



Existence of farthest points in Hilbert spaces

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Abstract

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$. In this paper, we introduce hybrid algorithms for generating cyclic, non-expansive mapping of H . Also, we discuss about necessary and sufficient conditions on subsets of Hilbert space to be remotal or uniquely remotal. Moreover, we give the basic concepts and theorems of farthest points of Bounded subsets of H . In the end, we will provide examples to illustrate our results.

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1. Introduction

Approximation theory, which mainly consists of theory of best approximation and theory of worst approximation [4, 8, 7, 10] is interesting topic in analysis. Some results in this area was introduced in 2003, by kirk[13]. Later, investigation in this area was continued by many researchers and obtained many results([1], [2], [5], [6], [12], [16], [18], [19]). In particular, Elderer and Yeeramani [3] proved some results about best proximity points of cyclic contraction maps. Farthest points at first time was introduced by B. Jessen [11]. The main purpose of this paper is to discuss about existence of farthest points for non-expansive maps and convex sets in Hilbert space H . This section reviews basic definitions, facts, and notation from set-valued analysis, and normed spaces that will be used throughout the paper.

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$. Recall that for nonempty subsets A and B of H and $x \in H$, we write

$$d(A, B) = \inf_{x \in A, y \in B} \|x - y\|$$

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and

$$d(x, B) = \inf_{y \in B} \|x - y\|.$$

Also we set

$$\delta(A, B) = \sup_{x \in A, y \in B} \|x - y\|$$

and

$$\delta_B(x) = \delta(x, B) = \sup_{y \in B} \|x - y\|,$$

for bounded subsets A, B of H . A mapping $T : A \cup B \rightarrow A \cup B$, $A, B \subset H$, is said to be non-expansive if $\|Tx - Ty\| \leq \|x - y\|$ holds for all $x, y \in A \cup B$. Suppose that B is a closed convex subset of H , the metric projection of some element x onto B , define as:

$$P_B(x) = \{x_0 \in B : \|x - x_0\| = d(x, B)\},$$

and denote by $F_B(x)$ set of all farthest points of x on bounded set B :

$$F_B(x) = \{x_0 \in B : \|x - x_0\| = \delta(x, B)\},$$

and $w_\omega(x_n) := \{x : \exists(x_{n_k}) \subset (x_n), x_{n_k} \xrightarrow{w} x\}$ denotes the weak ω -limit set of x_n . Bounded set B is said to be remotal, if $F_B(x)$ is nonempty for every $x \in H$.

In this article, we need to define the following definitions:

$$F(A^T) = \{a \in A : d(a, Ta) = \delta(A, B)\},$$

$$A^0 = \{a \in A : \|a - b\| = \delta(A, B), \text{ for some } b \in B\},$$

for closed, convex and bounded subsets A, B of H . It is quite national to see that $F(A^T) \subset A^0$.

Definition 1.1. [3] Let A and B be nonempty closed subsets of a metric space (X, d) . Then (A, B) is said to satisfy the UC property if $\{x_n\}$ and $\{z_n\}$ are sequences in A and $\{y_n\}$ is a sequence in B such that $\lim_{n \rightarrow \infty} d(x_n, y_n) = d(A, B)$ and $\lim_{n \rightarrow \infty} d(z_n, y_n) = d(A, B)$, then $\lim_{n \rightarrow \infty} d(x_n, z_n) = 0$.

Lemma 1.2. [15] For u, v in real Hilbert space H , we have:

$$\|u - v\|^2 = \|u\|^2 - \|v\|^2 - 2 \langle u - v, v \rangle,$$

$$\|\alpha u + (1 - \alpha)v\|^2 = \alpha\|u\|^2 + (1 - \alpha)\|v\|^2 - \alpha(1 - \alpha)\|u - v\|^2, \alpha \in [0, 1].$$

Lemma 1.3. [15]. Let B be a closed convex subset of a real Hilbert space H and let $T : B \rightarrow B$ be a non-expansive mapping such that $\text{Fix}(T) \neq \emptyset$. If a sequence $\{x_n\}$ in C is such that $x_n \xrightarrow{w} z$ and $\|x_n - Tx_n\| \rightarrow 0$, then $z = Tz$.

