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Rough continuity and rough separation axioms in G_m-closure approximation spaces

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Abstract

The theory of general topology view for continuous mappings is general version and is applied for topological graph theory. Separation axioms can be regard as tools for distinguishing objects in information systems. Rough theory is one of map the topology to uncertainty. The aim of this work is to presented graph, continuity, separation properties and rough set to put a new approaches for uncertainty. For the introduce of various levels of approximations, we introduce several levels of continuity and separation axioms on graphs in G_m-closure approximation spaces.

AMS Subject Classification: 54C05, 54D10, 05C10

Keywords: Rough sets, G_m-closure space, Approximation spaces, lower and upper approximation spaces, Rough continuity, Rough separation axioms.

1. Introduction

Rough sets, presented by author in [10], and introduce the data base property by incomplete and insufficient information. The notations (lower, upper) approximation (written as appxox-) in rough theory, hidden information in intelligent probably disintegrated and presented of decision systems. The closure operator is a tools in many parts of mathematics for example, in algebra theory [2, 4], topology theory [7, 8] and computer science theory [13, 17]. Several works introduce recently for example in structural analysis theory [14, 15], in chemistry science [16], and physics science [6]. In this paper we will put a version for the application of topological graph theory.

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2. Preliminaries

We introduce a review of some preliminaries of rough sets [3, 10, 11] and some preliminaries of G_m-closure spaces [1, 12, 14, 15, 18].

2.1. Some Preliminaries of Uncertainty

Rough set theory represent as a sub-sets of a universe set in terminology of equivalence classes (written as equival. clas.) of a partition of universe set. The partition introduce a topological space, is said to be approx-space and denoted by $\Omega = (\mathscr{X}, \mathscr{E})$ where $\mathscr{X}$ is said to be the universe set and $\mathscr{E}$ is an equival. relation [9, 11]. The equival. clas. of $\mathscr{E}$ is said to be the granules, elementary sets, we denote the equival. clas. containing $x \in \mathscr{X}$ by $\mathscr{E}_x \subseteq \mathscr{X}$. In the approx-space, the operators of the (upper, lower) aprox's of A ; $A \subseteq \mathscr{X}$, then the lower-approx- (resp. the upper-approx-) of $\mathscr{X}$ is define as

$$\mathscr{L}(A) = \{ x \in \mathscr{X} : \mathscr{E}_x \subseteq A \} \quad (resp. \mathscr{U}(A) = \{ x \in \mathscr{X} : \mathscr{E}_x \cap A \neq \phi \})$$

2.2. Some Preliminaries of G_m-Closure Spaces

Closure operators on digraphs are present and many property on the G_m-closure spaces are introduce.

Definition 2.1. *[14, 15] Let $G = (V_G, E_G)$ be a direct graph, $P(V_G)$ be the all direct subgraphs of G and $Cl_G : P(V_G) \to P(V_G)$ such that $Cl_G(V_H) \subseteq V_G$ is said to be closure subgraph, where $H = (V_H, E_H)$ a subgraph of G and define as:*

$$Cl_G(V_H) = V_H \cup \{ \varpi \in V_G - V_H; (h, \varpi) \in E_G \;\; forall \;\; h \in V_H \}$$

The mapping Cl_G is said to be direct graph closure operator and $(G, \mathscr{F}_G)$ is said to be G-closure space (written as G-cl-space), such that $\mathscr{F}_G$ is the collection of members of Cl_G. Clearly $Cl_G(V_H) = \cap\{ V_F; V_F \in \mathscr{F}_G \;\; and \;\; V_H \subseteq V_F \}$. The direct graph interior operator $Int_G : P(V_G) \to P(V_G)$ defined as $Int_G(V_H) = V_G - Cl_G(V_G - V_H)$, where $H \subseteq G¿$. Clear that the direct graph interior operator is the dual of direct graph closure operator. A collection of members of Int_G is said to be interior subgraph of H and written as $\mathscr{T}_G$, and have $(G, \mathscr{T}_G)$ is a topological space. Clearly $Int_G(V_H) = \cup\{ V_O; V_O \in \mathscr{T}_G \;\; and \;\; V_O \subseteq V_H \}$. Furthermore $Cl_G(V_H) = V_G - Int_G(V_G - V_H)$. A subgraph H of G_m-cl-space $(G, \mathscr{F}_G)$ is said to be closed subgraph if $Cl_G(V_H) = V_H$ and it is said to be open subgraph if its complement is closed subgraph, (i.e., $Cl_G(V_G - V_H) = V_G - V_H$ or $Int_G(V_H) = V_H$.

