Int. J. Nonlinear Anal. Appl. 12 (2021) No. 2, 729-741
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2020.19764.2099

# Dynamical analysis, stability and discretization of fractional-order predator-prey model with negative feedback on two species

Nader Biranvand[a], Esmail Babolian[b], Alireza Vahidi[a,*]

[a] *Department of Applied Mathematics, Yadegar-e-Imam Khomeini (RAH), Shahr-e-Rey Branch, Islamic Azad University, Tehran, Iran.*
[b] *Faculty of Mathematical Sciences and Computer, Kharazmi University, Tehran, Iran.*

*(Communicated by Jong Kyu Kim)*

**Abstract**

The Lotka-Volterra model is an important model being employed in biological phenomena to investigate the nonlinear interaction among existing species. In this work, we first consider an integer order predator-prey model with negative feedback on both prey and predator. Then by introducing a fractional model into the existing one, we give them a specified memory. We also obtain its discretized counterpart. Finally, along with giving the biological interpretation of the system, the stability and dynamical analysis of the proposed model are investigated and the results are illustrated as well.

*Keywords:* fractional calculus, predator-prey, Lotka-Volterra, stability, discretization.
*2010 MSC:* Primary 26A33; Secondary 34A08.

## 1. Introduction

Fractional calculus is a branch of mathematics which is the concept of convolution integral. In recent decades, fractional-order differentials are widely used in many research areas owing to its application in modeling and investigating complicated phenomena, such as neural network systems, viscoelastic systems, kinetic equations, dynamical and nonlinear systems, financial systems, and so on. This powerful mathematical approach provides researchers with the opportunity to derive further

*Corresponding author
*Email addresses:* nabiranvand@gmail.com (Nader Biranvand), babolian@khu.ac.ir (Esmail Babolian), Alrevahidi@yahoo.com (Alireza Vahidi)

*Received:* February 2020 *Accepted:* May 2020

and precise information about complex natural systems showing a nonlinear behavior of themselves compared to the ordinary calculus [29, 31, 4, 6, 10, 17, 32, 26].

Working with nonlinear systems is not always easy since their output might appear as a bifurcation and chaos system. For example, fractional-order Van der Pol oscillator, proposed in 1920 and several years later utilized to investigate biological systems such as heartbeat, firing process in neuron cells, and acoustic systems, showed a chaotic behavior [7, 8, 3]. Also, fractional-order Lorenz system obtained from Gory equations presents a chaotic behavior over time [21]. Likewise, some systems are having chaotic behavior e.g., fractional-order Chua's oscillator [33, 34], and fractional Duffing oscillator [20]. Chen's system [42], Lus system [12] Liu's system [9], Genesio-Tesiss system [23], Newton Leipnik system [36], financial system [24], Volta's system [33] and Lotka-Volterra system [37].

In 2002, Elaydi and Yakubu in [15] proved that in connected metric spaces $n$-cycles are not globally attracting (where $n \geq 2$). They applied this result to a two-species discrete-time Lotka-Volterra competition model with stocking. In particular, they showed that an $n$-cycle cannot be the ultimate life-history of the evolution of all population sizes. This solved Yakubu's conjecture. In 2007 and 2010, some dynamical behavior of the fractional-order Liu's system was investigated with the help of Routh-Hurwitz criteria and in [27], it was shown that the system with order less than 3 has a chaotic behavior. Also, to control chaos, a linear control technique was employed for a numerical visualization. In [28], by using Hsu and Kazarinoff theorem, periodic solutions and their stabilities about the equilibrium points were investigated. Furthermore, to stabilize the chaotic Lie system, linear feedback control techniques were employed, see e.g., [5] for more details.

From 2006 to 2007, a fractional-order logistic model, a fractional-order differential equation model for nonlocal epidemics and one fractional-order predator-prey and rabies model were proposed, and stability, existence, uniqueness and numerical solution of them were investigated [1, 2, 18].

Analytical approximate solutions and a predictor-corrector approach for the numerical solution of the fractional differential equation were investigated in [13, 39].

Investigation of how species in the natural world are connected and interacted is of great importance to keep species from extinction. So, this subject of research area has turned into an interesting topic for scholars to dig precisely into it and find a better understanding of it. One of the well-known mathematical models employed to investigate the dynamical behavior of biological systems is the Lotka-Volterra model. This model was independently proposed by Alfred Lotka in 1925 and Vito Volterra in 1926 [14].

In 2014, a predator-prey Lotka-Volterra equation with logistic behavior for two species was investigated [35]. Also, dynamical behaviors of predator-prey Lotka-Volterra with discretization was probed [17].

In this study, a new fractional model considering negative feedback on two species is proposed as follows and the biological interpretation of it is also presented.

$$\begin{aligned} \tfrac{dN(T)}{dT} &= N(T)\left(a - bN(T) - cP(T)\right), \\ \tfrac{dP(T)}{dT} &= P(T)\left(-f + ceN(T)\right) - kP^2(T), \end{aligned} \tag{1.1}$$

It is also worth mentioning that the same non-fractional model was numerically studied in [40]. Hence, we can say that the model proposed in this paper is an extension of the models considered in [17, 35].

In [41], biological control model of chaotic property on Lotka-Volterra with three species was extended and the stability concept of the model was also investigated by the Lyapunov theorem.

In 2017, dynamical analysis of linear feedback control and synchronization of an extended Lotka-Volterra model was studied and stability of the system, equilibrium points, existence, and uniqueness of Hopf bifurcation and simulation were also considered [16].

In 2017, the prediction of internet users in China with four models, Logistic, Bass, and Gompertz, the Lotka- Volterra was investigated. As a result, it was shown that Lotka- Volterra model can predict the system better than the other considered models [19].

