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On some maps into Banach algebras and Banach modules

Bahman Hayati

Faculty of Mathematical Sciences and Statistics, Malayer University, P.O. Box 65719-95863, Malayer, Iran

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Abstract

In this paper, we study some maps from a C^*-algebra into a Banach algebra or a Banach module. Under some conditions, by extending on unitization of Banach algebras, we prove that the maps defined into Banach algebras are homomorphisms and the others defined into Banach modules are derivations. Applications of our results in the context of C^*-algebras are also provided.

Keywords: Homomorphism; Derivation; Banach module; C^*-algebra.
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1. Introduction and preliminaries

Let A be a Banach algebra. It is well known that $A^\flat = A \oplus \mathbb{C}$, by the product $(a,\lambda)(b,\nu) = (ab + \lambda b + \nu a, \lambda\nu)$ is a Banach algebra with the following norm:

$$\|(a,\lambda)\| = \|a\| + |\lambda| \qquad (a \in A\ ,\ \lambda \in \mathbb{C}).$$

We set $A^\sharp$ (the unitization of A) to be A when A is unital, and to be $A^\flat$ otherwise. For a C^*-algebra such as $(A, *)$, one can easily see that $A^\sharp$ is a C^*-algebra with the following involution:

$$(a,\lambda)^* = (a^*, \bar{\lambda}) \qquad\qquad (a \in A, \lambda \in \mathbb{C}).$$

Let X be a A-module, X is said to be a Banach A-module if there exists a positive k such that

$$\|a \cdot x\| \le k\|a\|\|x\| \qquad\qquad (a \in A,\ x \in X).$$

Email address: hayati@malayeru.ac.ir, bn.hayati@gmail.com (Bahman Hayati)

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Let A be a Banach algebra and let X be a Banach A-module. Then X^* is a Banach A-module if for every $a \in A$, $x \in X$ and $x^* \in X^*$ we define

$$\langle a \cdot x^*, x\rangle = \langle x^*, x \cdot a\rangle, \qquad \langle x^* \cdot a, x\rangle = \langle x^*, a \cdot x\rangle.$$

Let A be a Banach algebra and X be a Banach A-module. A continuous linear map $D : A \longrightarrow X$ such that

$$D(ab) = D(a) \cdot b + a \cdot D(b) \qquad (a, b \in A)$$

is called a derivation from A into X. The space of all derivations of A into X is denoted by $\mathcal{Z}^1(A, X)$. For each $x \in X$, the map $a \mapsto a \cdot x - x \cdot a$ is a derivation, and these maps form the space $\mathcal{N}^1(A, X)$ of inner derivations. The quotient space $\mathcal{H}^1(A, X) = \mathcal{Z}^1(A, X)/\mathcal{N}^1(A, X)$ is the first cohomology group of A with coefficients in X. Let A be a Banach algebra, A is called amenable (contractible) if $\mathcal{H}^1(A, X^*) = \{0\}$ ($\mathcal{H}^1(A, X) = \{0\}$), for every Banach A-module X. Also, a Banach algebra A is said to be weakly amenable if every continuous derivation from A into A^* is inner. For a locally compact group G, $L^1(G)$ is a weakly amenable Banach algebra. Examples of weakly amenable Banach algebras include all C^*-algebras. Also, by a known result of Johnson, $L^1(G)$ is amenable if and only if G is amenable. For example, in case that G is abelian or compact, one can see that $L^1(G)$ is an amenable Banach algebra. Also all nuclear C^*-algebras are examples of amenable Banach algebras. For an example of contractible Banach algebras, one can see that a full matrix algebra M_n of $n \times n$ complex matrices is contractible (see [5, 8]).

Let A, B be two Banach algebras. For a given mapping, $f : A \to B$, we define $f^\sharp : A^\sharp \to B^\sharp$ as follows:

$$f^\sharp(a, \lambda) = (f(a), \lambda) \qquad (a \in A\ ,\ \lambda \in \mathbb{C}).$$

It is easily seen that X is a Banach $A^\sharp$-module by the following module actions:

$$(a, \lambda) \cdot x = a \cdot x + \lambda x, \qquad x \cdot (a, \lambda) = x \cdot a + \lambda x \qquad (a \in A,\ \lambda \in \mathbb{C},\ x \in X).$$

Moreover, for a given mapping $f : A \to X$, we define $f^\sharp : A^\sharp \to X$ by

$$f^\sharp(a, \lambda) = f(a) \qquad (a \in A,\ \lambda \in \mathbb{C}).$$

An almost homomorphism may become an exact homomorphism automatically. In fact, Bourgin [3] proved the following result. Suppose that A is a Banach algebra and B is a Banach algebra with unit. If $f : A \to B$ is a surjective mapping such that $\|f(x + y) - f(x) - f(y)\| \leq \varepsilon$ and $\|f(xy) - f(x)f(y)\| \leq \delta$ for some $\varepsilon, \delta \geq 0$ and all $x, y \in A$, then f is a ring homomorphism.

