Int. J. Nonlinear Anal. Appl. 12 (2021) No. 2, 537-545
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.5101

# Fuzzy equality co-neighborhood domination of graphs

Sahib Sh. Kahat[a], Ahmed A. Omran[b], Manal N. Al-Harere[c,*]

[a]Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq
[b]Department of Mathematics, College of Education for Pure Sciences, University of Babylon, Babylon, Iraq
[c]Department of Applied Sciences, University of Technology, Baghdad, Iraq

(Communicated by Madjid Eshaghi Gordji)

## Abstract

In that paper the fuzzy equality co-neighborhood domination and denoted by $\gamma_{en}(G)$ for a new definition of domination was described for the fuzzy graph. This new definition was studied in a strong fuzzy graph and constraints were found for many several graphs. Complementary strong fuzzy graphs of the same graphs were examined and studied in detail.

*Keywords:* Fuzzy equality co-neighborhood dominating set, fuzzy equality co-neighborhood domination number, strong fuzzy graph, complement fuzzy graph.

## 1. Introduction

Let $G = (V, E)$ be a simple and undirected graph and is devoid of single vertices. The actual subset $D$ of $V$ is equality co-neighborhood dominating set of $G$, if every vertex $v \in D$ has equally number of neighborhood vertices in the set $V$-$D$. Symbol $\gamma_{en}(G)$ is the minimum cardinality of the ENDS in $G$ indicates the domination number [16]. Domination topic in graph theory for the statement appealed many researchers, including [1, 2, 3, 4, 5, 6, 7, 8, 9, 11]and [10, 14, 22, 29, 35, 36] have set other condition on set $V$, also [12, 31] on domination number. In addition [25, 26, 27, 28] by setting some other terms on set $G - V$. As well as from research in dealing with polynomials for dominance [15, 16, 17, 18, 19] and [22], and from research in fuzzy graph [23, 30, 33, 34, 37, 38, 39] and others. In [13], C. Berg is the first to provide a criterion for dominance. The equality co-neighborhood, inverse

*Corresponding author
*Email addresses:* sgehet@gmail.com (Sahib Sh. Kahat), pure.ahmed.omran@uobabylon.edu.iq (Ahmed A. Omran), 100035@uotechnology.edu.iq (Manal N. Al-Harere)

*Received:* October 2020 *Accepted:* June 2021

equality co-neighborhood, and total equality co-neighborhood domination have been introduced in [20, 21, 24] respectively.

The fuzzy graph has been defined by some researchers by taking the minimum fuzzy cardinality to an all-dominating set as Mahioub and Soner [23]. While it was defined as by determining the minimum dominating set and taking the sum of its all members by Xavior et al.[37]. In this work, we followed the choice of the definition that has been put by Xavior et al. With some additions that appear to be the best adapted to our theory. Let $G(V, E)$ be an undirected, simple, and finite graph. A mapping $\sigma : V \to [0, 1]$ where V is a non-empty set of vertices called a fuzzy subset and $G = (\sigma, \mu)$ where $\mu : V \times V \to [0, 1]$ and $\mu(u, v) \leq \sigma(u) \wedge \sigma(v)$ is called a fuzzy graph. An edge $(u, v)$ is called effective if $\mu(u, v) = \sigma(u) \wedge \sigma(v)$ and the fuzzy graph is called strong if each edge belongs to it is effective. The $N_E(v) = \{u; (u, v) \text{ is an effective edge}\}$ is the open effective neighborhood and $N_E[v] = N_E(v) \cup v$ is the close effective neighborhood. Degree of a vertex italic v is the number of effective edges in $N_E(v)$ and denoted by $deg_E(v)$. In this work, we deal with the strong fuzzy graph, so every two adjacent vertices are joined by an effective edge.

**Definition 1.1.** *[24] Let G be a simple graph. A proper subset $D \subset V$ is called equality co-neighborhood dominating set of G (ENDS), if every vertex in set D is adjacent to equally number of vertices in $V - D$. The set D is called minimal ENDS( MENDS) if it has no proper ENDS. The equally domination number denoted by $\gamma_{en}(G)$ for simplicity $\gamma_{en}(G)$ is the minimum cardinality of a MENDS. The MENDS of cardinality $\gamma_{en}$ is called $\gamma_{en}$- set.*

**Definition 1.2.** *Let $G = (\sigma, \mu)$ be a fuzzy graph of a graph $G(V, E)$, if there is a set $D \subset V$ and D be dominating set of G and for all $v \in D$ has equally number of neighborhood vertices in the set $V - D$ there is a vertex u such that u is adjacent to v by effective edge (v dominates u), then D is called a fuzzy equality co-neighborhood dominating set (FENDS) on G.*

**Definition 1.3.** *A fuzzy equality co-neighborhood dominating set D in a fuzzy graph $G = (\sigma, \mu)$ is called minimum fuzzy equality co-neighborhood dominating (MFENDS) if the number of vertices of all equality co-neighborhood dominating set greater than or equal the number of vertices in D*

**Definition 1.4.** *Consider $W(D_i) = \{\sum \sigma(v): \forall v \in D_i : D_i$ is a minimum fuzzy equality co-neighborhood dominating set$\}$, then the fuzzy domination number of a fuzzy graph is $\gamma_{fen}(G) = min\{W(D_i) : D_i$ is a minimum fuzzy equality co-neighborhood dominating set$\}$*

To prove our main results we need the following results:

