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Fixed point in bicomplex valued metric spaces

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Abstract

In the paper we obtain sufficient conditions for the existence of common fixed point for a pair of contractive type mappings in bicomplex valued metric spaces.

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1. Introduction

Segre [18] made a pioneering attempt in the development of special algebras. He conceptualized commutative generalization of complex numbers as bicomplex numbers, tricomplex numbers, etc. as elements of an infinite set of algebras. Subsequently during the 1930s, other researchers also contributed in this area {cf.[19]-[9]}. But unfortunately the next fifty years failed to witness any advancement in this field. Afterward Price [16] developed the bicomplex algebra and function theory. Recently renewed interest in this subject finds some significant applications in different fields of mathematical sciences as well as other branches of science and technology. An impressive body of work has been developed by a number of researchers. Among them an important work on elementary functions of bicomplex numbers has been done by Luna-Elizarrarás et al. [15].

Recently Beg et al. [4] studied existence of common fixed points for maps on topological vector space valued cone metric spaces. Azam et al. [1] extended it to complex valued metric space and established a common fixed point theorem for a pair of self contracting mappings. Rouzkard & Imdad [10] generalized the result obtained by Azam et al. [1] and they proved another common fixed

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point theorem satisfying some rational inequality in complex valued metric spaces. The Banach contraction principle [3] is a popular and effective tool to solve the existence problems in many branches of mathematical analysis and is an active area of research. The celebrated Banach theorem states that: "Let (X,d) be a complete metric space and T be a mapping of X into itself satisfying $d(Tx, Ty) \leq kd(x, y), \forall x, y \in X$, where k is a constant in $(0, 1)$. Then T has a unique fixed point $x^* \in X$". Choudhury et al. [8] proved some fixed point results in partially ordered complex valued metric spaces for rational type expressions. Bhat et al. [6] proved fixed point of mapping satisfying rational inequality in complex valued metric spaces. Also one can see the attempts in {cf.[2],[7]}. Choi et al. [15] proved some common fixed point theorems in connection with two weakly compatible mappings in bicomplex valued metric spaces. Jebril et al. [12] proved some common fixed point theorems under rational contractions for a pair of mappings in bicomplex valued metric spaces. This article is in continuation of these works. We further investigate the bicomplex valued metric spaces and establish fixed point theorems for a pair of contractive type mappings satisfying a rational inequality.

Next we present some basic notions and notations for subsequent use.

We denote the set of real, complex and bicomplex numbers respectively as $\mathbb{C}_0, \mathbb{C}_1$ and $\mathbb{C}_2$. Segre [18] defined the bicomplex number as:

$$\xi = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2,$$

where $a_1, a_2, a_3, a_4 \in \mathbb{C}_0$, and the independent units i_1, i_2 are such that $i_1^2 = i_2^2 = -1$ and $i_1 i_2 = i_2 i_1$, we denote the set of bicomplex numbers $\mathbb{C}_2$ is defined as:

$$\mathbb{C}_2 = \{\xi : \xi = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2, a_1, a_2, a_3, a_4 \in \mathbb{C}_0\},$$

i.e.,

$$\mathbb{C}_2 = \{\xi : \xi = z_1 + i_2 z_2, z_1, z_2 \in \mathbb{C}_1\}.$$

where $z_1 = a_1 + a_2 i_1 \in \mathbb{C}_1$ and $z_2 = a_3 + a_4 i_1 \in \mathbb{C}_1$.

If $\xi = z_1 + i_2 z_2$ and $\eta = w_1 + i_2 w_2$ be any two bicomplex numbers then the sum is $\xi \pm \eta = (z_1 + i_2 z_2) \pm (w_1 + i_2 w_2) = (z_1 \pm w_1) + i_2 (z_2 \pm w_2)$ and the product is $\xi.\eta = (z_1 + i_2 z_2).(w_1 + i_2 w_2) = (z_1 w_1 - z_2 w_2) + i_2 (z_1 w_2 + z_2 w_1)$.

There are four idempotent elements in $\mathbb{C}_2$, they are $0, 1, e_1 = \frac{1+i_1 i_2}{2}$ and $e_2 = \frac{1-i_1 i_2}{2}$ out of which e_1 and e_2 are nontrivial such that $e_1 + e_2 = 1$ and $e_1 e_2 = 0$. Every bicomplex number $z_1 + i_2 z_2$ can uniquely be expressed as the combination of e_1 and e_2, namely

$$\xi = z_1 + i_2 z_2 = (z_1 - i_1 z_2) e_1 + (z_1 + i_1 z_2) e_2.$$

This representation of ξ is known as the idempotent representation of bicomplex number and the complex coefficients $\xi_1 = (z_1 - i_1 z_2)$ and $\xi_2 = (z_1 + i_1 z_2)$ are known as idempotent components of the bicomplex number ξ.