Lemma 1.4. [15] Let K be a closed convex subset of H . Let $\{x_n\}$ be a sequence in H and $x \in H$. Let $q = P_K(x)$. If $\{x_n\}$ is such that $w_\omega(x_n) \subset K$ and satisfies the condition

$$\|x_n - x\| \leq \|x - q\| \text{ for all } n \in N,$$

then $x_n \rightarrow q$.

In this paper, we discuss about necessary and sufficient conditions on subsets of Hilbert space to be remotal or uniquely remotal, also we give the basic concepts and theorems on Hilbert spaces for the existence of farthest points.

2. Remotal sets in Hilbert space

In this section we introduce the fundamental notion of the farthest points of a convex set. The key result is Proposition 2.1, which asserts that every nonempty closed convex subset C of H , which satisfy $S^{(x+y)/2}$ -property, is a unique remotal set, i.e., that every point in H possesses a unique farthest point from C , and which provides a characterization of this farthest points.

Definition 2.1. Let A and B be two closed subsets of a metric space (X, d) . Then A and B are said to satisfy the F -property if for $x_1, x_2 \in A^0$ and $y_1, y_2 \in B^0$ the following implication holds:
 $d(x_1, y_1) = d(x_2, y_2) = \delta(A, B) \implies d(x_1, x_2) = d(y_1, y_2)$

Definition 2.2. Let A and B be nonempty closed subsets of a metric space (X, d) . Then (A, B) is said to satisfy the FC property if $\{x_n\}$ and $\{z_n\}$ are sequences in A and $\{y_n\}$ is a sequence in B such that $\lim_{n \rightarrow \infty} d(x_n, y_n) = \delta(A, B)$ and $\lim_{n \rightarrow \infty} d(z_n, y_n) = \delta(A, B)$, then $\lim_{n \rightarrow \infty} d(x_n, z_n) = 0$.

Example 2.3. Let $X = R^2$ and consider $A = \{(x, y) | (x + 3)^2 + y^2 \leq 1\}$, $B = \{(x, y) | x^2 + y^2 \leq 1\}$ in X . It is clear that (A, B) satisfy in FC property.

Note that, every Hilbert space satisfies UC property, but this is not true for FC property, for example consider $A = \{(0, y) | 0 \leq y \leq 2\}$ and $B = \{(1, 1)\}$ in Euclidean space R^2 .

Definition 2.4. Let A and B be nonempty subsets of a Hilbert space H , $T : A \rightarrow B$. We say that T satisfies the FH -property if $x_n \overset{m}{\rightharpoonup} x \in A \cup B$, $\|x_n - Tx_n\| \rightarrow \delta(A, B)$ then $\|x - Tx\| = \delta(A, B)$. for $\{x_n\}_{n \geq 0} \in A \cup B$.

Definition 2.5. A Hilbert space H is said to be FH -convex if the following implication holds for all $x_1, x_2, p \in H, R > 0, \|x_i - p\| \geq R, i = 1, 2$ and $x_1 \neq x_2 \rightarrow \|(x_1 + x_2)/2 - p\| > R$.

Definition 2.6. Let C be a nonempty subset of H and $x, y \in H$. Then C is said to satisfy the $S^{(x+y)/2}$ -property, if there exist $S : C \rightarrow C$ such that for two sequence $\{y_n\}_{n \in N}$ and $\{z_n\}_{n \in N}$ in C , the following implication holds:

If $\|y_n - x\| \rightarrow \delta_C(x), \|z_n - y\| \rightarrow \delta_C(y)$ Then, as $n \uparrow \infty : \|(Sy_n + Sz_n)/2 - (x + y)/2\| \geq \delta_C((x + y)/2) - \epsilon, \forall \epsilon > 0$.

Example 2.7. Let $H = R^2, C = \{(x, y) : x^2 + y^2 \leq 1\}$. If $S(x, y) = (1, 0)$, then C is satisfying $S^{[(-2,0)+(-2,0)]/2}$ -property ($S^{(-2,0)}$ -property).