**Proposition 1.5.** *[24] $d_{en}(S_n, 1) = d_{en}(K_n, 1) = d_{en}(W_n, 1) = 1 \ \forall n \geq 2$.*

**Proposition 1.6.** *[24] Let G be a graph of order n, then*

1. *A cycle $C_n$, $n \geq 3$ has $\gamma_{en}(C_n) = \gamma_{bi}(C_n) = \lceil \frac{n}{3} \rceil$.*
2. *For path $P_n$, $n \geq 3$, $\gamma_{en}(P_n) = \lceil \frac{n}{3} \rceil$.*
3. *For complete $K_n$, wheel $W_n$, and star $S_n$, $\gamma_{en}(K_n) = \gamma_{en}(W_n) = \gamma_{en}(S_n) = 1$.*

**Proposition 1.7.** *[24] For a complete bipartite graph $K_{n,m}$,*

$$\gamma_{en}(K_{n,m}) = \min\{m, n, 2 + (\max\{m, n\} - \min\{m, n\}\}.$$

**Proposition 1.8.** *[22] Let $P_n$ be path with order n, and let D be ENDS with cardinality i, and let $k = |N(v_i)|$ in $V - D$, for all $v_i \in D$, then*

$k = 1$ *if* $i \geq \frac{n}{2}$.
$k = 2$ *if* $i < \frac{n}{2}$.

**Theorem 1.9.** *[24]For path $P_n, n \geq 4, \gamma_{en}(\overline{P_n}) = 2$.*

**Theorem 1.10.** *[24]For cycle $C_n, n \geq 4, \gamma_{en}(\overline{C_n}) = 2$.*

**Theorem 1.11.** *[24] For a complete bipartite graph $K_{n,m}$,*

$$\gamma_{en}(\overline{K_{n,m}}) = |m - n| + 2.$$

## 2. Fuzzy equality co-neighborhood dominating sets of graphs

*2.1. Fuzzy equality co-neighborhood dominating sets of certain graphs*

In this section we introduce FENDS of P(X), $S_n$, $K_n$, $W_n$ and $P_n$, for every $n \in Z^+$:

**Proposition 2.1.** *Let $S_n$ be a strong fuzzy star with order n, then*

$$\gamma_{fen}(S_n) = \sigma(v_1) \text{ for all } n > 2, \text{ where } v_1 \text{ is the center of } S_n.$$

**Proof .** Let $S_n$ be a star and let $V(S_n) = \{v_1, v_2, ..., v_n\}$ such that $v_1$ is the center of $S_n$, since $deg(v_1) = n - 1$ and $deg(v_i) = 1$, for all $i = 2, 3, \dots, n$, there are only $D_1 = \{v_1\}$ and $D_2 = V(S_n) - \{v_1\}$ that are ENDS, and since $\gamma_{en}(S_n) = 1$, $d_{en}(S_n, 1) = 1$, and $D_1 = \{v_1\}$ is $\gamma_{en}(S_n) - set$. Therefore $\gamma_{fen}(S_n) = \sigma(v_1)$. □

**Proposition 2.2.** *Let $K_n$ be a strong fuzzy complete with order n, then*

$$\gamma_{fen}(K_n) = \min\{\sigma(v_i) : i = 1, 2, \dots, n, \text{ for all } n \geq 2\}.$$

**Proof .** Let $K_n$ be a complete fuzzy graph with order $n$, since every vertex in $K_n$ is adjacent to every other vertex in $K_n$ and by Definition 1.1, every subset of $K_n$ is ENDS, then $d_{en}(K_n, i) = d(K_n, i) = \binom{n}{i}$ $\forall\ i < n \geq 2$. Since $\gamma_{en}(K_n) = 1$, $d_{en}(K_n, 1) = n$ and $\gamma_{en}(K_n) - set = \{v_i\}$, for all $i = 1, 2, ..., n$, therefore $\gamma_{fen}(K_n) = \min\{\sigma(v_i) : i = 1, 2, ..., n\}$. □

**Proposition 2.3.** *Let $W_n$ be a strong fuzzy wheel with order n, then $\gamma_{fen}(W_n) = \sigma(v_1)$, for all $n > 3$, where $v_1$ is the center of $W_n$.*

**Proof .** Let $W_n$ be a wheel fuzzy graph with order $n + 1$ and let $V(W_n) = \{v_1, v_2, ..., v_n, v_{n+1}\}$ such that $v_1$ is the center of $W_n$. Since $deg(v_1) = n$, it is clear that $D = \{v_1\}$ is ENDS, so $\gamma_{en}(W_n) = 1$, and $d_{en}(W_n, 1) = 1$ and $D = \{v_1\}$ is $\gamma_{en}(W_n) - set$ according to Proposition 1.5 and Proposition 1.6, therefore $\gamma_{fen}(W_n) = \sigma(v_1)$. □

**Theorem 2.4.** *Let $P_n$ G be a strong fuzzy path with order n, then*

**(i)** *if $n \equiv 0(mod3)$, then $\gamma_{fen}(P_n) = \sum_{j=0}^{\frac{n}{3}-1} \sigma(v_{2+3j})$.*

**(ii)** *if $n \equiv 2(mod3)$, then $\gamma_{fen}(P_n) = \min\{_{i=1}^{\frac{n-2}{3}} \sum_{j=0}^{\frac{n-2}{3}-1} \sigma(v_{l+3j}) + \sigma(v_{n-1})$ such that $l = 1$ if $j \geq i$ and $l = 2$ if $j < i$.*

**(iii)** *if $n \equiv 1(mod3)$, then $\gamma_{fen}(P_n) = min\{_{i=1}^{\frac{n-4}{3}} min\{_{m=1}^{\frac{n-1}{3}-i} \sum_{j=0}^{\frac{n-4}{3}} \sigma(v_{l+3j}) + \sigma(v_{n-1})$ such that $l = 2$ if $j < i$ and $l = 1$ if $j \geq i$ and $l = 0$ if $j \geq m + i$.*

**Proof .** Let $P_n$ be a strong fuzzy path with order $n$, and let $D$ be ENDS and $k = N(v_i)$ in $V - D$, for all $v_i \in D$, then $k = 2$ according to Proposition 1.8.