An element $\xi = z_1 + i_2 z_2 \in \mathbb{C}_2$ is said to be invertible if there exists another element η in $\mathbb{C}_2$ such that $\xi\eta = 1$ and η is said to be the inverse (multiplicative) of ξ. Consequently ξ is said to be the inverse (multiplicative) of η. An element which has an inverse in $\mathbb{C}_2$ is said to be the nonsingular element of $\mathbb{C}_2$ and an element which does not have an inverse in $\mathbb{C}_2$ is said to be the singular element of $\mathbb{C}_2$.

An element $\xi = z_1 + i_2 z_2 \in \mathbb{C}_2$ is nonsingular if and only if $|z_1^2 + z_2^2| \neq 0$ and singular if and only if $|z_1^2 + z_2^2| = 0$. The inverse of ξ is defined as

$$\xi^{-1} = \eta = \frac{z_1 - i_2 z_2}{z_1^2 + z_2^2}.$$

Zero is the only element in $\mathbb{C}_0$ which does not have multiplicative inverse and in $\mathbb{C}_1, 0 = 0 + i0$ is the only element which does not have multiplicative inverse. We denote the set of singular elements of $\mathbb{C}_0$ and $\mathbb{C}_1$ by O_0 and O_1 respectively. But there are more than one element in $\mathbb{C}_2$ which do not have multiplicative inverse; we denote this set by O_2 and clearly $O_0 = O_1 \subset O_2$.

A bicomplex number $\xi = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2 \in \mathbb{C}_2$ is said to be degenerated if the matrix $\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ is degenerated. In that case ξ^{-1} exists and it is also degenerated.

The norm $\|\cdot\|$ of $\mathbb{C}_2$ is a positive real valued function and $\|\cdot\| : \mathbb{C}_2 \to \mathbb{C}_0^+$ is defined by

$$\begin{aligned} \|\xi\| &= \|z_1 + i_2 z_2\| = \left\{|z_1|^2 + |z_2|^2\right\}^{\frac{1}{2}} \\ &= \left[\frac{|(z_1 - i_1 z_2)|^2 + |(z_1 + i_1 z_2)|^2}{2}\right]^{\frac{1}{2}} = \left(a_1^2 + a_2^2 + a_3^2 + a_4^2\right)^{\frac{1}{2}}, \end{aligned}$$

where $\xi = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2 = z_1 + i_2 z_2 \in \mathbb{C}_2$.

The linear space $\mathbb{C}_2$ with respect to defined norm is a norm linear space, also $\mathbb{C}_2$ is complete; therefore $\mathbb{C}_2$ is the Banach space. If $\xi, \eta \in \mathbb{C}_2$ then $\|\xi\eta\| \leq \sqrt{2}\,\|\xi\|\,\|\eta\|$ holds instead of $\|\xi\eta\| \leq \|\xi\|\,\|\eta\|$, therefore $\mathbb{C}_2$ is not the Banach algebra.

The partial order relation $\precsim_{i_2}$ on $\mathbb{C}_2$ is defined as:

Let $\mathbb{C}_2$ be the set of bicomplex numbers and $\xi = z_1 + i_2 z_2, \eta = w_1 + i_2 w_2 \in \mathbb{C}_2$ then $\xi \precsim_{i_2} \eta$ if and only if $z_1 \precsim w_1$ and $z_2 \precsim w_2$,

i.e., $\xi \precsim_{i_2} \eta$ if one of the following conditions is satisfied:

$$\begin{aligned} (1) \quad z_1 &= w_1,\ z_2 = w_2, \\ (2) \quad z_1 &\prec w_1,\ z_2 = w_2, \\ (3) \quad z_1 &= w_1,\ z_2 \prec w_2 \text{ and} \\ (4) \quad z_1 &\prec w_1,\ z_2 \prec w_2. \end{aligned}$$

In particular we can write $\xi \lnsim_{i_2} \eta$ if $\xi \precsim_{i_2} \eta$ and $\xi \neq \eta$ i.e. one of (2), (3) and (4) is satisfied and we will write $\xi \prec_{i_2} \eta$ if only (4) is satisfied.