Lemma 2.8. Let A be a bounded subset of the Banach space $X, x \in X$ and $w_0 \in A$. Then the following are equivalent:

- (i) $w_0 \in F_A(x)$,
- (ii) There exists an $f \in X^*$ such that f satisfies $\|f\| = 1$ and $|f(x - w_0)| \geq \delta(x, A)$.

Proof . (i) \rightarrow (ii). Suppose $w_0 \in F_A(x)$, then $\|x - w_0\| = \delta(x, A)$. By Hahn-Banach Theorem, there exists an $f \in X^*$ such that $\|f\| = 1$ and $|f(x - w_0)| = \|x - w_0\| = \delta(x, A)$.

(ii) \rightarrow (i). Suppose there exists an $f \in X^*$ such that f satisfies $\|f\| = 1$ and $|f(x - w_0)| \geq \delta(x, A)$. We have

$$\begin{aligned} \|x - w_0\| &= \|f\| \|x - w_0\| \\ &\geq |f(x - w_0)| \\ &\geq \delta(x, A) \\ &\geq \|x - w_0\|. \end{aligned}$$

□

Lemma 2.9. For x and y in H , the following hold:

$$(\forall \alpha \in [0, 1]) \|x\| \geq \|x - \alpha y\| \rightarrow (\forall \alpha \in R_+) \|x\| \geq \|x - \alpha y\| \rightarrow \langle x, y \rangle \geq 0.$$

Proof . Observe that

$$(\forall \alpha \in R) \|x - \alpha y\|^2 - \|x\|^2 = \alpha(\alpha \|y\|^2 - 2 \langle x, y \rangle).$$

If for every $\alpha \in (0, 1]$, $\|x\| \geq \|x - \alpha y\|$, then $\langle x, y \rangle \geq \alpha \|y\|^2 / 2$. As $\alpha \downarrow 0$, we obtain $\langle x, y \rangle \geq 0$.
 □

Lemma 2.10. [14] Let K be a closed and convex subset of real Hilbert space H . Then $z = P_K(x)$ if and only if the relation holds:

$$\langle x - z, y - z \rangle \leq 0, \text{ for all } y \in K.$$

Proposition 2.11. Let C be a closed convex subset of real Hilbert space H , which it satisfy in $S^{(x+y)/2}$ -property (in particular S^x -property), for every $x, y \in H$. Then C is unique remotal set, and for every $x, z \in H$, $z = F_C(x)$ imply that $\langle x - z, y - z \rangle \geq 0$, for all $y \in C$.

Proof . Let $x \in H$, by definition of δ_C , there exists a sequence $(y_n)_{n \in N}$ in C such that $\delta_C(x) = \lim \|y_n - x\|$, hence $\delta_C(x) = \lim \|S y_n - x\|$, because $\|S y_n - x\| = \|x - (S y_n + S y_n) / 2\| \geq \delta_C(x) - \epsilon, \forall \epsilon > 0$ by S^x -property of C . Now take m and n in N . Letting m and n go to $+\infty$, we obtain that $(S y_n)_{n \in N}$ is a Cauchy, from Apollonius's identity that

$$\begin{aligned} \|S y_n - S y_m\|^2 &= 2\|S y_n - x\|^2 + 2\|S y_m - x\|^2 - 4\|x - (S y_n + S y_m) / 2\|^2 \\ &\leq 2\delta_C^2(x) + 2\delta_C^2(x) - 4\delta_C^2(x). \end{aligned}$$

Since C is complete as a closed subset of H , it therefore converges to some point $p \in C$. Then $\|S y_n - x\| \rightarrow \|p - x\|$, by continuity of $\|\cdot - x\|$, hence $\delta_C(x) = \|p - x\|$. For uniqueness, suppose that $q \in C$ satisfies $\delta_C(x) = \|q - x\|$, therefore, there exist a sequence $(z_n)_{n \in N} \in C$ such that $\lim \|z_n - x\| = \|q - x\| = \delta_C(x)$, hence $\lim \|(S z_n + S y_n) / 2 - x\| = \|(p + q) / 2 - x\|$, therefore

$$\begin{aligned} \|(p + q) / 2 - x\| &= \lim \|(S z_n + S y_n) / 2 - x\| \\ &\geq \delta_C(x) - \epsilon. \end{aligned}$$