**(i)** Since $\gamma_{en}(P_n) = \lceil \frac{n}{3} \rceil$ according to Proposition 1.6, if $n \equiv 0(mod3)$, then $\gamma_{en}(P_n) = \frac{n}{3}$, hence for all three vertices in $P_n$ there is one vertex in $D$ dominate me two vertices in the $V - D$ and $v_1, v_n \notin D$ and must be $v_2, v_{n-1} \in D$, then in this case there is only one ENDS $D = \{v_{2+3j} : j = 0, 1, 2, ..., \frac{n}{3} - 1\}$, hence $d_{en}(P_n, \frac{n}{3})$= 1. Therefore, $\gamma_{fen}(P_n) = \sum_{j=0}^{\frac{n}{3}-1} \sigma(v_{2+3j})$

**(ii)** If $n \equiv 2(mod3)$, then $\gamma_{en}(P_n) = \frac{n+1}{3}$, then we have every vertex of $V - D$ that is dominated by one vertex of $D$ except for one vertex let $v_r$ be dominated by two vertices of $D$, moreover $v_1, v_n \notin D$ and must be $v_2, v_{n-1} \in D$, then in this case there must be $v_3, v_{n-2} \notin D$. we claim that $r = 3j$ such that $j = 1, 2, ..., \frac{n-2}{3}$, if $r = 4$, then $v_3 \in D$ which is a contradiction. And if r=5 then $v_4 \in D$ and $v_3 \notin D$, since $v_2 \in D$, then $r = 3$ and r=5 which is a contradiction there are only one vertex in $V - D$ dominated by two vertices of D. If r=7, $v_6, v_3 \in D$ which is a contradiction $v_3 \notin D$. As well as if $r = 8$, $v_7, v_4 \in D$, $r = 8$ and $r = 3$ which is a contradiction there are only one vertex in $V - D$ dominated by two vertices of $D$. Therefore, $r = 3j$, and $D_i = \{v_{l+3j} : j = 0, 1, ..., \frac{n-2}{3}\}$ is ENDS , such that $i = 1, 2, ..., \frac{n-2}{3}$ and $l = 1$ if $j \geq i$ and $l = 2$ if $j < i$. Then $\gamma_{fen}(P_n) = \min\{_{i=1}^{\frac{n-2}{3}} \sum_{j=0}^{\frac{n-2}{3}-1} \sigma(v_{l+3j}) + \sigma(v_{n-1})$.

**(iii)** If $n \equiv 1(mod3)$, then $\gamma_{en}(P_n) = \frac{n+2}{3}$, then we have every vertex of $V - D$ that is dominated by one vertex of $D$ except for two vertices let $v_{r1}$ and $v_{r2}$ dominated by two vertices of D, moreover $v_1, v_n \notin D$ and must be $v_2, v_{n-1} \in D$, so $v_3, v_{n-2} \notin D$. By the following formula $D_i^m = \{v_{l+3j} : j = 0, 1, 2, ..., \frac{n-4}{3}\} \cup \{v_{n-1}\}$, such that $i = 1, 2, ..., \frac{n-4}{3}$ and $m = 1, 2, ..., \frac{n-1}{3} - i$ and $l = 2$ if $j < i$ and $l = 1$ if $j \geq i$ and $l = 0$ if $j \geq m + i$, we get all possibilities for ENDS with cardinality $\frac{n+2}{3}$, then $\gamma_{fen}(P_n) = min\{_{i=1}^{\frac{n-4}{3}} min\{_{m=1}^{\frac{n-1}{3}-i} \sum_{j=0}^{\frac{n-4}{3}} \sigma(v_{l+3j}) + \sigma(v_{n-1})$ according to Definition 1.4.

□

**Theorem 2.5.** *Let $C_n$ be a strong fuzzy cycle with order n, then the following properties hold of $\gamma_{fen}(C_n)$, for every $3 \leq n \in Z^+$:*

1. *If $n \equiv 0(mod3)$, then $\gamma_{fen}(C_n) = \min\{_{i=1}^{3} \sum_{j=1}^{\frac{n}{3}} \sigma(v_{3j-i+1})$*
2. *If $n \equiv 2(mod3)$, then $\gamma_{fen}(C_n) = \min\{_{i=1}^{n} \sum_{j=0}^{\frac{n-5}{3}} \sigma(v_{3j+i+2}) + \sigma(v_i)$ such that $v_{n+x} \equiv v_x$ $\forall x \in Z^+$*
3. *If $n \equiv 1(mod3)$, then $\gamma_{fen}(C_n) = \min\{_{m=1}^{n} \min\{_{i=1}^{\frac{n-1}{3}} \sum_{j=0}^{\frac{n-7}{3}} \sigma(v_{l+3j}) + \sigma(v_m) + \sigma(v_{m+2})$ such that $l = m + 5$ if $j < i - 1$ and $l = m + 4$ if $j \geq i - 1$ and $v_{n+t} \equiv v_t$ $\forall t \leq n$.*