Badora [1, 2] gave generalizations of the above Bourgin's result for ring homomorphisms and ring derivations. Miura et al. [10] showed that if a Banach algebra A has an almost identity, or if A is commutative semisimple, then an almost ring derivation is an exact ring derivation. For more details see [4, 7] and references therein.

Let A be an algebra, B a Banach algebra and let $f : A \to B$ be a mapping. In this paper, we solve the following functional inequality

$$\left\|\nu f(x_1) + \sum_{j=2}^{n} f\left(\frac{(n-1)x_j - \nu x_1 - \sum_{i=2, i\neq j}^{n} x_i}{n}\right)\right\| \leq \left\|f\left(\frac{\nu x_1 + \sum_{i=2}^{n} x_i}{n}\right)\right\| + \varepsilon \qquad (1.1)$$

for some $\varepsilon \geq 0$, all $x_1, \ldots, x_n \in A$ and all $\nu \in V$, where V is a connected subset of $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ such that $1 \in V$ and $V \setminus \{1\} \neq \emptyset$. This is applied to show that this kind of maps, with a C^*- algebra as domain, are homomorphisms if they are defined into a Banach algebra and are continuous derivations if they are defined into a Banach module. We give some applications of our results in the context of C^*-algebras.

2. Main Results

In the rest of this paper, unless otherwise explicitly stated, we will assume that A is a C^*-algebra, X is a Banach A-module, $U(A^\sharp)$ is the set of unitary elements in $A^\sharp$, B is a Banach algebra, $Inv(B^\sharp)$ is the set of invertible elements of $B^\sharp$, $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ and V stands for a connected subset of $\mathbb{T}$ such that $1 \in V$ and $V \setminus \{1\} \neq \emptyset$ and n is a fixed integer with $n \geq 3$.

We introduce a useful result that can be easily derived from [4, Lemma 1].

Lemma 2.1. *Let $V \subset \mathbb{T}$ be a connected set containing at least two points. Let $f : A \to B$ be an additive mapping such that $f(\nu x) = \nu f(x)$ for all $x \in A$ and $\nu \in V$. Then f is $\mathbb{C}$-linear.*

Theorem 2.2. *Suppose that a mapping $f : A \to B$ satisfies the inequality* (1.1),

$$\lim_{k\to\infty} \frac{1}{(n-1)^k} f^\sharp\left((n-1)^k 1_{A^\sharp}\right) \in Inv\left(B^\sharp\right)$$

and

$$f^\sharp\left((n-1)^k ux\right) = f^\sharp\left((n-1)^k u\right) f^\sharp(x) \tag{2.1}$$

for all $u \in U(A^\sharp)$, all $x \in A^\sharp$ and all $k \in \mathbb{N}$. Then f is a homomorphism.

Proof . Letting $x_1 = (n-1)x$, $x_2 = \cdots = x_n = -x$ and $\nu = 1$ in the functional inequality (1.1), we get

$$\|f((n-1)x) + (n-1)f(-x)\| \leq \varepsilon \tag{2.2}$$

for all $x \in A$. Letting $x_1 = x$, $x_2 = -x$ and $x_3 = \cdots = x_n = 0$ in (1.1), we get

$$\|f(x) + f(-x)\| \leq (n-2)\|f(0)\| + \varepsilon \tag{2.3}$$

for all $x \in A$. It follows from (2.2) and (2.3) that

$$\left\|f(x) - \frac{1}{n-1} f((n-1)x)\right\| \leq (n-2)\|f(0)\| + \frac{n}{n-1}\varepsilon$$

for all $x \in A$. Hence

$$\begin{aligned} &\left\|\frac{1}{(n-1)^r} f((n-1)^r x) - \frac{1}{(n-1)^m} f((n-1)^m x)\right\| \\ &\leq \sum_{k=r}^{m-1} \frac{1}{(n-1)^k}\left((n-2)\|f(0)\| + \frac{n}{n-1}\varepsilon\right) \end{aligned} \tag{2.4}$$