For any two bicomplex numbers $\xi, \eta \in \mathbb{C}_2$ we can verify the followings:

(i) $\xi \precsim_{i_2} \eta \Rightarrow \|\xi\| \leq \|\eta\|$,

(ii) $\|\xi + \eta\| \leq \|\xi\| + \|\eta\|$,

(iii) $\|a\xi\| = a\,\|\xi\|$, where a is a non negative real number,

(iv) $\|\xi\eta\| \leq \sqrt{2}\,\|\xi\|\,\|\eta\|$ and the equality holds only when at least one of ξ and η is degenerated,

(v) $\|\xi^{-1}\| = \|\xi\|^{-1}$ if ξ is a degenerated bicomplex number with $0 \prec \xi$,

(vi) $\left\|\frac{\xi}{\eta}\right\| = \frac{\|\xi\|}{\|\eta\|}$, if η is a degenerated bicomplex number.

2. Bicomplex valued metric space

In this section we prove two lemmas on bicomplex valued metric spaces which will be needed in the sequel. Choi et al. [11] defined the bicomplex valued metric space as;

Definition 2.1. *[11] Let X be a nonempty set. Suppose the mapping $d : X \times X \to \mathbb{C}_2$ satisfies the following conditions:*

1. $0 \precsim_{i_2} d(x, y)$ for all $x, y \in X$,

2. $d(x,y)=0$ if and only if $x=y$,
3. $d(x,y)=d(y,x)$ for all $x,y\in X$ and
4. $d(x,y)\precsim_{i_2} d(x,z)+d(z,y)$ for all $x,y,z\in X$.
Then (X,d) is called the bicomplex valued metric space.

Definition 2.2. *[11] For a bicomplex valued metric space (X,d)*
(i). A sequence $\{x_n\}$ in X is said to be a convergent sequence and converges to a point x if for any $0\prec_{i_2} r\in\mathbb{C}_2$ there is a natural number $n_0\in\mathbb{N}$ such that $d(x_n,x)\prec_{i_2} r$, for all $n>n_0$ and we write $\lim\limits_{n\to\infty} x_n=x$ or $x_n\to x$ as $n\to\infty$.
(ii). A sequence $\{x_n\}$ in X is said to be a Cauchy sequence in (X,d) if for any $0\prec_{i_2} r\in\mathbb{C}_2$ there is a natural number $n_0\in\mathbb{N}$ such that $d(x_n,x_{n+m})\prec_{i_2} r$, for all $m,n\in\mathbb{N}$ and $n>n_0$.
(iii). If every Cauchy sequence in X is convergent in X then (X,d) is said to be a complete bicomplex valued metric space.

Lemma 2.3. *Let (X,d) be a bicomplex valued metric space and $\{x_n\}$ be a sequence in X . Then $\{x_n\}$ is convergent and converges to a point x if and only if $\lim\limits_{n\to\infty}\|d(x_n,x)\|=0$.*

Proof . Let $\{x_n\}$ is a convergent sequence and converges to a point x and let $\epsilon>0$ be any real number. Suppose

$$r=\frac{\epsilon}{2}+i_1\frac{\epsilon}{2}+i_2\frac{\epsilon}{2}+i_1i_2\frac{\epsilon}{2}$$

Then clearly $0\prec_{i_2} r\in\mathbb{C}_2$ and for this r there is a natural number $n_0\in\mathbb{N}$ such that $d(x_n,x)\prec_{i_2} r$ for all $n>n_0$

Therefore,

$$\|d(x_n,x)\|<\|r\|=\epsilon,\quad \forall\ n>n_0.$$

And this implies,

$$\lim_{n\to\infty}\|d(x_n,x)\|=0$$

Conversely let $\lim\limits_{n\to\infty}\|d(x_n,x)\|=0$.Then for $0\prec_{i_2} r\in\mathbb{C}_2$, there exists a real $\epsilon>0$, such that for all $\xi\in\mathbb{C}_2$

$$\|\xi\|<\epsilon\Rightarrow\xi\prec_{i_2} r.$$

Then for this $\epsilon>0$, there exists a natural number $n_0\in\mathbb{N}$ such that

$$\|d(x_n,x)\|<\epsilon,\quad\forall\ n>n_0.$$

Therefore,

$$d(x_n,x)\prec_{i_2} r,\quad\forall\ n>n_0.$$

Hence $\{x_n\}$ converges to a point x. □

Lemma 2.4. *Let (X,d) be a bicomplex valued metric space and $\{x_n\}$ be a sequence in X . Then $\{x_n\}$ is a Cauchy sequence in X if and only if $\lim\limits_{n\to\infty}\|d(x_n,x_{n+m})\|=0$.*