Example 2.2. *Let $G = (V_G, E_G)$ be a digraph such that : $V_G = \{ \varpi_1, \varpi_2, \varpi_3, \varpi_4 \}$,*
$E_G = \{ (\varpi_1, \varpi_2), (\varpi_1, \varpi_3), (\varpi_2, \varpi_1), (\varpi_2, \varpi_3), (\varpi_3, \varpi_3) \}$.

$\mathscr{F}_G = \{ V_G, \phi, \{ \varpi_3 \}, \{ \varpi_3, \varpi_4 \}, \{ \varpi_1, \varpi_2, \varpi_3 \} \}$
$\mathscr{T}_G = \{ V_G, \phi, \{ \varpi_4 \}, \{ \varpi_1, \varpi_2 \}, \{ \varpi_1, \varpi_2, \varpi_4 \} \}$

If we did not get the Cl_G-cl-space from step one, we redefine direct graph closure operator as follow :

Definition 2.3. *[14, 15] Let $G = (V_G, E_G)$ be a direct graph, and $Cl_{G_m} : P(V_G) \to P(V_G)$ an operator, so we have :*
(a) It is said to be G_m-cl-operator if $Cl_{G_m} = Cl_G(Cl_G(Cl_G....)$, m-times, where $H \subseteq G$
(b) It is said to be G_m-topo-cl-operator if $Cl_{G_{m+1}} = Cl_{G_m}$ for all$H \subseteq G$.
The space $(G, \mathscr{F}_{G_m})$ is said to be G_m-cl-space).

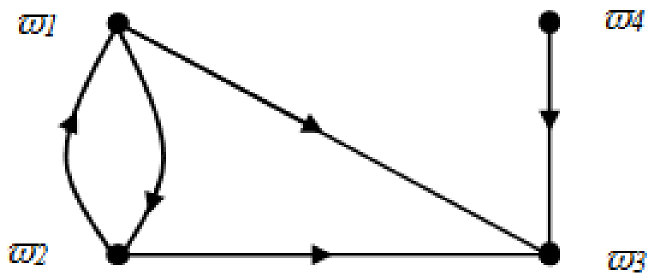

Figure 1: Graph G in Exam. 2.2.

Table 1: According to Example 2.2, Cl_G for all subgraph $H \subseteq G$.

V_H	$Cl_G(V_H)$	V_H	$Cl_G(V_H)$
V_G	V_G	$\{\varpi_1, \varpi_4\}$	V_G
ϕ	ϕ	$\{\varpi_2, \varpi_3\}$	$\{\varpi_1, \varpi_2, \varpi_3\}$
$\{\varpi_1\}$	$\{\varpi_1, \varpi_2, \varpi_3\}$	$\{\varpi_2, \varpi_4\}$	V_G
$\{\varpi_2\}$	$\{\varpi_1, \varpi_2, \varpi_3\}$	$\{\varpi_3, \varpi_4\}$	$\{\varpi_3, \varpi_4\}$
$\{\varpi_3\}$	$\{\varpi_3\}$	$\{\varpi_1, \varpi_2, \varpi_3\}$	V_G
$\{\varpi_4\}$	$\{\varpi_3, \varpi_4\}$	$\{\varpi_1, \varpi_2, \varpi_4\}$	V_G
$\{\varpi_1, \varpi_2\}$	$\{\varpi_1, \varpi_2, \varpi_3\}$	$\{\varpi_1, \varpi_3, \varpi_4\}$	V_G
$\{\varpi_1, \varpi_3\}$	$\{\varpi_1, \varpi_2, \varpi_3\}$	$\{\varpi_2, \varpi_3, \varpi_4\}$	V_G