for all $x \in A$ and integers $m > r \geq 0$. Thus it follows that a sequence $\left\{\frac{1}{(n-1)^m} f((n-1)^m x)\right\}$ is Cauchy in B and so it converges. Therefore we can define a mapping $\mathcal{L} : A \to B$ by

$$\mathcal{L}(x) := \lim_{m\to\infty} \frac{1}{(n-1)^m} f((n-1)^m x)$$

for all $x \in A$. From (1.1), we obtain

$$\begin{aligned}
&\lim_{m\to\infty} \frac{1}{(n-1)^m} \left\| \nu f((n-1)^m x_1) + \sum_{j=2}^{n} f\left(\frac{(n-1)^{m+1}x_j - \nu(n-1)^m x_1 - \sum_{i=2,i\neq j}^{n}(n-1)^m x_i}{n} \right) \right\| \\
&\quad \leq \lim_{m\to\infty} \frac{1}{(n-1)^m} \left\| f\left(\frac{\nu(n-1)^m x_1 + \sum_{i=2}^{n}(n-1)^m x_i}{n} \right) \right\| + \lim_{m\to\infty} \frac{\varepsilon}{(n-1)^m},
\end{aligned}$$

and hence

$$\left\| \nu\mathcal{L}(x_1) + \sum_{j=2}^{n} \mathcal{L}\left(\frac{(n-1)x_j - \nu x_1 - \sum_{i=2,i\neq j}^{n} x_i}{n} \right) \right\| \leq \left\| \mathcal{L}\left(\frac{\nu x_1 + \sum_{i=2}^{n} x_i}{n} \right) \right\| \tag{2.5}$$

for all $\nu \in V$ and all $x_1, \ldots, x_n \in A$. Letting $x_1 = \cdots = x_n = 0$ and $\nu = 1$ in (2.5), we get

$$\|n\mathcal{L}(0)\| \leq \|\mathcal{L}(0)\|.$$

Since $n \geq 3$, $\mathcal{L}(0) = 0$. Setting $x_1 = x$, $x_2 = -x$, $x_3 = \cdots = x_n = 0$ and $\nu = 1$ in (2.5), we have

$$\begin{aligned}
\|\mathcal{L}(x) + \mathcal{L}(-x) + (n-2)\mathcal{L}(0)\| &= \|\mathcal{L}(x) + \mathcal{L}(-x)\| \\
&\leq \|\mathcal{L}(0)\| \\
&= 0
\end{aligned}$$

for all $x \in A$. Hence $\mathcal{L}(-x) = -\mathcal{L}(x)$ for all $x \in A$. Putting $x_1 = x + y$, $x_2 = -x$, $x_3 = -y$, $x_4 = \cdots = x_n = 0$ and $\nu = 1$ in (2.5), we get

$$\begin{aligned}
\|\mathcal{L}(x+y) + \mathcal{L}(-x) + \mathcal{L}(-y) + (n-3)\mathcal{L}(0)\| &= \|\mathcal{L}(x+y) - \mathcal{L}(x) - \mathcal{L}(y)\| \\
&\leq \|\mathcal{L}(0)\| \\
&= 0,
\end{aligned}$$

which proves the additivity of $\mathcal{L}$. Letting $x_1 = x$, $x_2 = -\nu x$ and $x_3 = \cdots = x_n = 0$ in (2.5) and using the additivity of $\mathcal{L}$, we have

$$\nu\mathcal{L}(x) - \mathcal{L}(\nu x) = \nu\mathcal{L}(x) + \mathcal{L}(-\nu x) = 0.$$

So $\mathcal{L}(\nu x) = \nu\mathcal{L}(x)$ for all $\nu \in V$ and all $x \in A$. Now by using Lemma 2.1, we infer that the mapping $\mathcal{L} : A \to B$ is $\mathbb{C}-$linear.

It follows from (2.1) that

$$\mathcal{L}^\sharp(ux) = \lim_{k\to\infty} \frac{f^\sharp\left((n-1)^k ux\right)}{(n-1)^k} = \lim_{k\to\infty} \frac{f^\sharp\left((n-1)^k u\right)}{(n-1)^k} f^\sharp(x) = \mathcal{L}^\sharp(u) f^\sharp(x) \tag{2.6}$$

for all $u \in U(A^\sharp)$ and all $x \in A^\sharp$. On the other hand $\mathcal{L}^\sharp$ is linear. Then by (2.6), we have