A fractional B-spline collocation method has been recently proposed for fractional-order derivative. To discuss the efficiency of the proposed model, one predator-prey Lotka- Volterra model with variable coefficients was considered, and it was shown numerically that the method has not only high precision in the calculation but also has low calculation cost [30].

Also, the dynamical chaotic behavior of a discrete predator-prey with refuge [22], along with basic analysis of a fractional-order predator-prey in connection with prey refuge [11], and a fractional predator-prey with prey refuge and additional food for predators were numerically solved [38]. For more details, see [25]

In this paper, we first present some basic definitions of fractional calculus. Then, after proposing an integer model for predator-prey with negative feedbacks on both species, we fractionalize the model with two different orders. In the next step, the discretized counterpart of the model and the stability condition for the proposed system are also obtained. In the last section, the numerical results and the biological interpretation of the proposed model are given.

## 2. Basic Definition of Fractional Calculus

In this section, some basic definitions in fractional calculus are presented that are used later in the next sections[27].

***Fractional Caputo derivative:*** Let $f \in C^n[a,b]$ and $n-1<\alpha<n$, then the Caputo derivative is defined as follows:

$$ {}_a^C D_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha-n)} \int_a^t \frac{D^n f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau. \tag{2.1}$$

***Riemann-Liouville fractional integral:*** Let $\alpha \in R_+$. The operator $D_a^{-n}$ defined on $L_1[a,b]$ by

$$ D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau. \tag{2.2}$$

for $a<t<b$, is called the Riemann-Liouville fractional integral operator of order $\alpha$.

## 3. Fractional continuous Lotka-Volterra with negative feedback on two species

To investigate species interaction, quite a few models have been proposed so far. Although each model has its application and could be used for special purposes, they are not always capable of predicting species connection and behavior well enough. The reasons why such matters come up could be the system complexity and simplification of the model. Accordingly, in this section, a new fractional model with negative feedback on both prey and predator is proposed as equation (1.1).

In this model, we have

1. $N$ and $P$ are the population density of prey and predator, respectively.
2. $a$ and $f$ are the rate of growth for prey and predator, respectively.
3. $b$ and $k$ are the rates of death, caused by the interaction between two different species.

4. Preys would be injured when they are faced with predators, giving rise to an inclination of their population. Regarding that, a coefficient like $c$ is donated to the death rate of prey due to their interaction with predators. However, all preys faced with predators are not always haunted, and some of them would be able to escape and remain alive. Thus, a coefficient like $e$ is considered for preys that could not escape when they interact with predators. So the multiplication of these two parameters $(ce)$ represents feeding rate of cooperation attack which increases the predator's population density.

By considering the following parameters, the system (1.1) will be nondimensionalized as follows.

$$T = \frac{1}{a}t, \qquad N(T) = \frac{a}{b}X(t), \qquad P(T) = \frac{a}{c}Y(t),$$

$$\begin{aligned} \tfrac{dX}{dt} &= X(t)\left(1 - X(t) - Y(t)\right), \\ \tfrac{dY}{dt} &= Y(t)\left(-\beta - hY(t) + \varepsilon X(t)\right), \end{aligned} \tag{3.1}$$

where $\beta = \frac{f}{a}, \varepsilon = \frac{ce}{b}$ and $h = \frac{k}{c}$.

Since system (3.1) investigates the species behavior locally, we wish to extend the model in the next step by fractionalizing the system. Because the whole information of the system regarding species interaction and behavior are saved with a weight function $\omega(t) = t^{(\alpha-1)}/\Gamma(\alpha)$ over time, Eq. (3.2) indeed looks at the system from an extensive spectrum compared to Eq. (3.1). Thus,

$$\begin{aligned} D^\alpha X(t) &= X(t)\left(1 - X(t) - Y(t)\right), \\ D^\alpha Y(t) &= Y(t)\left(-\beta - hY(t) + \varepsilon X(t)\right), \end{aligned} \tag{3.2}$$

where $\alpha \in (0, 1]$ and $t > 0$.

## 4. Discretization of fractional Lotka-Volterra predator-prey model

In this section, we focus on the discretization of fractional Lotka- Volterra predator-prey model with negative feedback, Eq. (3.2). By considering uniform discretization with time steps, tis equation can be rewritten as follows.

$$\begin{aligned} D^\alpha X(t) &= X\left([\mathrm{t}/\mathrm{s}]s\right)\left[\left(1 - X\left([\mathrm{t}/\mathrm{s}]s\right)\right) - Y\left([\mathrm{t}/\mathrm{s}]s\right)\right], \\ D^\alpha Y(t) &= Y\left([\mathrm{t}/\mathrm{s}]s\right)\left[-\beta - hY\left([\mathrm{t}/\mathrm{s}]s\right) + \varepsilon X\left([\mathrm{t}/\mathrm{s}]s\right)\right]. \end{aligned} \tag{4.1}$$

Assume that $X(0) = 0, Y(0) = 0$, and $t \in [0, s)$, so $t/s \in [0, 1)$. Thus, we have

$$\begin{aligned} D^\alpha X_1 &= X_0\left[1 - X_0 - Y_0\right], \\ D^\alpha Y_1 &= Y_0\left[-\beta - hY_0 + \varepsilon X_0\right]. \end{aligned} \tag{4.2}$$

So, by the help of Riemann-Liouville integral, Eq. (4.2) is rewritten as follows:

$$\begin{aligned} X_1(t) &= X_0 + \tfrac{t^\alpha}{\Gamma(\alpha+1)}\left(X_0\left[1 - X_0 - Y_0\right]\right), \\ Y_1(t) &= Y_0 + \tfrac{t^\alpha}{\Gamma(\alpha+1)}\left(Y_0\left[-\beta - hY_0 + \varepsilon X_0\right]\right). \end{aligned} \tag{4.3}$$