**Proof .** Let $C_n$ be a cycle with order $n$, let $D$ be ENDS and let $k = |N(v_i)|$ in $V - D$, for all $v_i \in D$, then $k = 2$ according to Proposition 1.8

1. Since $\gamma_{en}(C_n) = \lceil \frac{n}{3} \rceil$ according to Proposition 1.6, if $n \equiv 0(mod3)$, then $\gamma_{en}(C_n) = \frac{n}{3}$, hence for every three consecutive vertices there is one vertex in $D$, then in this case there is only three ENDS $D_i = \{v_{i+3j} : j = 0, 1, 2, ..., \frac{n}{3} - 1\}$ such that i=1,2,3, hence $d_{en}(C_n, \frac{n}{3})$= 3. Therefore, $\gamma_{fen}(C_n) = \min\{_{i=1}^{3} \sum_{j=0}^{\frac{n}{3}-1} \sigma(v_{i+3j})$.
2. If $n \equiv 2(mod3)$, then $\gamma_{en}(C_n) = \frac{n+1}{3}$. Then every vertex of $V - D$ that is dominated by one vertex of $D$ except for one vertex let $v_r$ be dominated by two vertices of $D$, then it could be r=1,2,...,n. The following formula $D_i = \{v_{2+i+3j} : j = 0, 1, 2, ..., \frac{n-5}{3}\} \cup \{v_i\}$, i=1,2,...,n such that $v_{n+t} \equiv v_t$ $\forall t \leq n$ fulfills all possibilities for ENDS with cardinality $\frac{n+1}{3}$, so $d_{en}(C_n, \frac{n}{3})$= n. Therefore, $\gamma_{fen}(C_n) = \min\{_{i=1}^{n} \sum_{j=0}^{\frac{n-5}{3}} \sigma(v_{i+2+3j}) + \sigma(v_i)$.

3. When $n \equiv 1(mod3)$, then $\gamma_{en}(C_n) = \frac{n+2}{3}$, then we have every vertex of V-D that is dominated by one vertex of D except for two vertices let $v_{r_1}$ and $v_{r_2}$ are dominated by two vertices of D, then it could be $r_1 = 1, 2, ..., n$ and $r_2 = r_1 + 2 + 3j : j = 0, 1, ..., \frac{n+4}{3}$. By the following formula $D_i^m = \{v_{l+3j} : j = 0, 1, 2, ..., \frac{n-7}{3}\} \cup \{v_m, v_{m+2}\}$, such that $i = 1, 2, ..., \frac{n-1}{3}$ and $m = 1, 2, ..., n$ and $l = m+5$ if $j < i-1$ and $l = m+4$ if $j \geq i-1$ and $v_{n+t} \equiv v_t$ $\forall t \leq n$, we get all possibilities for ENDS with cardinality $\frac{n+2}{3}$. Therefore, $\gamma_{fen}(C_n) = \min\{_{m=1}^{n} min\{_{i=1}^{\frac{n-1}{3}} \sum_{j=0}^{\frac{n-7}{3}} \sigma(v_{l+3j}) + \sigma(v_m) + \sigma(v_{m+2})$.

□

In the following theorem, we obtain some properties of $\gamma_{fen}(K_{m,n})$ $m \geq n \geq 2$:

**Theorem 2.6.** *Let $K_{m,n}$ be a fuzzy strong complete bipartite graph with bipartition $(X, Y)$ such that $|V(X)| = n$, $|V(Y)| = m$ and $m \geq n \geq 2$, then*

$$\gamma_{fen}(K_{m,n}) = \min \begin{cases} \sum_{i=1}^{n} \sigma(v_i), \ \textit{where } v_i \in X \\ \min\{_{i=1}^{n} \sigma(v_i) + \sum_{k=1}^{m-n+1} \min \sigma(u_{j_k}) : j_k = 1, 2, ..., m \\ \textit{such that } u_{j_1} \neq u_{j_2} \neq ... \neq u_{j_{m-n+1}}, \ \textit{where } v_i \in X \textit{ and } u_{j_k} \in Y \end{cases}$$

**Proof .** We have $\gamma_{en}(K_{m,n}) = \min\{m, n, |m-n|+2\}$ according to Proposition 1.7, since $m \geq n \geq 2$, $\gamma_{en}(K_{m,n}) = \min\{n, m-n+2\}$, there are two cases that depend on $\gamma_{en}(K_{m,n})$ as follows:

Case 1. If $n < m-n+2$, then $X$ is $\gamma_{en} - set$, and $\gamma_{fen}(K_{m,n}) = \sum_{i=1}^{n} \sigma(v_i)$ where $v_i \in X$.

Case 2. If $m-n+2 < n$, then $\gamma_{en}(K_{m,n}) = m-n+2$, and $d_{en}(K_{m,n}, m-n+2) = \binom{m}{n-1}\binom{n}{1} = n\binom{m}{n-1}$.