Example 2.4. *Let $G = (V_G, E_G)$ be a digraph such that : $V_G = \{\varpi_1, \varpi_2, \varpi_3, \varpi_4\}$,* $E_G = \{(\varpi_1, \varpi_3), (\varpi_2, \varpi_1), (\varpi_2, \varpi_3), (\varpi_3, \varpi_4), (\varpi_4, \varpi_1)\}$.

$$\mathscr{F}_G = \{V_G, \phi, \{\varpi_1, \varpi_3, \varpi_4\}\}, \mathscr{T}_G = \{V_G, \phi, \{\varpi_2\}\}$$

Proposition 2.5. *[14] If $(G, \mathscr{F}_{G_m})$ is a G_m-cl-space. If $H, K \subseteq G$; $H \subseteq K \subseteq G$, then $Cl_{G_m}(V_H) \subseteq Cl_{G_m}(V_K)$ and $Int_{G_m}(V_H) \subseteq Int_{G_m}(V_K)$.*

Proposition 2.6. *[14] If $(G, \mathscr{F}_{G_m})$ is a G_m-cl-space. If $H, K \subseteq G$, then*
(a) $Cl_{G_m}(V_H \cup V_K) = Cl_{G_m}(V_H) \cup Cl_{G_m}(V_K)$,
(b) $Int_{G_m}(V_H \cap V_K) = Int_{G_m}(V_H) \cap Int_{G_m}(V_K)$,
(c) $Cl_{G_m}(V_H \cap V_K) \subseteq Cl_{G_m}(V_H) \cap Cl_{G_m}(V_K)$, and
(d) $Int_{G_m}(V_H) \cup Int_{G_m}(V_K) \subseteq Int_{G_m}(V_H \cup V_K)$.

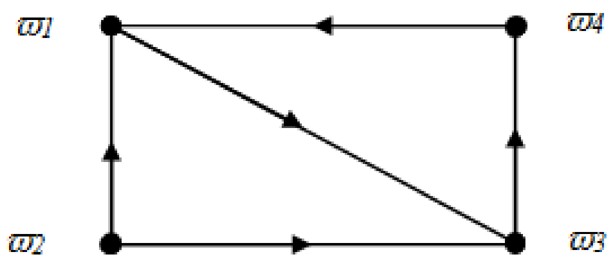

Figure 2: Graph G in Exam. 2.4.

Table 2: According to Example 2.4,Cl_G and Cl_{G_2} for all subgraph $H \subseteq G$

V_H	$Cl_G(V_H)$	$Cl_{G_2}(V_G)$	V_H	$Cl_G(V_H)$	$Cl_{G_2}(V_G)$
V_G	V_G	V_G	$\{\varpi_1, \varpi_4\}$	$\{\varpi_1, \varpi_3, \varpi_4\}$	$\{\varpi_1, \varpi_3, \varpi_4\}$
ϕ	ϕ	ϕ	$\{\varpi_2, \varpi_3\}$	V_G	V_G
$\{\varpi_1\}$	$\{\varpi_1, \varpi_3\}$	$\{\varpi_1, \varpi_3, \varpi_4\}$	$\{\varpi_2, \varpi_4\}$	V_G	V_G
$\{\varpi_2\}$	$\{\varpi_1, \varpi_2, \varpi_3\}$	V_G	$\{\varpi_3, \varpi_4\}$	$\{\varpi_1, \varpi_3, \varpi_4\}$	$\{\varpi_1, \varpi_3, \varpi_4\}$
$\{\varpi_3\}$	$\{\varpi_3, \varpi_4\}$	$\{\varpi_1, \varpi_3, \varpi_4\}$	$\{\varpi_1, \varpi_2, \varpi_3\}$	V_G	V_G
$\{\varpi_4\}$	$\{\varpi_1, \varpi_4\}$	$\{\varpi_1, \varpi_3, \varpi_4\}$	$\{\varpi_1, \varpi_2, \varpi_4\}$	V_G	V_G
$\{\varpi_1, \varpi_2\}$	$\{\varpi_1, \varpi_2, \varpi_3\}$	V_G	$\{\varpi_1, \varpi_3, \varpi_4\}$	$\{\varpi_1, \varpi_3, \varpi_4\}$	$\{\varpi_1, \varpi_3, \varpi_4\}$
$\{\varpi_1, \varpi_3\}$	$\{\varpi_1, \varpi_3, \varpi_4\}$	$\{\varpi_1, \varpi_3, \varpi_4\}$	$\{\varpi_2, \varpi_3, \varpi_4\}$	V_G	V_G