As a result, after $n$ steps, Eq. (4.4) will be obtained.

$$\begin{aligned} X_{n+1}(t) &= X_n(ns) + \tfrac{(t-ns)^\alpha}{\Gamma(\alpha+1)}\left(X_n(ns)\left[1 - X_n(ns) - Y_n(ns)\right]\right), \\ Y_{n+1}(t) &= Y_n(ns) + \tfrac{(t-ns)^\alpha}{\Gamma(\alpha+1)}\left(Y_n(ns)\left[-\beta - hY_n(ns) + \varepsilon X_n(ns)\right]\right). \end{aligned} \tag{4.4}$$

Also, let $t \in [ns, (n+1)s)$ when $t \longrightarrow (n+1)s$. So we have

$$\begin{aligned} X_{n+1} &= X_n + \tfrac{s^\alpha}{\Gamma(\alpha+1)} \left( X_n \left[ 1 - X_n - Y_n \right] \right), \\ Y_{n+1} &= Y_n + \tfrac{s^\alpha}{\Gamma(\alpha+1)} \left( Y_n \left[ -\beta - hY_n + \varepsilon X_n \right] \right). \end{aligned} \tag{4.5}$$

It should be noted that in the limiting case $\alpha \to 1$, the method is reduced to the Euler discretization method. The bifurcation diagrams of the above equation (4.5) for different values of $\alpha$ and $\beta$ are depicted in Fig 1. In fact, this figure illustrates the stability of the system in discretized case. It is seen that the lower $s$, more frequencies the solution has. The initial state of the system (4.5) is (0.25,0.2).

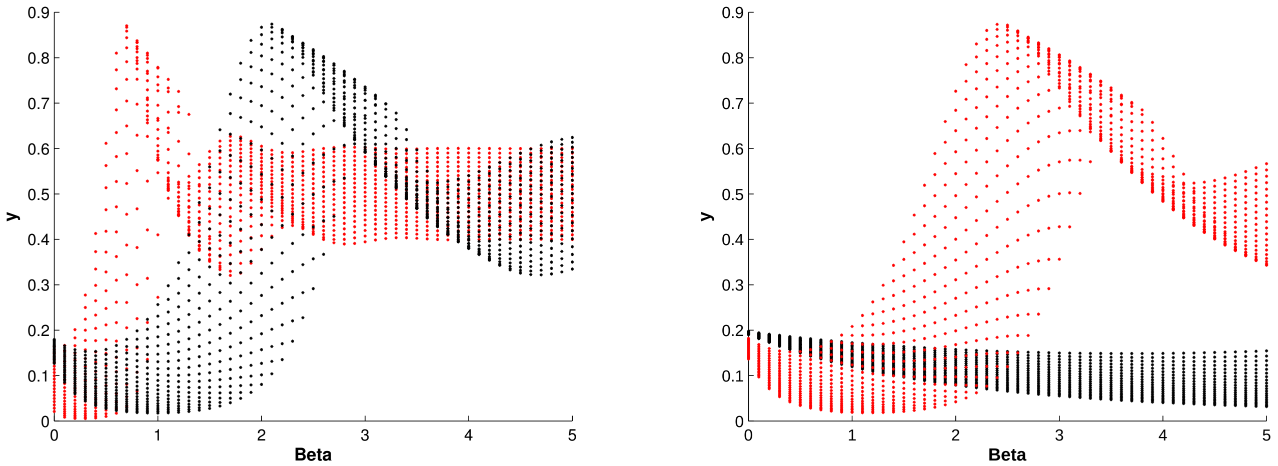


Figure 1: The bifurcation diagrams of the equation (4.5). The back and red points are refered to $\alpha = 0.9$ and 0.5, respectively. Also, in the left diagram $s = 0.1$ and in the right diagram, this value is 0.01.

## 5. Investigation of dynamical behavior of discrete and continuous fractional Lotka-Volterra predator-prey model

In this section, the dynamical behavior of fractional Lotka-Volterra predator-prey model, associated with four parameters $\beta, \alpha, \varepsilon$ and $h$, is investigated.
Consider non-linear fractional system (3.2). With $y_i$ as initial point ($i = 1, 2,..., n$).

$${}_0D_t^{\alpha_i} x_i(t) = f_i\left(x_1(t), x_2(t), ..., x_n(t), t\right), \qquad x_i(t) = y_i, \quad i = 1, 2, ..., n. \tag{5.1}$$

By rewriting the above equation, we will have [17]

$$D^\alpha x = f(x). \tag{5.2}$$

for $i = 1, 2,..., n, 0 < \alpha_i < 1, \alpha = [\alpha_1, \alpha_2, \cdots, \alpha_n]^T$ and $x \in R^n$. $E^* = (x_1^*, x_2^*, \cdots, x_n^*)$ is fixed point if and only if

$$f(x) = 0. \tag{5.3}$$

Also, Jacobian matrix of the system (3.2) is

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}. \tag{5.4}$$

**Theorem 1** [17]: When the following condition of the Jacobian matrix (5.4) at the fixed point $(x^*, y^*)$ is held, the fixed point will be stable.

$$Det(J) > 0, \quad trace(J) < 0.$$

**Lemma 1**[17]: Assume that $\lambda_1$ and $\lambda_2$ are eigenvalues of Jacobian matrix (5.4) in fixed point $(x^*, y^*)$, then

1. A fixed point $(x^*, y^*)$ is named a sink if $|\lambda_1| < 1$ and $|\lambda_2| < 1$. So the sink is locally asymptotically stable.
2. A fixed point $(x^*, y^*)$ is named a source if $|\lambda_1| > 1$ and $|\lambda_2| > 1$. So the source is unstable.
3. A fixed point $(x^*, y^*)$ is named a saddle if $|\lambda_1| < 1$ and $|\lambda_2| > 1$ or $|\lambda_1| > 1$ and $|\lambda_2| < 1$.
4. A fixed point $(x^*, y^*)$ is named a non-hyperbolic if either $|\lambda_1| = 1$ or $|\lambda_2| = 1$.