Proof . Let $\{x_n\}$ is a Cauchy sequence in X and let $\epsilon>0$ be any real number. Suppose

$$r=\frac{\epsilon}{2}+i_1\frac{\epsilon}{2}+i_2\frac{\epsilon}{2}+i_1i_2\frac{\epsilon}{2}.$$

Then clearly $0 \prec_{i_2} r \in \mathbb{C}_2$ and for this r there is a natural number $n_0 \in \mathbb{N}$ such that $d(x_n, x_{n+m}) \prec_{i_2} r$ for all $n > n_0$.

Therefore,

$$\|d(x_n, x_{n+m})\| < \|r\| = \epsilon, \quad \forall\ n > n_0.$$

And this implies,

$$\lim_{n\to\infty} \|d(x_n, x_{n+m})\| = 0$$

Conversely let $\lim_{n\to\infty} \|d(x_n, x_{n+m})\| = 0$.Then for $0 \prec_{i_2} r \in \mathbb{C}_2$, there exists a real $\epsilon > 0$, such that for all $\xi \in \mathbb{C}_2$

$$\|\xi\| < \epsilon \Rightarrow \xi \prec_{i_2} r.$$

Then for this $\epsilon > 0$, there exists a natural number $n_0 \in \mathbb{N}$ such that

$$\|d(x_n, x_{n+m})\| < \epsilon, \quad \forall\ n > n_0.$$

Therefore,

$$d(x_n, x_{n+m}) \prec_{i_2} r, \quad \forall\ n > n_0.$$

Hence $\{x_n\}$ is a Cauchy sequence. □

3. Main results

In this section we prove fixed point theorems on bicomplex valued metric spaces.

Theorem 3.1. *Let (X, d) be a complete bicomplex valued metric space with degenerated $1 + d\,(x, y)$ and $\|1 + d\,(x, y)\| \neq 0$ for all $x, y \in X$ and let $S, T : X \to X$ be mappings satisfying the condition*

$$d\,(Sx, Ty) \precsim_{i_2} ad\,(x, y) + \frac{bd\,(x, Sx)\, d\,(y, Ty)}{1 + d\,(x, y)} \tag{3.1}$$

for all $x, y \in X$, where a, b are non-negative real numbers with $a + \sqrt{2}b < 1$. Then S, T have a unique common fixed point.

Proof . Let x_0 be an arbitrary point in X. We construct a sequence $\{x_n\}$ such that

$$x_{2k+1} = Sx_{2k,} \quad x_{2k+2} = Tx_{2k+1,} \quad k = 0, 1, 2, ...$$

Then we have

$$\begin{aligned} d\,(x_{2k+1}, x_{2k+2}) &= d\,(Sx_{2k}, Tx_{2k+1}) \\ &\precsim_{i_2} ad\,(x_{2k}, x_{2k+1}) + \frac{bd\,(x_{2k}, Sx_{2k})\, d\,(x_{2k+1}, Tx_{2k+1})}{1 + d\,(x_{2k}, x_{2k+1})} \\ &\precsim_{i_2} ad\,(x_{2k}, x_{2k+1}) + \frac{bd\,(x_{2k}, x_{2k+1})\, d\,(x_{2k+1}, x_{2k+2})}{1 + d\,(x_{2k}, x_{2k+1})} \end{aligned}$$

Therefore,

$$\begin{aligned} \|d\,(x_{2k+1}, x_{2k+2})\| &\leq a\,\|d\,(x_{2k}, x_{2k+1})\| \\ &\quad + \sqrt{2}b \frac{\|d\,(x_{2k}, x_{2k+1})\|}{\|1 + d\,(x_{2k}, x_{2k+1})\|}\, \|d\,(x_{2k+1}, x_{2k+2})\| \end{aligned}$$

Also $\|d(x_{2k},x_{2k+1})\| \leq \|1+d(x_{2k},x_{2k+1})\|$.
Thus

$$\begin{aligned} \|d(x_{2k+1},x_{2k+2})\| &\leq a\|d(x_{2k},x_{2k+1})\| + \sqrt{2}b\|d(x_{2k+1},x_{2k+2})\| \\ i.e., \ \left(1-\sqrt{2}b\right)\|d(x_{2k+1},x_{2k+2})\| &\leq a\|d(x_{2k},x_{2k+1})\| \\ i.e., \ \|d(x_{2k+1},x_{2k+2})\| &\leq \frac{a}{(1-\sqrt{2}b)}\|d(x_{2k},x_{2k+1})\| \end{aligned}$$