In G_m-cl-space $(G, \mathscr{F}_{G_m})$ the direct subggraph $H \subseteq G$ is said to be [14]:
(a) Regular open (written as $r - osg$) if $V_H = Int_{G_m}(Cl_{G_m}(V_H))$.
(b) Semi-open (written as $s - osg$) if $V_H \subseteq Cl_{G_m}(Int_{G_m}(V_H))$.
(c) Pre-open (written as $p - osg$) if $V_H \subseteq Int_{G_m}(Cl_{G_m}(V_H))$.
(d)γ-open (written as $\gamma - osg$) if $V_H \subseteq Cl_{G_m}(Int_{G_m}(V_H)) \cup Int_{G_m}(Cl_{G_m}(V_H))$.
(e) α-open (written as $\alpha - osg$) if $V_H \subseteq Int_{G_m}(Cl_{G_m}(Int_{G_m}(V_H))$.
(f) β-open (written as $\beta - osg$) if $V_H \subseteq Cl_{G_m}(Int_{G_m}(Cl_{G_m} V_H))$.
The complement of above $j-osg$ is said to be j-closed subgraph (written as $j-csg$) and the collection of all $j - osg's$ of $(G, \mathscr{F}_{G_m})$ is written as $j - O_{G_m}(G)$ where $j = r, s, p, \gamma, \alpha, \beta$. Also, all of $j - O_{G_{m(G)}}$ are bigger than $\mathscr{T}_{G_m}$ and closed under union property where $j = r, s, p, \gamma, \alpha, \beta$. The collection of all $j - csg's$ of $(G, \mathscr{F}_{G_m})$ is written as $j - C_{G_m}(G)$ where $j = r, s, p, \gamma, \alpha, \beta$. The j-closure (resp. j-interior) of $H \subseteq G$ in a G_m-cl-space $(G, \mathscr{F}_{G_m})$ is written as $Cl^j_{G_m}(V_H)$ (resp. $Int^j_{G_m}(V_H)$) and defined by

$$\begin{aligned} Cl^j_{G_m}(V_H) &= \cap\{V_F; V_F isj - csg \text{ and } V_H \subseteq V_F\} \\ (resp.Int^j_{G_m}(V_H) &= V_G - Cl^j_{G_m}(V_G - V_H)) where j = r, s, p, \gamma, \alpha, \beta. \end{aligned}$$

Proposition 2.7. *[14] If $(G, \mathscr{F}_{G_m})$ is G_m-cl-space, we have the following statements.*
(a) $r - O_{G_m}(G) \subseteq \mathscr{T}_{G_m} \subseteq \alpha - O_{G_m}(G) \subseteq s - O_{G_m}(G) \subseteq \gamma - O_{G_m}(G) \subseteq \beta - O_{G_m}(G)$,
(b) $\alpha - O_{G_m}(G) \subseteq p - O_{G_m}(G) \subseteq \gamma - O_{G_m}(G)$.

3. Generalization of Pawlak Approximation Spaces

The approx-space $G_m = (G, Cl_{G_m})$ with Cl_{G_m} on G is G_m-cl-space $(G, \mathscr{F}_{G_m})$; $\mathscr{F}_{G_m}$ is the G_m-cl-space to G_m. So We have:

Definition 3.1. *If $G_m = (G, Cl_{G_m})$ is an approx-space ; G is a nonempty universe direct graph, Cl_{G_m} is define on G_m, and $\mathscr{F}_{G_m}$ is the G_m-cl-space to G_m. Then $\mathscr{G}_m = (G, Cl_{G_m}, \mathscr{F}_{G_m})$ is said to be a G_m-cl-approx-space.*

We present the definitions of lower (resp. near lower) and upper (resp. near upper) approx's in a G_m-cl-approx-space $\mathscr{G}_m = (G, Cl_{G_m}, \mathscr{F}_{G_m})$.