**Lemma 2**[17]: fixed points in $\alpha = \alpha_1 = \alpha_2 = ... = \alpha_n$ is asymptotically stable if Jacobian matrix (5.4) for all eigenvalues $\lambda_i (i = 1, 2, ..., n)$ in each point of the stability point $E^*$ satisfies the following equation.

$$|\arg(eig(i))| = |\arg(\lambda_i)| > \frac{\alpha\pi}{2}, \quad i = 1, 2, ..., n. \tag{5.5}$$

Based on the above Lemma, stability and unstability region in the complex surface are depicted in Fig 2.

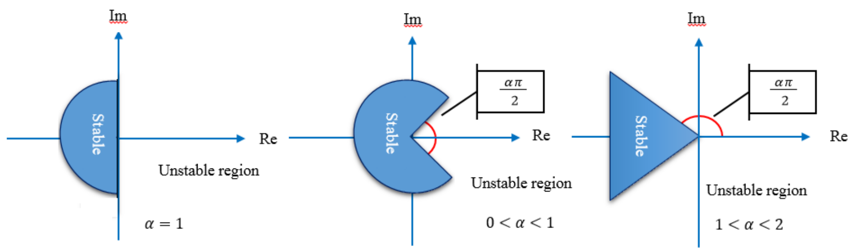


Figure 2: Stability and unstability region of the linearized fractional-order system.

## 6. Stability of the continuous model

Assume that $E^* = (x^*, y^*)$ is the fixed point of Eq. (3.1). Based on Jacobian matrix, we have

$$J(x^*, y^*) = \begin{pmatrix} 1 - 2x^* - y^* & -x^* \\ \varepsilon y^* & -\beta - 2hy^* + \varepsilon x^* \end{pmatrix}. \tag{6.1}$$

Also, it can be shown that the following points are the fixed points of system (3.1).

$$E_1 = (0, 0), \quad E_2 = (1, 0), \quad E_3 = (\frac{h + \beta}{h + \varepsilon}, \frac{\varepsilon - \beta}{h + \varepsilon}).$$

**Proposition:** The fixed points of system (3.2), $E_1, E_2$, and $E_3$, have the following properties:

i. $E_1$ is an unstable point.

ii. The sufficient and efficient situation for the stability of point $E_2$ is $\varepsilon < \beta$. Otherwise, $E_2$ will be a non-hyperbolic point.

iii. $E_3$ is a stable point when $\beta < \varepsilon$ and $h \leq \beta$.

**Proof:**

**i**. By putting $E_1$ in Eq. (6.1), the eigenvalues 1 and $-\beta$ will be obtained, which means that the considered system (3.2) is always unstable in $E_1$.

**ii.** The eigenvalues of the matrix (6.1) at $E_2$ are (-1) and $\varepsilon - \beta$. Thus, if $\varepsilon < \beta$, then the system (3.2) will be stable. On the other hand, if $\varepsilon > \beta$, then based on Lemma 1, $E_2$ will be a non-hyperbolic point.

**iii.** By putting $E_3$ in the matrix (6.1), we have

$$J(E_3) = \begin{pmatrix} \frac{-\beta-h}{\varepsilon+h} & -\frac{\beta+h}{\varepsilon+h} \\ \frac{\varepsilon(\varepsilon-\beta)}{(\varepsilon+h)} & \frac{2h(\beta-\varepsilon)}{\varepsilon+h} \end{pmatrix}, \tag{6.2}$$

thus $trace(J(E_3)) < 0$ if and only if $\beta < \varepsilon$. So it can be quickly deduced that $Det(J(E_3)) > 0$ when $h \leq \beta$. Hence, under the obtained conditions and theorem 1, the point $E_3$ is stable. This is supported for some numerical parameters given in the caption of in Fig. 3 lading to the point $E_3 = (0.2, 0.8)$ which is the equilibrium as seen in the right figure.

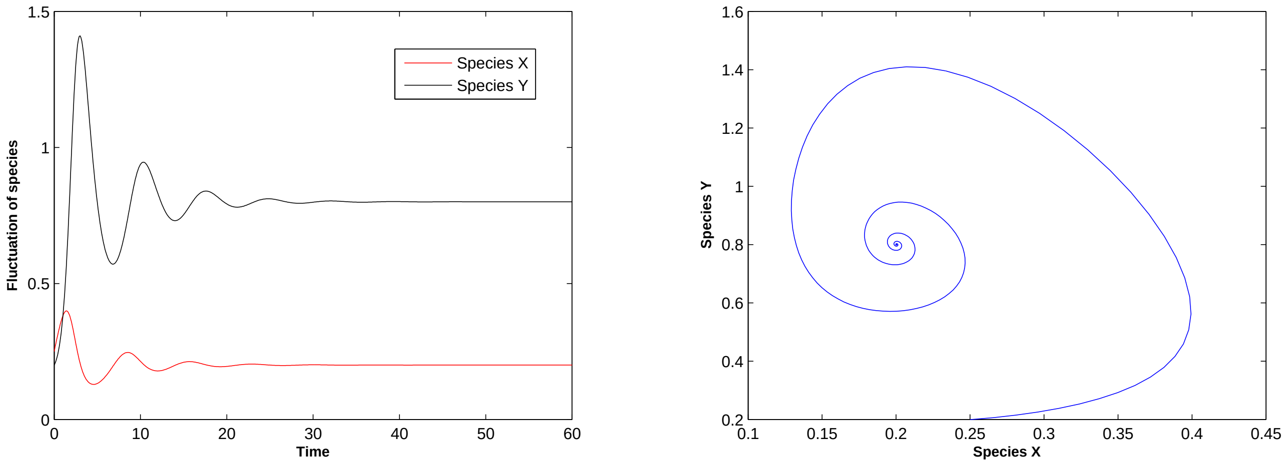


Figure 3: Stability analysis of the system (3.2) at fixed point $E_3$ with initial states (0.25,0.2) and also $\beta = 1, \varepsilon = 5, h = 0$ and $\alpha = 0.95$.