Now by Apollonius's identity,

$$\begin{aligned} \|p - q\|^2 &= 2\|p - x\|^2 + 2\|q - x\|^2 - 4\|x - (p + q) / 2\|^2 \\ &= 4\delta_C^2(x) - 4(\delta_C(x) - \epsilon)^2 \leq 0. \end{aligned}$$

This implies that $p = q$ and shows uniqueness. Finally, for every $y \in C$ and $\alpha \in [0, 1]$, set $y_\alpha = \alpha y + (1 - \alpha)z$, which belongs to C by convexity. Lemma 2.9 yields

$$\begin{aligned} \|x - z\| = \delta_C(x) &\implies (\forall y \in C)(\forall \alpha \in [0, 1]) \|x - z\| \geq \|x - y_\alpha\| \\ &\implies (\forall y \in C)(\forall \alpha \in [0, 1]) \|x - z\| \geq \|x - z - \alpha(y - z)\| \\ &\implies (\forall y \in C) \langle x - z, y - z \rangle \geq 0. \end{aligned}$$

□

Corollary 2.12. Let C be a closed subset of real Hilbert space H , which it satisfy in S^x -property. Then C is remotal set.

3. Farthest points

Let H be a real Hilbert space and suppose that A and B are a closed, convex and bounded subset of H , which B satisfying $S^{(x+y)/2}$ -property. Under assumption of non-expansive map definition, the following algorithm produces two sequences $\{x_n\} \subset A$ and $\{y_n\} \subset B$ as:

Algorithm 3.1. $x_0 \in A^0$ arbitrarily,
 $y_n = \alpha_n F_B(x_n) + (1 - \alpha_n)T(x_n), n \in N \cup \{0\}$
 $C_n = \{z \in A^0 : \|y_n - z\| \geq \delta(A, B) - \|x_n - z\|\}$
 $Q_n = \{z \in A^0 : \langle x_n - z, x_n - x_0 \rangle \leq 0\}$
 $x_{n+1} = P_{(C_n \cap Q_n)}(x_0)$.
 where $\alpha_n \in [0, 1], \alpha_n \rightarrow 0$.

We start with some basic properties.

Lemma 3.2. C_n generated in Algorithm 3.1 is convex.

Proof . By definition of C_n , we have

$$\|y_n - z\|^2 \geq (\delta(A, B))^2 + \|x_n - z\|^2 - 2\|x_n - z\|\delta(A, B).$$

Using lemma 1.2, we get

$$\begin{aligned} \|y_n - x_n\|^2 &= \|y_n - z\|^2 - \|x_n - z\|^2 - 2 \langle y_n - x_n, x_n - z \rangle \\ &\geq (\delta(A, B))^2 + \|x_n - z\|^2 - 2\|x_n - z\|\delta(A, B) \\ &\quad - \|x_n - z\|^2 - 2 \langle y_n - x_n, x_n - z \rangle \\ &\geq -(\delta(A, B))^2 - 2 \langle y_n - x_n, x_n - z \rangle . \end{aligned}$$

Hence, we have

$$\|y_n - x_n\|^2 + \delta(A, B))^2 + 2 \langle y_n - x_n, x_n - z \rangle \geq 0.$$

Now, consider $z_1, z_2 \in C_n$, for $\lambda \in [0, 1]$, we have

$$\begin{aligned} \|y_n - x_n\|^2 &+ (\delta(A, B))^2 \\ &+ 2 \langle y_n - x_n, x_n - (\lambda z_1 + (1 - \lambda)z_2) \rangle \\ &= \|y_n - x_n\|^2 + (\delta(A, B))^2 \\ &+ 2\lambda \langle y_n - x_n, x_n - z_1 \rangle + 2(1 - \lambda) \langle y_n - x_n, x_n - z_2 \rangle \\ &\geq \lambda(-\|y_n - x_n\|^2 - (\delta(A, B))^2) + (1 - \lambda)(-\|y_n - x_n\|^2 - (\delta(A, B))^2) \\ &+ \|y_n - x_n\|^2 + \delta(A, B))^2 \\ &= 0. \end{aligned}$$

Therefore

$$\lambda z_1 + (1 - \lambda)z_2 \in C_n.$$

□

Lemma 3.3. For $x \in A^0$, we have $\|F_B(x) - x\| = \delta(A, B)$.

