	(16,1)	(10,1.5)	(14,1)

According to Theorem 4.1 the model (4.1) is equivalent to the following model

$$\begin{cases} \min \quad Q(x) = \dfrac{\sum_{i=1}^{3}\sum_{j=1}^{5}\mu_{ij}x_{ij}}{\sum_{i=1}^{3}\sum_{j=1}^{5}\mu'_{ij}x_{ij}} \\ \text{subject to} \\ \quad \displaystyle\sum_{j=1}^{5} x_{ij} \le \mu_i, \quad i = 1,2,3, \\ \quad \displaystyle\sum_{i=1}^{3} x_{ij} \ge \mu'_j, \quad j = 1,2,3,4,5, \\ \quad x_{ij} \ge 0, \quad i = 1,2,3, \quad j = 1,2,3,4,5. \end{cases} \tag{5.1}$$

This model is a linear fractional transportation problem and its feasible region is a convex polyhedron. Using the transportation simplex method for linear fractional transportation problem, we can solve this problem [1]. The optimal transportation plan is

$$\begin{cases} x_{11}=0 \quad x_{12}=10 \quad x_{13}=0 \quad x_{14}=0 \quad x_{15}=0 \\ x_{21}=0 \quad x_{22}=0 \quad x_{23}=22 \quad x_{24}=10 \quad x_{25}=0 \\ x_{31}=12 \quad x_{32}=0 \quad x_{33}=0 \quad x_{34}=0 \quad x_{35}=0 \end{cases}$$

The optimal objective function value for this model is $Q(x^*) = 0.484271$.

Using model (4.2) and Theorem 4.2, the corresponding equivalent model is as follows

$$\begin{cases} \min \qquad Q(x) = \dfrac{\sum_{i=1}^{3}\sum_{j=1}^{5}\mu_{ij}x_{ij}}{\sum_{i=1}^{3}\sum_{j=1}^{5}\mu'_{ij}x_{ij}} \\ \text{subject to} \\ \qquad \displaystyle\sum_{j=1}^{5} x_{ij} \le \mu_i + \frac{\sigma_i\sqrt{3}}{\pi}\ln\frac{1-\alpha_i}{\alpha_i}, \quad i=1,2,3, \\ \qquad \displaystyle\sum_{i=1}^{3} x_{ij} \ge \mu'_j + \frac{\sigma'_j\sqrt{3}}{\pi}\ln\frac{\beta_j}{1-\beta_j}, \quad j=1,2,3,4,5, \\ \qquad x_{ij}\ge 0, \quad i=1,2,3, \quad j=1,2,3,4,5. \end{cases} \tag{5.2}$$

Assume the confidence levels are $\alpha_i = \beta_j = 0.9$ for $i = 1,2,3$, and $j = 1,2,3,4,5$. The optimal transportation plan for this model is

$$\begin{cases} x_{11}=0 & x_{12}=11.2114 & x_{13}=0.2968 & x_{14}=0 & x_{15}=0 \\ x_{21}=1.4513 & x_{22}=0 & x_{23}=16.9145 & x_{24}=11.8171 & x_{25}=0 \\ x_{31}=12.3658 & x_{32}=0 & x_{33}=0 & x_{34}=0 & x_{35}=15.2114 \end{cases}$$

The optimal objective function value for this model is $Q(x^*) = 0.487961$.

If the decision maker prefers treating the problem under the chance constraints, then we use model (4.3). Assume the confidence levels $\alpha = \alpha_i = \beta_j = 0.9$, where $i = 1,2,3$ and $j = 1,2,3,4,5$. According to Theorem 4.3, the chance-constrained programming model for this example is equivalent to the following model

$$\begin{cases} \min \qquad Q(x) = \dfrac{\sum_{i=1}^{3}\sum_{j=1}^{5}\left(\mu_{ij} + \frac{\sigma_{ij}\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)x_{ij}}{\sum_{i=1}^{3}\sum_{j=1}^{5}\left(\mu'_{ij} + \frac{\sigma'_{ij}\sqrt{3}}{\pi}\ln\frac{1-\alpha}{\alpha}\right)x_{ij}} \\ \text{subject to} \\ \qquad \displaystyle\sum_{j=1}^{5} x_{ij} \le \mu_i + \frac{\sigma_i\sqrt{3}}{\pi}\ln\frac{1-\alpha_i}{\alpha_i}, \quad i=1,2,3, \\ \qquad \displaystyle\sum_{i=1}^{3} x_{ij} \ge \mu'_j + \frac{\sigma'_j\sqrt{3}}{\pi}\ln\frac{\beta_j}{1-\beta_j}, \quad j=1,2,3,4,5, \\ \qquad x_{ij}\ge 0, \quad i=1,2,3, \quad j=1,2,3,4,5. \end{cases} \tag{5.3}$$

The optimal transportation plan for this model is

$$\begin{cases} x_{11}=1.4513 & x_{12}=11.2114 & x_{13}=0 & x_{14}=0 & x_{15}=0 \\ x_{21}=0 & x_{22}=0 & x_{23}=18.3658 & x_{24}=11.8171 & x_{25}=0 \\ x_{31}=12.3658 & x_{32}=0 & x_{33}=0 & x_{34}=0 & x_{35}=15.2114 \end{cases}$$

The optimal objective function value for this model is $Q(x^*) = 0.581786$.

6. Conclusions

In this paper, we have formulated uncertainty version of linear fractional transportation problem. It was converted into a deterministic model by three approaches. (a) taking expected value on the objective function and constraints, (b) taking expected value on the objective function and confidence level on the constraint functions and (c) chance constrained model. A numerical example was given and its optimal solution was also found by the transportation simplex method to show the efficiency of the model for the linear fractional transportation problem.

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