Since every $\gamma_{en} - set$ has one vertex from $X$ and $m-n+1$ vertices from $Y$. Let $S_p$ such that $p = 1, 2, ..., \binom{m}{n-1}$ be any subset of $Y$ with order $m-n+1$, then the following formula $D_i^p = \{v_i\} \cup S_p$ for all $i = 1, 2, ..., n$ and for all $p = 1, 2, ..., \binom{m}{n-1}$ fulfills all possibilities for ENDS with cardinality $m-n+1$. Therefore, $\gamma_{fen}(K_{m,n}) = \min\{_{i=1}^{n} \sigma(v_i) + \sum_{k=1}^{m-n+1} \min \sigma(u_{j_k}) : j_k = 1, 2, ..., m$ such that $u_{j_1} \neq u_{j_2} \neq ... \neq u_{j_{m-n+1}}$, and $\sigma(u_{j_1}) \leq \sigma(u_{j_2}) \leq, ..., \sigma(u_{j_{m-n+1}})$, where $v_i \in X$ and $u_{j_k} \in Y$. By minimum of case 1 and case 2 the proof is done. □

**Theorem 2.7.** *Let $G = (V, E)$ be graph and $X = \{a_1, a_2, ...., a_n\}$, for all $n > 3$ such that $V(G) = P(X)$ be a collection of all proper subsets of $X$, when $A \cap B = \phi$ such that $A \neq B \neq \phi$, then $AB \in E(G)$. If $G$ is a strong fuzzy, then $\gamma_{fen}(G) = \sum_{i=1}^{n} \sigma(\{a_i\})$ .*

**Proof .** We have $X \backslash_{\{a_i\}} \cap \{a_i\} = \phi$, for all $a_i \in X$, and $X \backslash_{\{a_i\}} \cap A \neq \phi$, for all $A \neq \{a_i\}$, therefore $deg(X \backslash_{\{a_i\}})$= 1 $\forall i = 1, 2, ..., n$, i. e., $X \backslash_{\{a_i\}}$ is adjacent only with $\{a_i\}$, then every dominating set must contain $X \backslash_{\{a_i\}}$ or $\{a_i\}$, then $\gamma(G) \geq n$. We have the set $\{a_i\}$ is adjacent with every subset has not $a_i$, then $deg(a_i)$ is equal $\forall i = 1, 2, ..., n$ and only subset $\{a_i : i = 1, 2, ..., n\}$ is the minimum dominating set of $G$. Therefore, $\gamma(G) = \gamma_{en}(G) = n$ $\forall\, n > 3$. It is clear that $\gamma_{fen}(G) = \sum_{i=1}^{n} \sigma(\{a_i\})$.
□

**Example 2.8.** *Let n=4, and let $X = \{1, 2, 3, 4\}$, then $P(X)$= $\{\{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 4\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, X, \phi\}$ $E(G) = \{\{\{1\}, \{2\}\}, \{\{1\}, \{3\}\}, \{\{2\}, \{3\}\}, \{\{\{1\}, \{4\}\}, \{\{2\}, \{4\}\}, \{\{3\}, \{4\}\}, \{\{1\}, \{2, 3\}\}, \{\{1\}, \{2, 4\}\}, \{\{1\}, \{3, 4\}\}, \{\{2\}, \{1, 3\}\}, \{\{2\}, \{1, 4\}\}, \{\{2\}, \{3, 4\}\}, \{\{3\}, \{1, 2\}\}, \{\{3\}, \{1, 4\}\}, \{\{3\}, \{2, 4\}\}, \{\{4\}, \{1, 2, 3\}\}, \{\{3\}, \{1, 2, 4\}\}, \{\{2\}, \{1, 3, 4\}\}, \{\{1\}, \{2, 3, 4\}\}\}$, we have $\{\{1\}, \{2\}, \{3\}, \{4\}\}$ is minimum dominating set of G by Theorem 2.12, see Figure 1.*

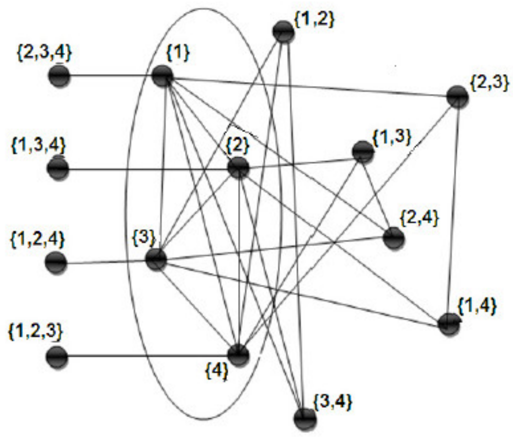


Figure 1: $\{\{1\}, \{2\}, \{3\}, \{4\}\}$ is minimum dominating set of $G$

*2.2. Fuzzy equality Co-Neighborhood Dominating sets of complement of some graphs $G_n$*

We shall investigate FENDS of complement of $P_n$, $C_n$ and $K_{m,n}$.