Definition 3.2. *If $\mathscr{G}_m = (G, Cl_{G_m}, \mathscr{F}_{G_m})$ is a G_m-cl-approx-space and $H \subseteq G$. The lower approx- (resp. the upper approx-) of H is denoted by $\mathscr{L}(V_H)$ (resp. $\mathscr{U}(V_H)$) and is defined by*

$$\mathscr{L}(V_H) = Int_{G_m}(V_H)(resp.\mathscr{U}(V_H) = Cl_{G_m}(V_H)).$$

Definition 3.3. *If $\mathscr{G}_m = (G, Cl_{G_m}, \mathscr{F}_{G_m})$ is a G_m-cl-approx-space and $H \subseteq G$. The near lower approx- "written as j-lower-approx- (resp. near upper approx- "written as j-upper-approx-") of H is denoted by $\mathscr{L}^j(V_H)$ (resp. $\mathscr{U}^j(V_H)$) and is defined by*

$$\mathscr{L}^j(V_H) = Int^j_{G_m}(V_H)$$

$$(resp.\mathscr{U}^j(V_H) = Cl^j_{G_m}(V_H)), where j = r, s, p, \gamma, \alpha, \beta.$$

Proposition 3.4. *If $\mathscr{G}_m = (G, Cl_{G_m}, \mathscr{F}_{G_m})$ is a G_m-cl-approx-space and $H \subseteq G$, then $\mathscr{L}(V(H)) \subseteq \mathscr{L}^j(V(H)) \subseteq V_H \subseteq \mathscr{U}^j(V(H) \subseteq \mathscr{U}(V_H)$, for all $j \in \{ s, p, \gamma, \alpha, \beta \}$.*
Proof . *The proofs are similar for the five cases; So, we will only prove the case $j = S$: Now,*

$$\begin{aligned} & \mathscr{U}(V_H) = Cl_{G_m}(V_H) \\ = \ & \cap\{ V_F; V_F \in \mathscr{F}_{G_m} \text{ and } V_H \subseteq V_F \} \\ \supseteq \ & \cap\{ V_F; V_F \in s - C_{G_m} \text{ and } V_H \subseteq V_F \} since \mathscr{F}_{G_m} \subseteq s - C_{G_m}(G) \\ = \ & Cl^s_{G_m}(V_H) = \mathscr{U}^s(V_H) \supseteq V_H \end{aligned} \tag{3.1}$$

$$\begin{aligned} & \mathscr{L}(V_H) = Int_{G_m}(V_H) \\ = \ & V_G - Cl_{G_m}(V_G - V_H) \subseteq V_G - Cl^s_{G_m}(V_G - V_H) \text{ since } \mathscr{T}_{G_m} \subseteq s - O_{G_m}(G) \\ = \ & Int^s_{G_m}(V_H) = \mathscr{L}^s(V_H) \subseteq V_H \end{aligned} \tag{3.2}$$

From 3.1) and (3.2) we get
$\mathscr{L}(V_H) \subseteq \mathscr{L}^s(V_H \subseteq V_H \subseteq \mathscr{U}^s(V_H \subseteq \mathscr{U}(V_H)$ □