Similarly,

$$\begin{aligned} d(x_{2k+2},x_{2k+3}) &= d(Tx_{2k+1},Sx_{2k+2}) = d(Sx_{2k+2},Tx_{2k+1}) \\ &\precsim_{i_2} ad(x_{2k+2},x_{2k+1}) + \frac{bd(x_{2k+2},Sx_{2k+2})\,d(x_{2k+1},Tx_{2k+1})}{1+d(x_{2k+2},x_{2k+1})} \\ &\precsim_{i_2} ad(x_{2k+2},x_{2k+1}) + \frac{bd(x_{2k+2},x_{2k+3})\,d(x_{2k+1},x_{2k+2})}{1+d(x_{2k+2},x_{2k+1})} \end{aligned}$$

Therefore we obtain that

$$\begin{aligned} \|d(x_{2k+2},x_{2k+3})\| &\leq a\|d(x_{2k+2},x_{2k+1})\| \\ &\quad +\sqrt{2}b\frac{\|d(x_{2k+1},x_{2k+2})\|}{\|1+d(x_{2k+1},x_{2k+2})\|}\|d(x_{2k+2},x_{2k+3})\| \end{aligned}$$

$$i.e., \|d(x_{2k+2},x_{2k+3})\| \leq a\|d(x_{2k+2},x_{2k+1})\| + \sqrt{2}b\|d(x_{2k+2},x_{2k+3})\|,$$
$$\text{as } \|d(x_{2k+1},x_{2k+2})\| \leq \|1+d(x_{2k+1},x_{2k+2})\|$$

$$\begin{aligned} i.e., \ \left(1-\sqrt{2}b\right)\|d(x_{2k+2},x_{2k+3})\| &\leq a\|d(x_{2k+2},x_{2k+3})\| \\ i.e., \ \|d(x_{2k+2},x_{2k+3})\| &\leq \frac{a}{(1-\sqrt{2}b)}\|d(x_{2k+2},x_{2k+3})\| \end{aligned}$$

Suppose $\alpha = \frac{a}{1-\sqrt{2}b}$. Then $0 \leq \alpha < 1$ and

$$\begin{aligned} \|d(x_{n+1},x_{n+2})\| &\leq \alpha\|d(x_n,x_{n+1})\| \\ &\leq \alpha^2\|d(x_{n-1},x_n)\| ..... \leq \alpha^{n+1}\|d(x_0,x_1)\|. \end{aligned}$$

Also for any two positive integers m,n with $m>n$ we get that

$$d(x_n,x_m) \precsim_{i_2} d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + ... + d(x_{m-1},x_m)$$

Therefore,

$$\begin{aligned} \|d(x_n,x_m)\| &\leq \|d(x_n,x_{n+1})\| + \|d(x_{n+1},x_{n+2})\| + ... + \|d(x_{m-1},x_m)\| \\ &\leq \left[\alpha^n + \alpha^{n+1} + ... + \alpha^{m-1}\right]\|d(x_0,x_1)\| \\ i.e., \ \|d(x_n,x_m)\| &\leq \alpha^n\left[1+\alpha+\alpha^2 ...... + \alpha^{m-n-1}\right]\|d(x_0,x_1)\| \end{aligned}$$

Since, $0 \leq \alpha < 1$, then $1+\alpha+\alpha^2 ...... + \alpha^{m-n-1} \leq \frac{1}{1-\alpha}$.
Hence

$$\|d(x_n,x_m)\| \leq \frac{\alpha^n}{1-\alpha}\|d(x_0,x_1)\|$$

Again since $\frac{\alpha^n}{1-\alpha} \longrightarrow 0$ as $n \longrightarrow \infty$. then for any $\varepsilon > 0$ there exists a positive integer n_0 such $\|d(x_n, x_m)\| < \varepsilon$ for all $m, n > n_0$. Hence $\{x_n\}$ is a Cauchy sequence in X. Also X is a complete bicomplex valued metric space. Then there exists $u \in X$ such that $\lim_{n \longrightarrow \infty} x_n = u$.