Proof . From the definition of $F_B(x)$, we get

$$\|F_B(x) - x\| = \delta(x, B).$$

Also, by definition of A^0 , we have

$$\|x - b\| = \delta(A, B), \text{ for some } b \in B.$$

Therefore

$$\begin{aligned} \delta(A, B) &\geq \|F_B(x) - x\| \\ &= \delta(x, B) \\ &\geq \|x - b\| \\ &= \delta(A, B). \end{aligned}$$

□

Proposition 3.4. *Let A and B be nonempty closed, convex and bounded subsets of a Hilbert space H satisfy F -property and $T : A^0 \rightarrow B^0$ be a cyclic non-expansive map such that $F(A^T)$ is a nonempty convex subset of A^0 . Also, B satisfy $S^{(x+y)/2}$ -property for $S : B \rightarrow B$. For $x_0 \in A^0$, the sequences $\{x_n\}$ and $\{y_n\}$ are generated algorithm 3.1. Then x_n is bounded and $\|x_n - Tx_n\| \rightarrow d(A, B)$ and $\|y_n - x_n\| \rightarrow d(A, B)$.*

Proof . Choose $x_0 \in A_0$ arbitrarily. It is clear that C_n and Q_n are closed, convex and bounded subsets of A . Now we show that C_n is nonempty subset of A_0 . Let $u \in F(A^T)$, therefore $\|u - T(u)\| = \delta(A, B)$. Using the F -property we obtain that $\|F_B(x_n) - T(u)\| = \|x_n - u\|$, by $\|F_B(x_n) - x_n\| = \|u - T(u)\| = \delta(A, B)$. Hence

$$\begin{aligned} \|y_n - Tu\| &= \|\alpha_n F_B(x_n) + (1 - \alpha_n)T(x_n) - Tu\| \\ &\leq \|(1 - \alpha_n)(Tu - T(x_n))\| + \|\alpha_n(F_B(x_n) - Tu)\| \\ &\leq (1 - \alpha_n)\|x_n - u\| + \alpha_n\|x_n - u\| \\ &= \|x_n - u\|. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \|y_n - u\| &\geq \|u - Tu\| - \|y_n - Tu\| \\ &\geq \delta(A, B) - \|x_n - u\|. \end{aligned}$$

Using the induction principle and lemma 2.10, it is quite natural to see that $F(A^T) \subset Q_n$. It is clear that $F(A^T) \subset A^0$. Let $Q_0 = A^0$ and we assume that $F(A^T) \subset Q_n$ is true. By Lemma 2.10 we have that $\langle x_{n+1} - z, x_{n+1} - x_0 \rangle \leq 0$ for all $z \in C_n \cap Q_n$ in particular for all $F(A^T)$, because $x_{n+1} = P_{C_n \cap Q_n}(x_0)$. Hence, $F(A^T) \subset Q_{n+1}$.

Since $x_{n+1} = P_{C_n \cap Q_n}(x_0)$ and $F(A^T) \subset C_n \cap Q_n$, we obtain that

$$\|x_{n+1} - x_0\| \leq \|q - x_0\|,$$

where $q \in P_{F(A^T)}(x_0)$. Since, $x_{n+1} \in Q_n$ and by using Lemma 1.2, we get

$$\|x_{n+1} - x_n\|^2 \leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2,$$

which in turn gives that

$$\sum_0^\infty \|x_{n+1} - x_n\|^2 \leq \|q - x_0\|^2 - \|x_1 - x_0\|^2.$$

Hence

$$\|x_{n+1} - x_n\| \rightarrow 0,$$

by the definition of C_n , we have

$$\begin{aligned} \|y_n - x_n\| &\geq \|y_n - x_{n+1}\| - \|x_{n+1} - x_n\| \\ &\geq \delta(A, B) - \|x_{n+1} - x_n\| - \|x_{n+1} - x_n\|. \end{aligned}$$