From the proposition in section 6, it can be quickly deduced that the negative feedback results in the stability of the system (3.2). Indeed, as it is depicted in Fig 4, if $\varepsilon = 0$, then the system will be unstable. Moreover, as the negative feedback on predators increases, preys will be stabilized sooner compared to the predators.

## 7. Stability of the discontinuous model

In this section, the stability condition of the discontinuous model is investigated.

**Proposition:** $E_1, E_2$, and $E_3$ in system (4.5) have the following properties:

i. $E_1$ is a saddle point.

ii. $E_2$ is a sink point.

iii. Under the following condition $E_3$ is a stable point

$$\beta \geq \varepsilon, \quad s > (2\Gamma(\alpha+1))^{\frac{1}{\alpha}}, \quad h > \frac{\varepsilon}{2(\varepsilon-\beta)-1}.$$

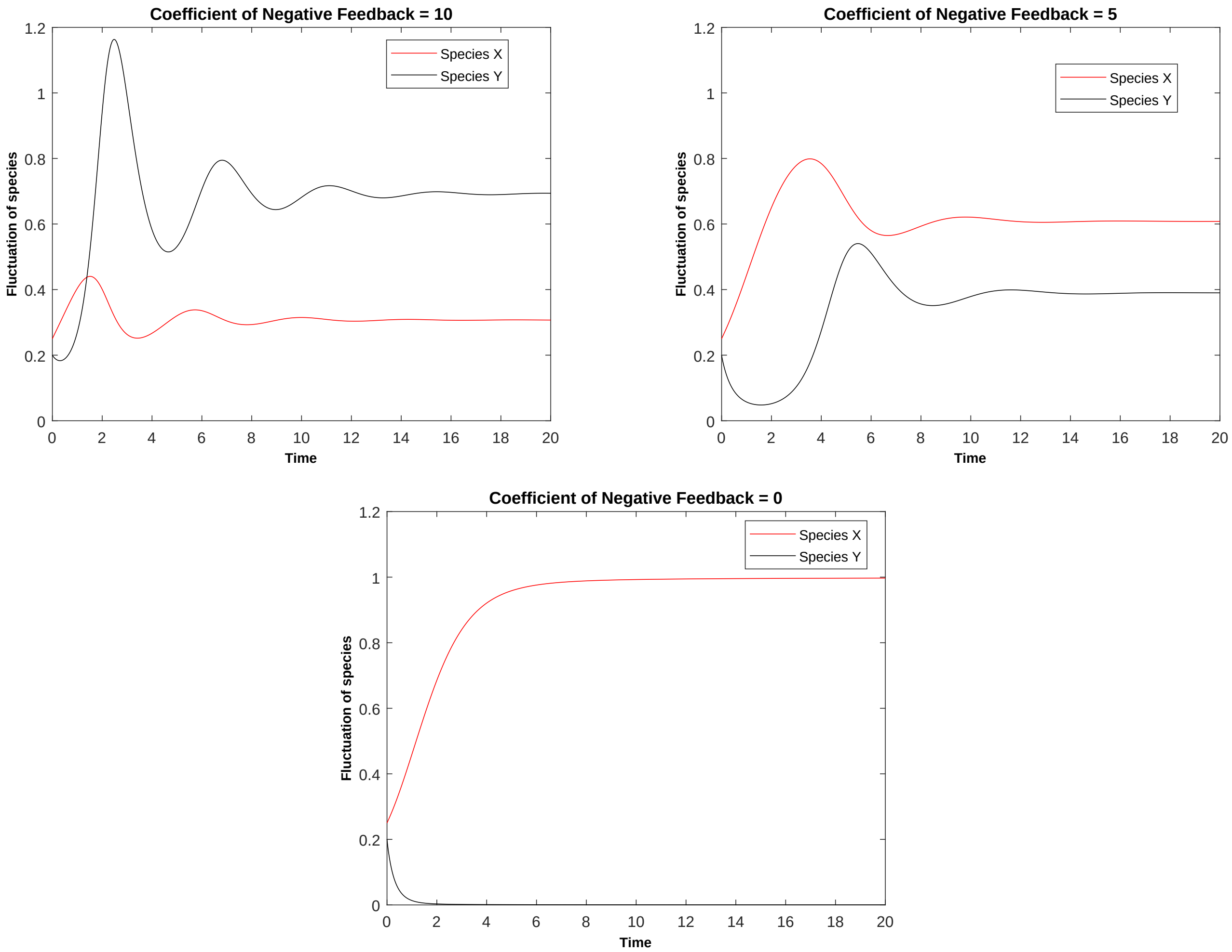


Figure 4: Alteration of the number of species over time with the initial states (0.25, 0.2). The system (3.2) is unstable when there is not any negative feedback on the predators. That is $\varepsilon = 0$. The effect of negative feedback on predators. As $\varepsilon$ increases, fluctuation in the number of prey damps during time down, and the system stabilizes sooner compared to predators (parameter values are $\varepsilon = 5$ *and* $10, h = 0.1, \beta = 3$ and fractional-orders: $\alpha = 0.95$).