Now we show that $u = Su$. If not then there exists an $0 \prec_{i_2} \xi \in \mathbb{C}_2$ such that $d(u, Su) = \xi$. Therefore we have

$$\begin{aligned}
\xi &= d(u, Su) \\
&\precsim_{i_2} d(u, x_{2k+2}) + d(x_{2k+2}, Su) \\
&\precsim_{i_2} d(u, x_{2k+2}) + d(Tx_{2k+1}, Su) \\
&\precsim_{i_2} d(u, x_{2k+2}) + ad(x_{2k+1}, u) + \frac{bd(x_{2k+1}, Tx_{2k+1})\, d(u, Su)}{1 + d(x_{2k+1}, u)} \\
i.e.,\ \xi &\precsim_{i_2} d(u, x_{2k+2}) + ad(x_{2k+1}, u) + \frac{bd(x_{2k+1}, x_{2k+2})\, \xi}{1 + d(x_{2k+1}, u)}
\end{aligned}$$

Hence

$$\|\xi\| \leq \|d(u, x_{2k+2})\| + a\|d(x_{2k+1}, u)\| + \sqrt{2}\frac{b\|d(x_{2k+1}, x_{2k+2})\|\,\|\xi\|}{\|1 + d(x_{2k+1}, u)\|}.$$

Since $\lim_{n \longrightarrow \infty} x_n = u$, taking limit on both sides as $n \longrightarrow \infty$ we get that $\|\xi\| \leq 0$, which is a contradiction. Therefore $\|\xi\| = 0 \Rightarrow \|d(u, Su)\| = 0 \Rightarrow u = Su$. Similarly, we can show that $u = Tu$.

Hence S and T have a common fixed point.

Now we show that S and T have unique common fixed point. If possible suppose $u^* \in X$ be another common fixed point of S and T.

Then

$$d(u, u^*) = d(Su, Tu^*) \precsim_{i_2} ad(u, u^*) + \frac{bd(u, Su)\, d(u^*, Tu^*)}{1 + d(u, u^*)}$$

$$\begin{aligned}
i.e.,\ \|d(u, u^*)\| &\leq a\|d(u, u^*)\| + \sqrt{2}\frac{b\|d(u, Su)\|\,\|d(u^*, Tu^*)\|}{\|1 + d(u, u^*)\|} \\
i.e.,\ \|d(u, u^*)\| &\leq a\|d(u, u^*)\| \\
i.e.,\ \|d(u, u^*)\| &= 0 \\
i.e.,\ u &= u^*.
\end{aligned}$$

This completes the proof of the theorem. □

Corollary 3.2. *Let (X, d) be a complete bicomplex valued metric space with degenerated $1 + d(x, y)$ and $\|1 + d(x, y)\| \neq 0$ for all $x, y \in X$ and $S : X \to X$ be any mapping satisfying the condition*

$$d(Sx, Sy) \precsim_{i_2} ad(x, y) + \frac{bd(x, Sx)\, d(y, Sy)}{1 + d(x, y)} \tag{3.2}$$

for all $x, y \in X$, where a, b are non-negative real numbers with $a + \sqrt{2}b < 1$. Then S has a unique fixed point.

Proof . We can easily prove this result by applying the Theorem 3.1 and taking $T = S$. □

Corollary 3.3. *Let (X, d) be a complete bicomplex valued metric space with degenerated $1 + d(x, y)$ and $\|1 + d(x, y)\| \neq 0$ for all $x, y \in X$ and let $S : X \to X$ be any mapping satisfying the condition*

$$d(S^n x, S^n y) \precsim_{i_2} ad(x, y) + \frac{bd(x, S^n x)\, d(y, S^n y)}{1 + d(x, y)} \tag{3.3}$$

for all $x, y \in X$, where a, b are non-negative real numbers with $a + \sqrt{2}b < 1$. Then S has a unique fixed point.

Proof . By Corollary 3.2 there exists a unique point $u \in X$ such that

$$\begin{aligned} Su &= u \\ i.e.,\ S^2 u &= Su = u \\ &\quad \ldots\ldots\ldots \\ i.e.,\ S^n u &= u. \end{aligned}$$

Therefore,

$$d(Su, u) = d(SS^n u, S^n u) = d(S^n Su, S^n u) \precsim_{i_2} ad(Su, u) + \frac{bd(Su, S^n Su)\, d(u, S^n u)}{1 + d(Su, u)}$$

$$i.e.,\ d(Su, u) \precsim_{i_2} ad(Su, u) + \frac{bd(Su, S^n Su)\, d(u, u)}{1 + d(Su, u)}$$

$$i.e.,\ d(Su, u) \precsim_{i_2} ad(Su, u)$$

$$\begin{aligned} i.e.,\ \|d(Su, u)\| &\leq a\|d(Su, u)\| \\ i.e.,\ \|d(Su, u)\| &= 0 \\ i.e.,\ u &= u^*. \end{aligned}$$