Therefore as $n \rightarrow \infty$, we get that

$$\|y_n - x_n\| \rightarrow \delta(A, B).$$

Also,

$$\begin{aligned} y_n - x_n &= \alpha_n F_B(x_n) + (1 - \alpha_n)T(x_n) - x_n \\ &= \alpha_n(F_B(x_n) - T(x_n)) + (T(x_n) - x_n). \end{aligned}$$

Therefore, we obtain

$$\|T(x_n) - x_n\| \geq \|y_n - x_n\| - \alpha_n \|F_B(x_n) - T(x_n)\|.$$

Hence, $\|T(x_n) - x_n\| \rightarrow \delta(A, B)$ as $n \rightarrow \infty$. \square

Theorem 3.5. *Let A and B be nonempty closed, convex and bounded subsets of a Hilbert space H which satisfy F -property and $T : A^0 \rightarrow B^0$ be a non-expansive map. Let $F(A^T)$ is a nonempty convex subset of A^0 . Also, B satisfy $S^{(x+y)/2}$ -property for $S : H \rightarrow B$. Then there exists $(x, y) \in A \times B$ such that*

$$\|x - y\| = \delta(A, B).$$

Proof . Since $\|x_n - Tx_n\| \rightarrow \delta(A, B)$ and

$$\begin{aligned} y_n - x_n &= \alpha_n F_B(x_n) + (1 - \alpha_n)T(x_n) - x_n \\ &= \alpha_n(F_B(x_n) - T(x_n)) + (T(x_n) - x_n), \end{aligned}$$

we obtain

$$\|T(x_n) - x_n\| \leq \|y_n - x_n\| + \alpha_n \|F_B(x_n) - T(x_n)\|.$$

Hence

$$\|y_n - x_n\| \geq \delta(A, B) \geq \delta(B, x_n),$$

therefore

$$y_n \in F_B(x_n)$$

and $\delta(A, B) = \delta(B, x_n)$, for any $n > N$, for some $N \in \mathbb{N}$. For $x_0 \in A^0$, the sequences $\{x_n\}$ is generated by algorithm 3.1 is bounded, by boundedness of A and B . Since A and B are closed and convex, they are weakly closed. Since H is reflexive and A is weakly closed, the sequence $\{x_n\}$ has

a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \xrightarrow{w} x \in A$ as $k \rightarrow \infty$. Also B is weakly closed, hence the sequence $\{y_n\}$ has a subsequence $\{y_{n_k}\}$ such that $y_{n_k} \xrightarrow{w} y \in B$ as $k \rightarrow \infty$. Since $x_{n_k} - y_{n_k} \xrightarrow{w} x - y \neq 0$ as $k \rightarrow \infty$, from lemma 2.8 there exists a bounded linear functional $f : X \rightarrow [0, \infty)$ such that $\|f\| = 1$ and

$$\begin{aligned} \delta(A, B) &= \delta(B, x_n) \\ &\leq |f(x_{n_k} - y_{n_k})| \longrightarrow |f(x - y)| \\ &\leq \|f\| \|x - y\| = \|x - y\|, \end{aligned}$$

because $y_n \in F_B(x_n)$ for any $n > N$, for some $N \in \mathbb{N}$. So $\|x - y\| = \delta(A, B)$. \square

Theorem 3.6. *Let A and B be nonempty closed, convex and bounded subsets of a Hilbert space H which satisfy F -property and $T : A^0 \rightarrow B^0$ be a non-expansive map. Let $F(A^T)$ is a nonempty convex subset of A^0 . Also, (A, B) satisfy in FC property and B satisfy $S^{(x+y)/2}$ -property for $S : H \rightarrow B$. Then the sequence (x_n, y_n) generated by Algorithm 3.1 converges to a proximity pair in $A \times B$. In particular (x_n, y_n) converges to $(q, T(q))$, where $q = P_{F(A^T)}(x_0)$.*

Proof . From proposition 3.4, we have

$$\|T(x_n) - x_n\| \rightarrow \delta(A, B) \text{ as } n \rightarrow \infty.$$

Using FC property of (A, B) and $\|F_B(x_n) - x_n\| \rightarrow \delta(A, B)$, we get $\|F_B(x_n) - T(x_n)\| \rightarrow 0$, as $n \rightarrow \infty$. Hence $\|F_A(Tx_n) - x_n\| \rightarrow 0$, as $n \rightarrow \infty$, by F -property, because $\|F_A(Tx_n) - Tx_n\| \rightarrow \delta(A, B)$, as $n \rightarrow \infty$.