**Remark.** $\overline{S}_n$, $\overline{K}_n$ and $\overline{W}_n$ there are no FENDS, because $\overline{S}_n$, $\overline{K}_n$ and $\overline{W}_n$ have isolated vertices, so there are no ENDS

**Proposition 2.9.** *Let $P_n$ be a strong fuzzy path graph, and let $\overline{P}_n$ be complement of $P_n$, then*

$$\gamma_{fen}(\overline{P}_n) = \min \begin{cases} \sigma(v_1) + \sigma(v_n), \\ \min\{_{i=2}^{n-1}\sigma(v_i) + \{_{j=2}^{n-1}\sigma(v_j) such\ that\ i \neq j and\ i \neq j_{+}^{-}2. \end{cases}$$

**Proof .** Since $\gamma_{en}(\overline{P_n}) = 2$ according to Theorem 1.9, and $deg(v_1) = deg(v_n) = n-2$ and $deg(v_i) = deg(v_j) = n-3$ in $\overline{P_n}$ for all $i = 2, 3, ..., n-1$ and $j = 2, 3, ..., n-1$,

a. $(v_1)$ is adjacent to all vertices in $\overline{P_n}$ except for support vertex of $v_1$, and so $v_n$, then the subset $D_1 = \{v_1, v_n\}$ is MEDNS of $\overline{P_n}$.

b. For $D_i = \{v_i, v_{i+2}\}$, a vertex $v_{i-1}$ is not adjacent to $v_i$ nor to $v_{i-2}$, then $D_i^{i+2} = \{v_i, v_{1+2}\}$ are not MEDNS of $\overline{P_n}$.

c. Every vertex $v_i$ adjacent to all vertices in $\overline{P_n}$ except for two vertices $v_{i+1}$ and $v_{i-1}$, then the subsets $D_i^j = \{v_i, v_j\}$ such that $i \neq j$ and $i \neq j_{+}^{-}2$ are MEDNS of $\overline{P_n}$ for all $i = 2, 3, ..., n-1$ and $j = 2, 3, ..., n-1$.

From a, b and c we get

$$\gamma_{fen}(\overline{P}_n) = \min \begin{cases} \sigma(v_1) + \sigma(v_n), \\ \min\{_{i=2}^{n-1}\sigma(v_i) + \{_{j=2}^{n-1}\sigma(v_j) \text{such that } i \neq j \text{and } i \neq j_{+}^{-}2. \end{cases}$$

□

**Proposition 2.10.** *Let $C_n$ be strong fuzzy cycle graph, and let $\overline{C}_n$ be complement of $C_n$, then $\gamma_{fen}(\overline{C}_n) = \min\{_{i=1}^{n}\sigma(v_i) + \{_{j=1}^{n}\sigma(v_j)$ such that $i \neq j$ and $i \neq j_{+}^{-}2$.*

**Proof .** Since $\gamma_{en}(\overline{C_n}) = 2$ according to Theorem 1.10, and $deg(v_i) = deg(v_j) = n-3$ in $\overline{C_n}$ for all $i = 1, 2, ..., n$ and $j = 1, 2, ..., n$,

a. For $D_i = \{v_i, v_{i+2}\}$, a vertex $v_{i-1}$ is not adjacent to $v_i$ nor to $v_{i-2}$, then $D_i^{i+2} = \{v_i, v_{1+2}\}$ are not MEDNS of $\overline{C_n}$.

b. Every vertex $v_i$ adjacent to all vertices in $\overline{C_n}$ except for two vertices $v_{i+1}$ and $v_{i-1}$, then the subsets $D_i^j = \{v_i, v_j\}$ such that $i \neq j$ and $i \neq j_+^- 2$ are MEDNS of $\overline{C_n}$ for all $i = 1, 2, ..., n$ and $j = 1, 2, ..., n$.

From a and b we get $\gamma_{fen}(\overline{C}_n)$=min$\{_{i=1}^{n}\sigma(v_i)$+$\{_{j=1}^{n}\sigma(v_j)$ such that $i \neq j$ and $i \neq j_+^- 2$. □

**Theorem 2.11.** *Let $K_{m,n}$ be a strong fuzzy complete bipartite graph with bipartition $(X, Y)$ such that $|V(X)| = n$, $|V(Y)| = m$ and $m \geq n \geq 2$, and let $\overline{K_{m,n}}$ be complement of $K_{m,n}$, then*

$$\gamma_{fen}(\overline{K_{m,n}}) = \min\{_{i=1}^{n}\sigma(v_i) + \sum_{k=1}^{m-n+1} \min \sigma(u_{j_k}) : j_k = 1, 2, ..., m$$

*such that $u_{j_1} \neq u_{j_2} \neq ... \neq u_{j_{m-n+1}}$, where $v_i \in X$ and $u_{j_k} \in Y$.*

**Proof .** Since $\gamma_{en}(\overline{K_{m,n}}) = m - n + 2$ according to Theorem 1.11, and since $\overline{K_{m,n}} = K_n \cup K_m$, every $\gamma_{en} - set$ has one vertex from X and $m - n + 1$ vertices from Y, in addition to $d_{en}(\overline{K_{m,n}}, m - n + 2) = \binom{m}{n-1}\binom{n}{1} = n\binom{m}{n-1}$. Let $S_p$ such that $p = 1, 2, ..., \binom{m}{n-1}$ be any subset of $Y$ with order $m - n + 1$, then the following formula $D_i^p = \{v_i\} \cup S_p$, for all $i = 1, 2, ..., n$ and for all $p = 1, 2, ..., \binom{m}{n-1}$ fulfills all possibilities for ENDS with cardinality $m - n + 1$. Therefore, $\gamma_{fen}(\overline{K_{m,n}})$= min$\{_{i=1}^{n}\sigma(v_i) + \sum_{k=1}^{m-n+1}$ min $\sigma(u_{j_k}) : j_k = 1, 2, ..., m$ such that $u_{j_1} \neq u_{j_2} \neq ... \neq u_{j_{m-n+1}}$, and $\sigma(u_{j_1}) \leq \sigma(u_{j_2}) \leq, ..., \sigma(u_{j_{m-n+1}})$, where $v_i \in X$ and $u_{j_k} \in Y$. □