This completes the proof of the corollary. $\square$

Remark 3.4. *The following example ensures the validity of the Corollary3.2*

Example 3.5. *Consider $X = \left\{0, \frac{1}{2}, 2\right\}$, define a bicomplex valued metric $d : X \times X \to \mathbb{C}_2$ by $d(x, y) = (1 + i_2)|x - y|$, $\forall x, y \in X$.*

From the above definition of d one can easily verify that (X, d) is a bicomplex valued metric space.

Now we consider the mapping $S : X \to X$ defined by

$$S(0) = 0, \quad S\left(\frac{1}{2}\right) = 0 \quad and \quad S(2) = \frac{1}{2}$$

Let $a = \frac{1}{3}$ and $b = \frac{1}{6}$, then clearly $a + \sqrt{2}b = \frac{1}{3} + \sqrt{2}\frac{1}{6} < 1$. Also the condition (3.2) of the Corollary3.2 is satisfied, clearly $x = 0$ is the unique fixed point of S.

Again, notice that $S^n x = 0$, $\forall x \in X$. so that $0 = d(S^n x, S^n y) \precsim_{i_2} ad(x, y) + \frac{bd(x, S^n x) d(y, S^n y)}{1 + d(x, y)}$, therefore $S^n x$ satisfies the condition (3.3) of the Corollary3.3 and clearly $x = 0$ is the unique fixed point of S.

Theorem 3.6. *Let (X,d) be a complete bicomplex valued metric space and let the mappings $S,T : X \to X$ satisfy the condition*

$$d(Sx,Ty) \precsim_{i_2} \frac{a\left[d(x,Sx)\,d(x,Ty)+d(y,Ty)\,d(y,Sx)\right]}{d(x,Ty)+d(y,Sx)}$$

for all $x,y \in X$ and if $\|d(x,Ty)+d(y,Sx)\| \neq 0$ and $d(x,Ty)+d(y,Sx)$ is degenerated, where 'a' is a non-negative real number with $0 \leq a < 1$. Then S,T have a unique common fixed point.

Proof . Let x_0 be an arbitrary point in X. We consider a sequence $\{x_n\}$ in X such that

$$x_{n+1} = Sx_n, \text{ and } x_{n+2} = Tx_{n+1} \text{ for all } n = 0,1,2,.....$$

Then

$$\begin{aligned}
d(x_{n+1},x_{n+2}) &= d(Sx_n,Tx_{n+1}) \\
&\precsim_{i_2} \frac{a\left[d(x_n,Sx_n)\,d(x_n,Tx_{n+1})+d(x_{n+1},Tx_{n+1})\,d(x_{n+1},Sx_n)\right]}{d(x_n,Tx_{n+1})+d(x_{n+1},Sx_n)} \\
&\precsim_{i_2} \frac{a\left[d(x_n,x_{n+1})\,d(x_n,x_{n+2})+d(x_{n+1},x_{n+2})\,d(x_{n+1},x_{n+1})\right]}{d(x_n,x_{n+2})+d(x_{n+1},x_{n+1})} \\
&\precsim_{i_2} \frac{ad(x_n,x_{n+1})\,d(x_n,x_{n+2})}{d(x_n,x_{n+2})} \\
&\precsim_{i_2} ad(x_n,x_{n+1}).
\end{aligned}$$

Therefore for all $n \geq 0$ we get that

$$d(x_{n+1},x_{n+2}) \precsim_{i_2} ad(x_n,x_{n+1}) \precsim_{i_2} a^2 d(x_{n-1},x_n) \precsim_{i_2} .... \precsim_{i_2} a^{n+1} d(x_0,x_1)$$

Thus for any two positive integers m,n with $m > n$ we have

$$\begin{aligned}
d(x_n,x_m) &\precsim_{i_2} d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+.....+d(x_{m-1},x_m) \\
&\precsim_{i_2} a^n d(x_0,x_1)+a^{n+1}d(x_0,x_1)+.......+a^{m-1}d(x_0,x_1) \\
&\precsim_{i_2} \left[a^n+a^{n+1}+......+a^{m-1}\right] d(x_0,x_1) \\
&\precsim_{i_2} a^n\left[1+a+a^2......+a^{m-n-1}\right] d(x_0,x_1) \\
&\precsim_{i_2} \frac{a^n}{1-a} d(x_0,x_1).
\end{aligned}$$