**Proof:** Assume that $E^* = (x^*, y^*)$ is a fixed point of the system (4.5). So we have

$$J(x^*, y^*) = \begin{pmatrix} 1 + \frac{s^\alpha}{\Gamma(\alpha+1)}(1 - 2x^* - y^*) & \frac{s^\alpha}{\Gamma(\alpha+1)}\varepsilon y^* \\ -\frac{s^\alpha}{\Gamma(\alpha+1)}x^* & 1 + \frac{s^\alpha}{\Gamma(\alpha+1)}(-\beta - 2hy^* + \varepsilon x^*) \end{pmatrix}. \tag{7.1}$$

where $\alpha, \beta > 0$, $s > 0$.

i. By putting $E_1$ in Eq. (7.1), we have

$$J(E_1) = \begin{pmatrix} 1 + \frac{s^\alpha}{\Gamma(\alpha+1)} & 0 \\ 0 & 1 - \frac{\beta s^\alpha}{\Gamma(\alpha+1)} \end{pmatrix}. \tag{7.2}$$

Thus

$$\lambda_1 = 1 + \frac{s^\alpha}{\Gamma(\alpha+1)}, \qquad \lambda_2 = 1 - \frac{\beta s^\alpha}{\Gamma(\alpha+1)}.$$

By referring to Lemma (1), we can claim that $E_1$ is a saddle point.

ii. By putting $E_2$ in Eq. (7.1), we have

$$J(E_2) = \begin{pmatrix} 1 - \frac{s^\alpha}{\Gamma(\alpha+1)} & 0 \\ -\frac{s^\alpha}{\Gamma(\alpha+1)} & 1 + \frac{s^\alpha}{\Gamma(\alpha+1)}(-\beta+\varepsilon) \end{pmatrix}. \tag{7.3}$$

Thus

$$\lambda_1 = 1 - \frac{s^\alpha}{\Gamma(\alpha+1)}, \qquad \lambda_2 = 1 + \frac{s^\alpha}{\Gamma(\alpha+1)}(-\beta+\varepsilon).$$

By referring to Lemma (1), we can claim that $E_2$ is a sink point.

iii. By putting $E_3$ in Eq. (7.1), we have

$$J(E_3) = \begin{pmatrix} 1 - \frac{s^\alpha}{\Gamma(\alpha+1)}\frac{\beta+h}{\varepsilon+h} & \frac{s^\alpha \varepsilon}{\Gamma(\alpha+1)}\frac{\varepsilon-\beta}{\varepsilon+h} \\ -\frac{s^\alpha}{\Gamma(\alpha+1)}\frac{h+\beta}{\varepsilon+h} & 1 + \frac{s^\alpha}{\Gamma(\alpha+1)}\frac{h(\beta-\varepsilon)}{\varepsilon+h} \end{pmatrix}. \tag{7.4}$$

Thus, we have

$$trace(J(E_3)) = (\frac{2\Gamma(\alpha+1)}{s^\alpha} - 1)(\varepsilon+h) + (\beta-\varepsilon)(h-1). \tag{7.5}$$

Based on Theorem 1, point $E_3$ is stable if $trace(J(E_3)) < 0$ and $Det(J(E_3)) > 0$. From Eq. (7.5), it can be quickly deduced that when $trace(J(E_3)) < 0$ if $\beta > \varepsilon, s > (2\Gamma(\alpha+1))^{\frac{1}{\alpha}}$. Thus, the first condition is held.

Now it is enough to show that under what situation $Det(J(E_3)) > 0$. According to matrix (7.1), we have

$$Det(J(E_3)) = \left[(\frac{\Gamma(\alpha+1)}{s^\alpha})(\varepsilon+h) - (\beta+h)\right]\left[(\frac{\Gamma(\alpha+1)}{s^\alpha})(\varepsilon+h) + h(\beta-\varepsilon)\right] + \varepsilon(\varepsilon-\beta)(h+\beta).$$

Considering $\beta > \varepsilon$, if the following condition is satisfied, $Det(J(E_3)) > 0$.

$$\left[(\frac{\Gamma(\alpha+1)}{s^\alpha})\frac{(\varepsilon+h)}{\varepsilon-\beta} - h\right] > 0. \tag{7.6}$$

Eq. (7.6) will be held if $h > \frac{\varepsilon}{2(\varepsilon-\beta)-1}$. So under the obtained conditions the proof is completed.

## 8. Summary and discussion

In this section, numerical results of the proposed model is discussed and biological interpretation of the model based on the numerical results is also presented.

According to the concepts presented above, it can be claimed that the closer fractional-order to integer number one, the higher memory the fractional system has, and also the system sees an opposite trend when the order tends to zero.

The rationale behind this claim is that the derivative employed in the system (3.2) is Caputo; which means that information of the system is being saved by weight function $\omega(t) = t^{\alpha-1}/\Gamma(\alpha)$. Thus, if $\alpha$ leads towards one, $\omega$ will lead to one. Meanwhile, when the fractional-order leads to zero, $\omega$ leads to zero. Hence, it can be concluded that by leading fractional-order towards one, information of the function is saved by a coefficient around one.

In the real world, when two species (predator and prey) have an interaction with each other, it is expected that the more increase in nourishment, the higher population for prey and as a result more supply of food would be available for predators and they can feed more food. Hence, the increase in the population of one species results in the decrease of the other species.