**Theorem 2.12.** *Let $G = (V, E)$ be graph and $X = \{a_1, a_2, ...., a_n\}$ for all $n > 3$ such that $V(G) = P(X)$ be a collection of all proper subsets of $X$, when $A \cap B = \phi$ such that $A \neq B \neq \phi$, then $AB \in E(G)$. If $\overline{G}$ is a strong fuzzy, then $\gamma_{fen}(\overline{G})$ = min$\{_{i=1}^{n}\sigma(X\backslash_{\{a_i\}})$ + min$\{_{j=1}^{n}\sigma(X\backslash_{\{a_j\}})$, such that $i \neq j$ and $\sigma(X\backslash_{\{a_i\}}) \leq \sigma(X\backslash_{\{a_j\}})$.*

**Proof .** Since $V(G) = V(\overline{G})$ and $vu \notin E(\overline{G})$ if $vu \in E(G)$ and vice versa, $V(\overline{G}) = P(X)$ when $A \cap B \neq \phi$ such that $A \neq B$, then $AB \in E(\overline{G})$. We have $X\backslash_{\{a_i\}} \cap \{a_i\} = \phi$, for all $a_i \in X$, and $X\backslash_{\{a_i\}} \cap A \neq \phi$, for all $A \neq \{a_i\}$, then $X\backslash_{\{a_i\}}$ is adjacent to every subset except $\{a_i\}$ and $X\backslash_{\{a_j\}}$ is adjacent to every subset except $\{a_j\}$, for all $i = 1, 2, ..., n$ and for all $j = 1, 2, ..., n$, such that $i \neq j$, then $D = \{X\backslash_{\{a_i\}}, X\backslash_{\{a_j\}}\}$ is MENDS of $\overline{G}$ and $deg(X\backslash_{\{a_i\}})$ is equal for all $i = 1, 2, ..., n$, hence $\gamma_{en}(G) = 2$. Therefore, $\gamma_{fen}(\overline{G})$ = min$\{_{i=1}^{n}\sigma(X\backslash_{\{a_i\}})$ + min$\{_{j=1}^{n}\sigma(X\backslash_{\{a_j\}})$, (see Figure 2). □

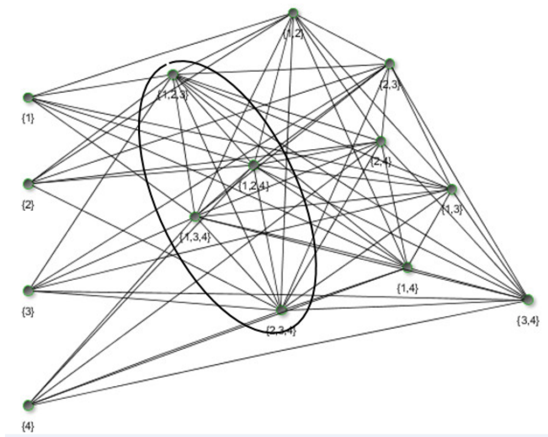


Figure 2: $\{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$ is minimum dominating set of $\overline{G}$

## References


[1] M. A. Abdlhusein and M. N. Al-Harere, *Total pitchfork domination and its inverse in graphs*, Dis. Math. Algor. Appl. (2021) 2150038.

[2] M. A. Abdlhusein and M. N. Al-Harere, *New parameter of inverse domination in graphs*, Indian J. Pure Appl. Math. (accepted to appear)(2021).

[3] M. A. Abdlhusein and M. N. Al-Harere, *Doubly connected pitchfork domination and its inverse in graphs*, TWMS J. App. Eng. Math. (accepted to appear) (2021).

[4] M. A. Abdlhusein, *Stability of inverse pitchfork domination*, Int. J. Nonlinear Anal. Appl. 12(1) (2021) 1009–1016.

[5] M. A. Abdlhusein, *Doubly connected bi-domination in graphs*, Dis. Math. Algor. Appl. 13(2) (2021) 2150009.

[6] M. N. Al-Harere and M. A. Abdlhusein, *Pitchfork domination in graphs*, Dis. Math. Algor. Appl. 12(2)(2020) 2050025.

[7] M.N. Al-Harere, A. A. Omran, and A. T. Breesam, *Captive domination in graphs*, Dis. Math. Algor. Appl. 12(6) (2020) 2050076.

[8] M. N. Al-Harere and P. A. Khuda Bakhash, *Tadpole domination in graphs*, Baghdad Sci. J. 15 (2018) 466–471.

[9] M. N. Al-Harere, P. A. Khuda, *Tadpole domination in duplicated graphs*, Dis. Math. Algor. Appl. 13(2) (2021) 2150003.

[10] M. N. Al-Harere, R. J. Mitlf and F. A. Sadiq, *Variant domination types for a complete h-ary tree*, Baghdad Sci. J. 18(1) (2021) 2078–8665.

[11] M. N. Al-Harere and A. A. Omran, *On binary operation graphs*, Bol. Soc. Paranaense Mat. 38(7) (2020) 59–67 .

[12] L. K. Alzaki, M. A. Abdlhusein and A. K. Yousif, *Stability of (1,2)-total pitchfork domination*, Int. J. Nonlinear Anal. Appl. 12(2) (2021) 265–274.

[13] C. Berge, *The Theory of Graphs and its Applications*, Methuen and Co, London, 1962.