Since $0 \leq \alpha < 1$, then $1+a+a^2......+a^{m-n-1} \leq \frac{1}{1-a}$.
Hence

$$\|d(x_n,x_m)\| \leq \frac{a^n}{1-a}\|d(x_0,x_1)\|.$$

Again since $\frac{a^n}{1-a} \longrightarrow 0$ as $n \longrightarrow \infty$. then for any $\varepsilon > 0$ there exists a positive integer n_0 such $\|d(x_n,x_m)\| < \varepsilon,$ for all $m,n > n_0$. Hence $\{x_n\}$ is a Cauchy sequence in X. Also X is a complete bicomplex valued metric space. Therefore there exists $u \in X$ such that $\lim\limits_{n\longrightarrow\infty} x_n = u$.

Now we show that $u = Su$. If not then there exists $0 \prec_{i_2} \xi \in \mathbb{C}_2$ such that $d(u,Su) = \xi$.

Therefore,

$$\begin{aligned}
\xi &= d(u, Su) \\
&\precsim_{i_2} d(u, x_{n+2}) + d(x_{n+2}, Su) \\
&\precsim_{i_2} d(u, x_{n+2}) + d(Su, Tx_{n+1}) \\
&\precsim_{i_2} d(u, x_{n+2}) + \frac{a\left[d(u, Su)\, d(u, Tx_{n+1}) + d(x_{n+1}, Tx_{n+1})\, d(x_{n+1}, Su)\right]}{d(u, Tx_{n+1}) + d(x_{n+1}, Su)} \\
&\precsim_{i_2} d(u, x_{n+2}) + \frac{a\left[\xi d(u, x_{n+2}) + d(x_{n+1}, x_{n+2})\, d(x_{n+1}, Su)\right]}{d(u, x_{n+2}) + d(x_{n+1}, Su)},
\end{aligned}$$

which yields that

$$\|\xi\| \leq \|d(u, x_{n+2})\| + \sqrt{2}\frac{a\left[\|\xi\|\, \|d(u, x_{n+2})\| + \|d(x_{n+1}, x_{n+2})\|\, \|d(x_{n+1}, Su)\|\right]}{\|d(u, x_{n+2}) + d(x_{n+1}, Su)\|}$$

Taking limit as $n \to \infty$ we get that $\|\xi\| \leq 0$, which is a contradiction. Therefore $\|\xi\| = 0 \Rightarrow \|d(u, Su)\| = 0 \Rightarrow u = Su$. Similarly, we can show that $u = Tu$.

Hence S and T have a common fixed point.

Now we show that S and T have unique common fixed point. If possible suppose that $u^* \in X$ be another common fixed point of S and T.

Then

$$d(u, u^*) = d(Su, Tu^*) \precsim_{i_2} \frac{a\left[d(u, Su)\, d(u, Tu^*) + d(u^*, Tu^*)\, d(u^*, Su)\right]}{d(u, Tu^*) + d(u^*, Su)}$$

$$\begin{aligned}
\|d(u, u^*)\| &\leq \sqrt{2}\frac{a\left[\|d(u, Su)\|\, \|d(u, Tu^*)\| + \|d(u^*, Tu^*)\|\, \|d(u^*, Su)\|\right]}{\|d(u, Tu^*) + d(u^*, Su)\|} \\
i.e.,\ \|d(u, u^*)\| &\leq 0
\end{aligned}$$

which is a contradiction, Therefore,

$$\begin{aligned}
i.e.,\ \|d(u, u^*)\| &= 0 \\
i.e.,\ u &= u^*.
\end{aligned}$$

This completes the proof of the theorem.□

4. Open problem

In this work we started the study of bicomplex valued metric spaces and established fixed point results for a pair of contractive type mappings satisfying a rational inequality in complete bicomplex valued metric spaces. Ran and Reurings [17] generalized Banach's contraction principle to partially ordered metric spaces. Beg and Butt [5] further generalized results of Ran and Reurings [17]. Recently Khalehoghli et al. [13, 14] introduced R-metric spaces and obtained a generalization of Banach fixed point theorem. It is an interesting open problem to study the R-bicomplex valued metric spaces and obtain fixed point results on complete R-bicomplex valued metric spaces. .

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