Proposition 3.5. *If $\mathscr{G}_m = (G, Cl_{G_m}, \mathscr{F}_{G_m})$ is a G_m-cl-approx-space and $H \subseteq G$, then the following holds for $j = s, p, \gamma, \alpha, \beta$.*
(a) $\mathscr{L}(V_H) \subseteq \mathscr{L}^\alpha(V_H) \subseteq \mathscr{L}^s(V_H) \subseteq \mathscr{L}^\gamma(V_H) \subseteq \mathscr{L}^\beta(V_H)$,
(b) $\mathscr{L}^\alpha(V_H) \subseteq \mathscr{L}^p(V_H) \subseteq \mathscr{L}^\gamma(V_H)$.
Proof . *By Proposition (3.1), we have $\mathscr{L}(V_H) \subseteq \mathscr{L}^\alpha(V_H)$. To prove $\mathscr{L}^\alpha(V_H) \subseteq \mathscr{L}^s(V_H)$. Now,*
$\mathscr{L}^\alpha(V_H) = Int^\alpha_{G_m}(V_H) = V_G - Cl^\alpha_{G_m}(V_G - V_H)$
$\subseteq V_G - Cl^s_{G_m}(V_G - V_H)$ *since* $\alpha_{G_m}(G) \subseteq s - O_{G_m}(G)$. *Thus*
$\mathscr{L}^\alpha(V_H) = Int^\alpha_{G_m}(V_H) \subseteq Int^s_{G_m}(V_H) = \mathscr{L}^s(V_H)$. □

The prove of the other cases are similarly.

4. Rough Continuous Mappings in G_m-Closure Approximation Spaces

The main goal of this part is to give one of the G_m-topological applications that represented by the concept of rough continuous mappings. This notion has a great importance in the theory of rough set, since this type of continuity can make different approximation spaces to be related to each others. The following definition introduces rough continuous mappings between two G_m-cl-approx-space's.

Definition 4.1. *Let $\mathscr{G}^1_m = (G^1, Cl^1_{G_m}, \mathscr{F}^1_{G_m})$ and $\mathscr{G}^2_m = (G^2, Cl^2_{G_m}, \mathscr{F}^2_{G_m})$ be two G_m-cl-approx-spaces. Then a mapping $f : \mathscr{G}^1_m \to \mathscr{G}^2_m$ is said to be rough continuous if $f^{-1}(\mathscr{L}_2(V_H) \subseteq \mathscr{L}_1(f^{-1}(V_H))$ for every subgraph H in G.*

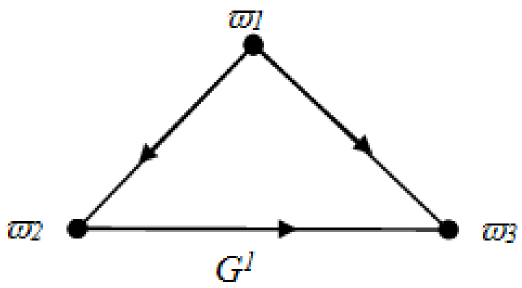

Figure 3: Graphs G^1 and G^2 in Exam. 4.2.

Example 4.2. *Let* $\mathscr{G}^1_m = (G^1, Cl^1_{G_m}, \mathscr{F}^1_{G_m})$ *and* $\mathscr{G}^2_m = (G^2, Cl^2_{G_m}, \mathscr{F}^2_{G_m})$ *be two* G_m*-cl-approx-spaces's where :*
$G^1 = (V_{G^1}, E_{G^1}); V_{G^1} = \{ \varpi_1, \varpi_2, \varpi_3 \}, E_{G^1} = \{ (\varpi_1, \varpi_2), (\varpi_1, \varpi_3), (\varpi_2, \varpi_3) \}$,
$G^2 = (V_{G^2}, E_{G^2}); V_{G^2} = \{ u_1, u_2, u_3 \}, E_{G^2} = \{ (u_1, u_2), (u_1, u_3), (u_2, u_3) \}$.

$\mathscr{F}^1_{G_1} = \{ V_{G^1}, \phi, \{ \varpi_3 \}, \{ \varpi_2, \varpi_3 \} \}, \mathscr{T}^1_{G_1} = \{ V_{G^1}, \phi, \{ \varpi_1 \}, \{ \varpi_1, \varpi_2 \} \}$ and $\mathscr{F}^2_{G_1} = \{ V_{G^2}, \phi, \{ u_3 \}, \{ u_2, u_3 \} \}, \mathscr{T}^2_{G_1} = \{ V_{G^2}, \phi, \{ u_1 \}, \{ u_1, u_2 \} \}$.