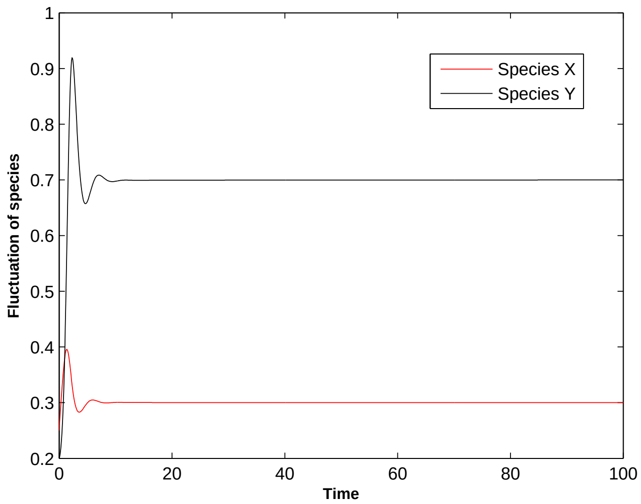


Figure 5: A schematic scheme of interaction among species over time with fractional-order $\alpha = 0.9$ and parameters $\varepsilon = 9, h = 1, \beta = 2$, with the initial state (0.25,0.2).

Regardless of unexpected natural catastrophie and overhunting by human beings, it is expected that a stable condition is established among species as time goes by, and this stability condition will come up sooner if species have a memory and learn about the surrounding area in which they are living. Therefore, according to the mentioned matters, the system (3.2) must be stabilized sooner when fractional-order ($\alpha$) leads towards one as we can see it in Fig 6.

It is also worth noting that we made a comparison between the species with different negative feedbacks on the predators. It is indeed shown that negative feedback is a necessary condition for the stability of the system. Also, from Fig 4, it can be quickly deduced that when the negative feedback on predators ($\varepsilon$) increases, fluctuation in the number of the predator increases too. However, the opposite trend is true for the prey (Fig 4.).

However, we know that every species has a specific brain by which they can remember and respond to the matters that happen around them. Although the system (3.2) denotes a memory concept to the model to investigate species interaction, it is not truly realistic. To address the risen flaw, different fractional-orders are allocated to the system (3.2), that is, preys are saving the information of their environment by $\alpha$ order, and predators memory is concerned with $\beta$ order. The numerical result of this model is depicted in Fig 7.

In systems without negative feedback, the solution in the phase plane is located in a class of infinite cycles. The cycle solution is a desirable property of the interactions between the species that are seen frequently in nature. Since a small change in the parameters may lead to a big structural and fundamental change in the behavior of the solutions, the addition of the negative feedback as a positive coefficient into the model helps to reach solutions with more stability in the system.

## 9. Conclusion

Interaction among species is of great importance to control the creatures from being virtually extinct. Considering this, scholars have proposed various models to study such sophisticated phenomena. However, the models sometimes fail to probe the system properly due to the complexity of the system. In this conducted research, we extend a predator-prey model on two species with negative feedback on both prey and predator. Then, to look at the system more realistically, with the help