$$\mathcal{L}^\sharp(ux) = \frac{\mathcal{L}^\sharp\left(u((l-1)^k x)\right)}{(l-1)^k} = \mathcal{L}^\sharp(u) \frac{f^\sharp\left((l-1)^k x\right)}{(l-1)^k}$$

for all $u \in U(A^\sharp)$ and all $x \in A^\sharp$. By letting $k \to \infty$ in the last equality, we get

$$\mathcal{L}^\sharp(ux) = \mathcal{L}^\sharp(u)\mathcal{L}^\sharp(x) \tag{2.7}$$

for all $u \in U(A^\sharp)$ and all $x \in A^\sharp$. By putting $u = 1_{A^\sharp}$ in (2.6) and (2.7), we get

$$\mathcal{L}^\sharp(1_{A^\sharp})\mathcal{L}^\sharp(x) = \mathcal{L}^\sharp(1_{A^\sharp}) f^\sharp(x)$$

for all $x \in A^\sharp$. By hypothesis, we have $\mathcal{L}^\sharp(1_{A^\sharp}) \in Inv(B^\sharp)$. Then we have

$$\mathcal{L}^\sharp(x) = f^\sharp(x)$$

for all $x \in A^\sharp$. It follows that $\mathcal{L} = f$. Now, let $x \in A^\sharp$. By Theorem 4.1.7 of [9], x is a finite linear combination of unitary elements of $A^\sharp$, i.e., $x = \sum_{j=1}^n c_j u_j \quad \left(c_j \in \mathbb{C}, u_j \in U(A^\sharp)\right)$. Since $f^\sharp$ is $\mathbb{C}-$linear, it follows from (2.7) that

$$\begin{aligned} f^\sharp(xa) &= f^\sharp\left((\sum_{j=1}^n c_j u_j)a\right) = \sum_{j=1}^n c_j f^\sharp(u_j a) \\ &= \sum_{j=1}^n c_j \mathcal{L}^\sharp(u_j a) = \sum_{j=1}^n c_j \mathcal{L}^\sharp(u_j)\mathcal{L}^\sharp(a) \\ &= \sum_{j=1}^n c_j f^\sharp(u_j) f(a) = f^\sharp\left(\sum_{j=1}^n c_j u_j\right) f^\sharp(a) \\ &= f^\sharp(x) f^\sharp(a) \end{aligned}$$

for all $a \in A^\sharp$. It follows that $f^\sharp : A^\sharp \to B^\sharp$ is a homomorphism. Thus $f : A \to B$ is a homomorphism. □

Let A be a Banach algebra. There are many classical theorems asserting that all derivations from A into all, or some, Banach A-modules are automatically continuous under a variety of conditions on A. The fact that every derivation from a C^*-algebra into a Banach module is continuous was proved by Ringrose in [11].

Theorem 2.3. *Suppose that a mapping $f : A \to X$ satisfies the inequality* (1.1) *and*

$$f^\sharp\left((n-1)^k ux\right) = f^\sharp\left((n-1)^k u\right)x + \left((n-1)^k u\right)f^\sharp(x) \tag{2.8}$$

for all $u \in U\left(A^\sharp\right)$, all $x \in A^\sharp$ and all $k \in \mathbb{N}$. Then f is a continuous derivation.

Proof . It is easy to show that $X \oplus_1 A^\sharp$ is a unital Banach algebra equipped with the following ℓ_1-norm

$$\|(x,a)\| = \|x\| + \|a\| \qquad (a \in A^\sharp, x \in X),$$

and the product

$$(x_1,a_1)(x_2,a_2) = (x_1 \cdot a_2 + a_1 \cdot x_2, a_1 a_2) \qquad (a_1, a_2 \in A^\sharp, x_1, x_2 \in X).$$

We refer the reader [6, 7] for details. We define the mapping $\varphi_f : A \to X \oplus_1 A$ by $a \mapsto (f(a), a)$. It is easy to show that $\varphi_f^\sharp(1_{A^\sharp}) = (0, 1_{A^\sharp}) = 1_{X\oplus_1 A^\sharp}$. It follows from (2.8) that

$$\begin{aligned} \varphi_f^\sharp\left((l-1)^k ux\right) &= (f^\sharp\left((l-1)^k ux\right), (l-1)^k ux\big) \\ &= \left(f^\sharp\left((l-1)^k u\right)x + (l-1)^k u f^\sharp(x), (l-1)^k ux\right) \\ &= \left(f^\sharp\left((l-1)^k u\right), (l-1)^k u\right) + \left(f^\sharp(x), x\right) \\ &= \varphi_f^\sharp\left((l-1)^k u\right)\varphi_f^\sharp(x) \end{aligned}$$

for all $u \in U(A^\sharp), x \in A^\sharp$, and all $k \in \mathbb{N}$. It follows that $\varphi_f^\sharp : A^\sharp \to X \oplus_1 A^\sharp$ satisfies (2.8).