Define $\phi : A^0 \rightarrow A^0$ as $\phi(x) = F_A(Tx)$ for all $x \in A^0$. We get ϕ is non-expansive and $F(A^T) = Fix(\phi)$, by F -property, because

$$\|F_A(Tx) - Tx\| = \delta(A, B), \|F_A(Ty) - Ty\| = \delta(A, B).$$

Therefore, by Lemma 1.3 we obtain $w_\omega(x_n) \subset Fix(\phi)$. Hence, by Lemma 1.4, $\{x_n\}$ converges strongly to a fixed point of Φ (for example p), because

$$\|x_{n+1} - x_0\| \leq \|q - x_0\|,$$

where $q \in P_{F(A^T)}(x_0)$. Therefore, $\{x_n\}$ converges to the point $p \in A^0$ which satisfies $d(p, T(p)) = \delta(A, B)$. Therefore $p \in F(A^T)$, and hence,

$$\|x_0 - p\| \geq \|x_0 - q\| = d(x_0, F(A^T)).$$

Also, using equation $\|x_n - x_0\| \leq \|q - x_0\|$, we get

$$\|x_0 - p\| = \lim_{n \rightarrow \infty} \|x_0 - x_n\| \leq \|q - x_0\| \leq d(x_0, F(A^T)).$$

Therefore, $\|x_0 - p\| = d(x_0, F(A^T))$, hence $p = q$, and this completes the proof. \square

The following theorem gives a condition that guarantees the existence of unique farthest points.

Theorem 3.7. *Let A and B be nonempty closed, convex and bounded subsets of a Hilbert space H which satisfy FH -property and $T : A^0 \rightarrow B^0$ be a non-expansive map. Let $F(A^T)$ is a nonempty convex subset of A^0 . Also, B satisfy $S^{(x+y)/2}$ -property for $S : H \rightarrow B$, then there exists unique $x \in A$ such that*

$$\|x - T(x)\| = \delta(A, B). \tag{3.1}$$

provided that one of the following conditions is satisfied

- (a) T is weakly continuous on A and B .
- (b) T satisfy the F -property.

Proof . Since $\|x_n - y_n\| \rightarrow \delta(A, B)$ and

$$\begin{aligned} y_n - x_n &= \alpha_n F_B(x_n) + (1 - \alpha_n)T(x_n) - x_n \\ &= \alpha_n(F_B(x_n) - T(x_n)) + (T(x_n) - x_n). \end{aligned}$$

From the above equality, we obtain

$$\|y_n - x_n\| \leq \|T(x_n) - x_n\| + \alpha_n \|F_B(x_n) - T(x_n)\|.$$

Hence

$$\|T(x_n) - x_n\| \geq \delta(A, B) \geq \delta(B, x_n),$$

therefore

$$T(x_n) \in F_B(x_n)$$

and $\delta(A, B) = \delta(B, x_n)$, for any $n > N$, for some $N \in \mathbb{N}$.

For $x_0 \in A^0$, the sequences $\{x_n\}$ is generated by algorithm 3.1 is bounded, by boundedness of A and B. Since A and B are closed and convex, they are weakly closed. Since H is reflexive and A is weakly closed, From weak continuity of T, $Tx_{n_k} \xrightarrow{w} Tx \in B$ as $k \rightarrow \infty$. So $x_{n_k} - Tx_{n_k} \xrightarrow{w} x - Tx \neq 0$ as $k \rightarrow \infty$. Since $x_{n_k} - Tx_{n_k} \xrightarrow{w} x - Tx \neq 0$ as $k \rightarrow \infty$, from Lemma 2.8 there exists a bounded liner functional $f : X \rightarrow [0, \infty)$ such that $\|f\| = 1$ and