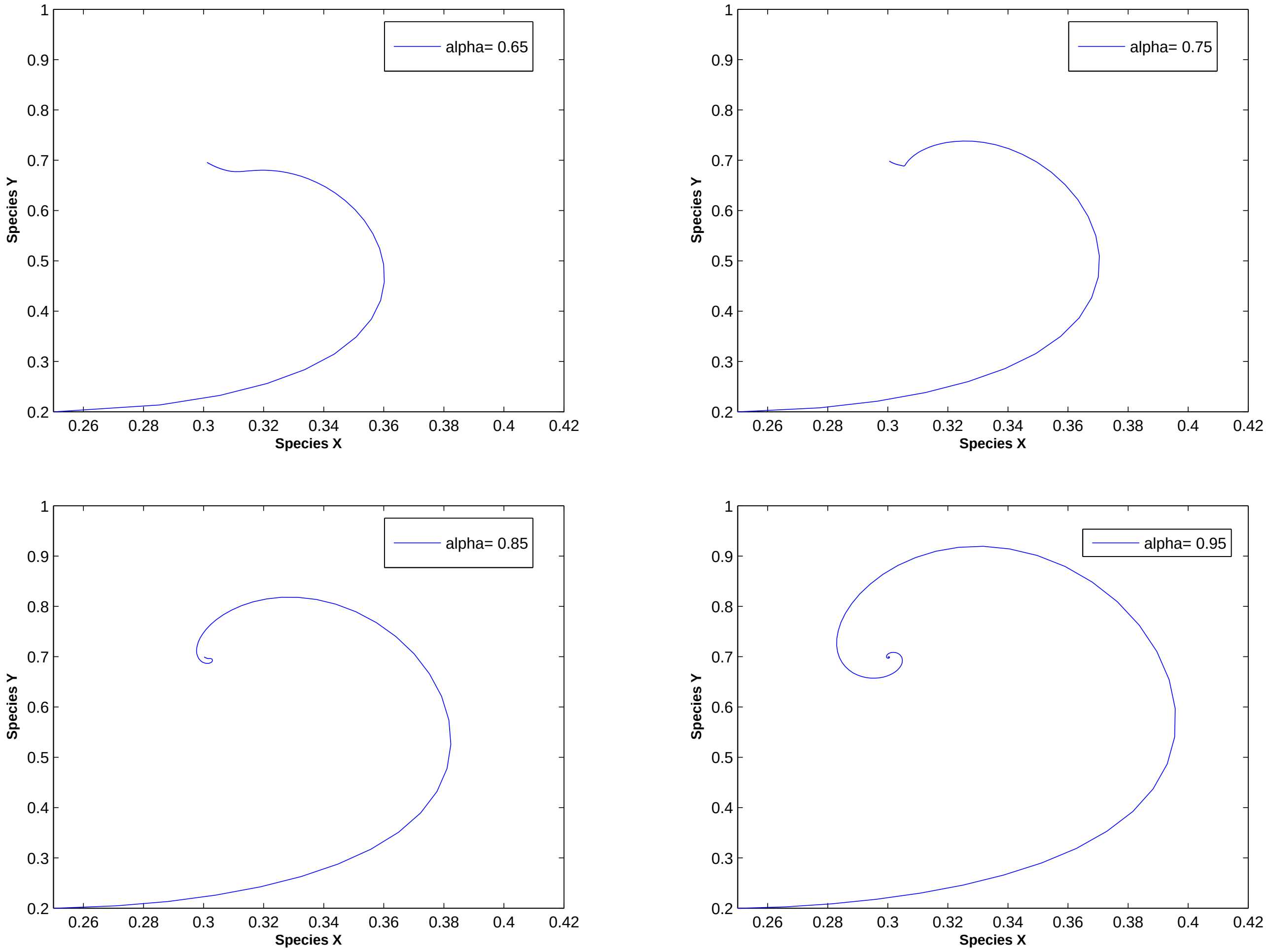


Figure 6: Phase diagrams for system (3.2) with the initial states (x(0), y(0)) = (0.25, 0.2), and parameter values $\varepsilon = 9, h = 1, \beta = 2$ and fractional-orders: $\alpha = 0.65, 0.75, 0.85, 0.95$.

of fractional calculus, both predators and preys are given different memories as well. The negative feedback in the model facilitates the equilibrium of the system. The stability and dynamical analysis of the suggested model is obtained and its discretized counterpart is also calculated. Finally, according to biological interpretation, it is numerically shown that the stability of the model is increased by donating memory to the system in both continuous and discretization states is increased.

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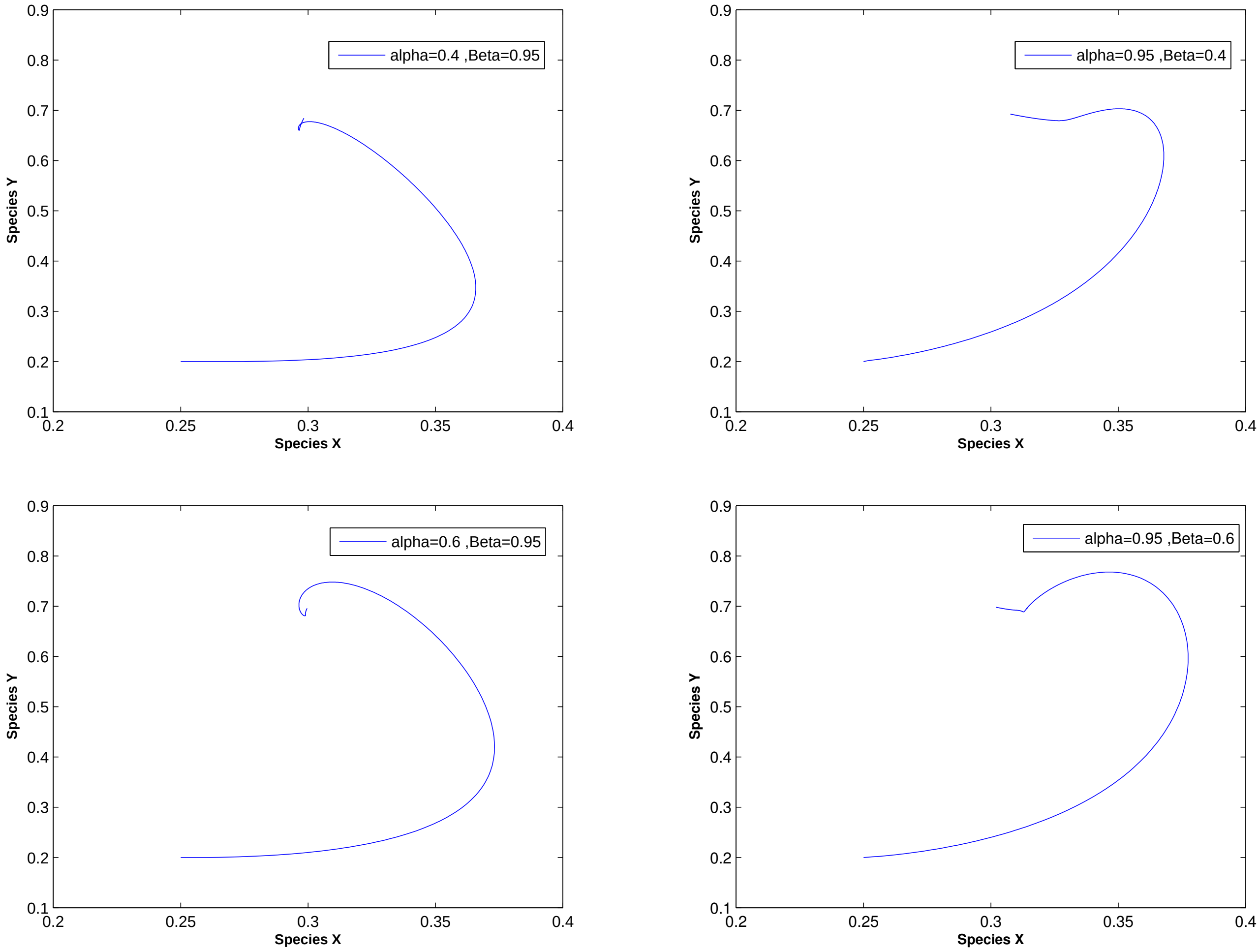


Figure 7: Phase diagrams for system (3.2) with the initial states (x(0), y(0)) = (0.25, 0.2), and parameter values $\varepsilon = 9, h = 1$ and different fractional-orders.


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