[14] A. A. Jabor and A. A. Omran, *Hausdorff topological of path in graph*, IOP Conf. Ser.: Mater. Sci. Eng. 928 (2020) 042008, doi:10.1088/1757-899X/928/4/042008.

[15] S. Sh. Kahat, A. M. Khalaf, R. Hasni, *Dominating sets and domination polynomials of stars*, Aust. J. Basic Appl. Sci. 8(6) (2014) 383–386.

[16] S. Sh. Kahat, A. M. Khalaf, *Dominating sets and domination polynomial of complete graphs with missing edges*, J. Kufa Math. Comp. 2(1) (2014) 64–68.

[17] S. Sh. Kahat, A. M. Khalaf, R. Hasni, *Dominating sets and domination polynomial of wheels*, Asian J. Appl. Sci. 2(3) (2014) 287–290.

[18] S. Sh. Kahat, A. M. Khalaf, *Dominating Sets and Domination polynomial of kr-gluing of Graphs*, J. Dirasat Tarbawiya, to appear.

[19] S. Sh. Kahat, A. M. Khalaf, *Dominating Sets and Domination polynomial of kr-gluing of Graphs II*, Dirasat Tarbawiya, 12(48) (2019) 358–367.

[20] S. Sh. Kahat, M. N. Al-Harere, *Inverse Equality Co-Neighborhood Domination in Graphs*, journal (IHICPAS). By "Ibn Al-Haitham 2nd. International Conference for Pure and Applied Science" accepted for publication (2020).

[21] S. Sh. Kahat, M. N. Al-Harere, *Total Equality Co-Neighborhood Domination in Graphs*, AIP (ISSN: 0094-243X). ICCEPS-2021. By Sixth National Scientific/ Third International Conference accepted for publication.

[22] S. Sh. Kahat, M. N. Al-Harere, *Dominating sets and polynomial of equality co-neighborhood domination of graphs*, AIP (ISSN: 0094-243X).ICARPAS. By 1st International Conference on Advanced Research in Pure and Applied Science accepted for publication (2021).

[23] Q. M. Mahioub and N. D. Soner, *The split domination number of fuzzy graphs*, Far East J. Appl. Math. 30 (2008) 125–132.

[24] A. A. Omran, M. N. Al-Harere, and Sahib Sh. Kahat, *Equality co-neighborhood domination in graphs*, Disc. Math. Algor. Appl. (2021). https://doi.org/10.1142/S1793830921500981

[25] A. A. Omran and Y. Rajihy, *Some properties of frame domination in graphs*, J. Eng.Appl. Sci. 12 (2017) 8882–8885.

[26] A. A. Omran and M. M. Shalaan, *Inverse co-even domination of graphs*, IOP Conf. Ser.: Mater. Sci. Eng. 928 (2020) 042025.

[27] A. A. Omran and T. Swadi, *Observer domination number in graphs*, J. Adv. Res. Dyn. Cont. Syst. 11 (01-Special Issue) (2019) 468–495.

[28] A. A. Omran and H. Hamed Oda, *Hn domination in graphs*, Baghdad Sci. J. 16(1) (2019) 242–247.

[29] A. A. Omran and M. M. Shalaan, *Inverse co-even domination of graphs*, IOP Conf. Ser.: Mater. Sci. Eng. 928 (2020) 042025. doi:10.1088/1757-899X/928/4/042025.

[30] A. A. Omran and T. A. Ibrahim, *Fuzzy co-even domination of strong fuzzy graphs*, Int. J. Nonlinear Anal. Appl. 12(1) (2021) 727–734.

[31] S. J. Radhi, M. A. Abdlhusein and A. E. Hashoosh, *The arrow domination in graphs*, Int. J. Nonlinear Anal. Appl. 12(1) (2021) 473–480.

[32] M. M. Shalaan and A. A. Omran, *Co-even domination number in some graphs*, IOP Conf. Ser.: Mater. Sci. Eng. 928 (2020) 042015. doi:10.1088/1757-899X/928/4/042015.

[33] A. Somasundaram and S. Somasundaram, 1975, *Domination in fuzzy graphs-I*, Pattern Rec. Lett.19 (1998) 787–791.

[34] A. Somasundaram, *Domination in fuzzy graphs-II*, J. Fuzzy Math. 13 (2005) 281–288.

[35] S. H. Talib, A. A. Omran , and Y. Rajihy, *Additional properties of frame domination in graphs*, J. Phys.: Conf. Ser. 1664 (2020) 012026, doi:10.1088/1742-6596/1664/1/012026.

[36] S. H. Talib, A. A. Omran , and Y. Rajihy , *Inverse frame domination in graphs*, IOP Conf. Ser.: Mater. Sci. Eng. 928 (2020) 042024, doi:10.1088/1757-899X/928/4/042024.

[37] D. A. Xavior, F. Isido and V. M. Chitra, *On domination in fuzzy graphs*, Int. J. Comput. Algor. 2 (2013) 248–250.

[38] H. J. Yousif and A. A. Omran, *The Split Anti Fuzzy Domination in Anti Fuzzy Graphs*, J. Phys., Conf. Ser. 1591 (2020) 012054. doi:10.1088/1742-6596/1591/1/012054.

[39] H. J. Yousif and A. A. Omran, *2-anti fuzzy domination in anti fuzzy graphs*, IOP Conf. Ser.: Mater. Sci. Eng. 928 (2020) 042027.