$$\begin{aligned} \delta(A, B) &= \delta(B, x_n) \\ &\leq |f(x_{n_k} - Tx_{n_k})| \rightarrow |f(x - Tx)| \\ &\leq \|f\| \|x - Tx\| \\ &= \|x - Tx\|, \end{aligned}$$

because $Tx_n \in F_B(x_n)$ for any $n > N$, for some $N \in \mathbb{N}$. So $\|x - Tx\| = \delta(A, B)$. From (b) and Proposition 3.4,

$$\|x_{n_k} - Tx_{n_k}\| \rightarrow d(A, B) \text{ as } k \rightarrow \infty.$$

So $d(A, B) = \|x - Tx\|$. For the uniqueness of x , suppose that there exists $a \in A$ such that $\|a - Ta\| = d(A, B)$. By the FH -convexity of H, and convexity of A and B, we have

$$\|(x + a)/2 - (Tx + Ta)/2\| = \|(x - Tx)/2 + (a - Ta)/2\| > \delta(A, B). \tag{3.2}$$

which is a contraction. This shows that x is unique. \square

We conclude this section with a following examples.

Example 3.8. Consider $X = R^2$ with the usual metric. Let $A = \{(2, y) : 1 \leq y \leq 2\}$, $B = \{(3, y) : 1 \leq y \leq 3/2\}$, $\alpha_n = 1/(n + 1)$ and $T : A \rightarrow B$, $S : B \rightarrow B$ be given by $T(2, y) = (3, 3 - y)$ and $S(3, y) = (3, y)$. Now, we choose an arbitrary element $x_0 = (2, 3/2)$, it is clear that B satisfying in $S^{(x+y)/2}$ -property, T is a non-expansive mapping and $(2, 2)$ is the farthest point of T, that it is unique.

Example 3.9. Consider the Euclidean ordered space $X = R^2$ with the usual metric. Let $A = \{(x, y) : -x - 3 \leq y \leq x + 3, -3 \leq x \leq -1\}$, $B = \{(x, y) : x - 3 \leq y \leq -x + 3, 1 \leq x \leq 3\}$, $\alpha_n = 1/(n + 1)$ and $T : A \rightarrow B$, $S : B \rightarrow B$ are defined by $T(x, y) = (-x, y)$ and $S(x, y) = (3, 0)$. Let us choose an arbitrary element $x_0 = (-3, 0)$, it is clear that B satisfying in $S^{(x+y)/2}$ -property, T is a non-expansive mapping and $(-3, 0)$ is the unique farthest point of T.

Note that, $S^{(x+y)/2}$ -property is a sufficient condition for the farthest points, but not a necessary condition. Of course, it is needed to be unique.

Example 3.10. Let $X = R^2$ with the usual metric. Let $A = \{(2, y) : 1 \leq y \leq 2\}$, $B = \{(3, y) : 1 \leq y \leq 2\}$, $\alpha_n = 1/(n+1)$ and $T : A \rightarrow B$, $S : B \rightarrow B$ be defined by $T(2, y) = (3, 3 - y)$ and $S(3, y) = (3, 3 - y)$. Now, us choose an arbitrary element $x_0 = (2, 2)$, it is clear that T is a non-expansive mapping and B not satisfying the $S^{(x+y)/2}$ -property. But $(2, 2), (2, 1)$ are the farthest points of T , that farthest point is not unique.

Example 3.11. Consider $X = R^2$ with the usual metric. Let $A = \{(x, y) : x^2 + y^2 = 1, x \leq 0\}$, $B = \{(x, y) : x^2 + y^2 = 1, x \geq 0\}$, $\alpha_n = 1/(n+1)$ and $T : A \rightarrow B$, $S : B \rightarrow B$ be given by $T(x, y) = (-x, -y)$ and $S(x, y) = (x, y)$. Now, we choose an arbitrary element $x_0 = (0, 1)$, it is clear that T is a non-expansive mapping and B not satisfying the $S^{(x+y)/2}$ -property and the set of all points in a A are the farthest point of T .

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