On the other hand by (1.1), we have

$$\begin{aligned}
&\left\| \nu\varphi_f(x_1) + \sum_{j=2}^{n} \varphi_f\left(\frac{(n-1)x_j - \nu x_1 - \sum_{i=2, i\neq j}^{n} x_i}{n}\right)\right\| \\
&= \left\| \nu(f(x_1), x_1) + \sum_{j=2}^{n}\left(f\left(\frac{(n-1)x_j - \nu x_1 - \sum_{i=2, i\neq j}^{n} x_i}{n}\right)\right.\right. \\
&\qquad\qquad \left.\left., \left(\frac{(n-1)x_j - \nu x_1 - \sum_{i=2, i\neq j}^{n} x_i}{n}\right)\right)\right\| \\
&= \left\| \nu f(x_1) + \sum_{j=2}^{n} f\left(\frac{(n-1)x_j - \nu x_1 - \sum_{i=2, i\neq j}^{n} x_i}{n}\right)\right\| + \left\|\left(\frac{\nu x_1 + \sum_{i=2}^{n} x_i}{n}\right)\right\| \\
&\leq \left\| f\left(\frac{\nu x_1 + \sum_{i=2}^{n} x_i}{n}\right)\right\| + \left\|\left(\frac{\nu x_1 + \sum_{i=2}^{n} x_i}{n}\right)\right\| + \varepsilon \\
&= \left\| (f\left(\frac{\nu x_1 + \sum_{i=2}^{n} x_i}{n}\right), \left(\frac{\nu x_1 + \sum_{i=2}^{n} x_i}{n}\right))\right\| + \varepsilon \\
&= \left\| \varphi_f\left(\frac{\nu x_1 + \sum_{i=2}^{n} x_i}{n}\right)\right\| + \varepsilon.
\end{aligned}$$

This means that φ_f satisfies (1.1). Then by Theorem 2.2, $\varphi_f : A \to X \oplus_1 A$ is a homomorphism from A into $X \oplus_1 A$. On the other hand

$$\begin{aligned}
(a_1 f(a_2) + a_2 f(a_1), a_1a_2) &= (f(a_1), a_1)\,(f(a_2), a_2) \\
&= \varphi_f\,(a_1)\,\varphi_f\,(a_2) \\
&= \varphi_f\,(a_1a_2) \\
&= (a_2 f(a_1) + a_1 f(a_2), a_1a_2)
\end{aligned}$$

for all $a_1, a_2 \in A$. It follows that $f : A \to X$ is a derivation. Hence, from the Ringrose's result [11], we see that f is a continuous derivation. □

Some significant applications of the results are as follows.

Corollary 2.4. *Let A be finite dimension and let $f : A \to X$ be a mapping satisfying the inequalities* (1.1) *and* (2.8). *Then there exists $x \in X$ such that $f(a) = ax - xa$ for all $a \in A$. Furthermore, if X is symmetric, then f is identically zero.*

Proof . We know that every finite dimensional C^*-algebra is contractible (see [5]). Then every continuous derivation from A into X is inner. On the other hand, by Theorem 2.3, f is a continuous derivation. Then f is an inner derivation. Thus there exists $x \in X$ such that $f(a) = ax - xa$ for all $a \in A$. □

It is known that a C^*-algebra A is amenable if and only if it is nuclear. Thus, by Theorem 2.3, we deduce the following result.

Corollary 2.5. *Let A be nuclear and let $f : A \to X^*$ be a mapping satisfying the inequalities* (1.1) *and* (2.8). *Then there exists $x' \in X^*$ such that $\langle f(a), x\rangle = \langle x', xa - ax\rangle$ for all $a \in A$ and all $x \in X$. Furthermore, if X is symmetric, then f is identically zero.*

Since every C^*-algebra is weakly amenable [5], by Theorem 2.3, we obtain the following result.

Corollary 2.6. *Let $f : A \to A^*$ be a mapping satisfying the inequalities* (1.1) *and* (2.8)*. Then there exists $a' \in A^*$ such that $\langle f(a), b\rangle = \langle a', ba - ab\rangle$ for all $a, b \in A$. Furthermore, if A is commutative, then f is identically zero.*

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