Define a mapping $f : \mathscr{G}^1_m \to \mathscr{G}^2_m$ such that $f(\varpi_1) = f(\varpi_2) = u_2, f(\varpi_3) = u_3$.
Hence f is rough continuous since $f^{-1}(\mathscr{L}_2(V_H) \subseteq \mathscr{L}_1(f^{-1}(V_H))$ for every subgraph H in $\mathscr{G}^2_1$.

Example 4.3. *Let* $\mathscr{G}^1_m = (G^1, Cl^1_{G_m}, \mathscr{F}^1_{G_m})$ *and* $\mathscr{G}^2_m = (G^2, Cl^2_{G_m}, \mathscr{F}^2_{G_m})$ *be two* G_m*-cl-approx-spaces wheres:*
$G^1 = (V_{G^1}, E_{G^1}); V_{G^1} = \{ \varpi_1, \varpi_2, \varpi_3 \}, E_{G^1} = \{ (\varpi_1, \varpi_2), (\varpi_1, \varpi_3), (\varpi_2, \varpi_3) \}$,
$G^2 = (V_{G^2}, E_{G^2}); V_{G^2} = \{ u_1, u_2, u_3 \}, E_{G^2} = \{ (u_2, u_3), (u_3, u_2) \}$.

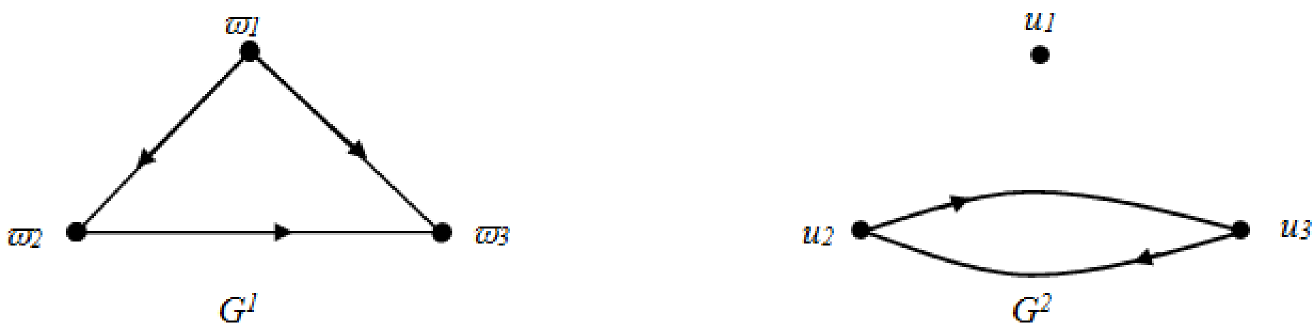

Figure 4: Graphs G^1 and G^2 given in Exam. 4.3.

$\mathscr{F}^1_{G_1} = \{ V_{G^1}, \phi, \{ \varpi_3 \}, \{ \varpi_2, \varpi_3 \} \}, \mathscr{T}^1_{G_1} = \{ V_{G^1}, \phi, \{ \varpi_1 \}, \{ \varpi_1, \varpi_2 \} \}$ and $\mathscr{F}^2_{G_1} = \{ V_{G^2}, \phi, \{ u_1 \}, \{ u_2, u_3 \} \}, \mathscr{T}^2_{G_1} = \{ V_{G^2}, \phi, \{ u_1 \}, \{ u_2, u_3 \} \}$.

Define a mapping $f : \mathscr{G}^1_m \to \mathscr{G}^2_m$ such that $f(\varpi_1) = u_1, f(\varpi_2) = u_2, f(\varpi_3) = u_3$.
Let $H = (V_H, E_H); V_H = \{ u_2, u_3 \}, E_H = \{ (u_2, u_3), (u_3, u_2) \}$ be a subgraph of G^2. Then, $f^{-1}(\mathscr{L}_2(V_H) = f^{-1}(\{ u_2, u_3 \}) = \{ \varpi_2, \varpi_3 \}$, but $\mathscr{L}_1(f^{-1}(V_H)) = \mathscr{L}_1(\{ \varpi_2, \varpi_3 \}) = \phi$.
Hence there exists a subgraph H of G^2 such that $f^{-1}(\mathscr{L}_2(V_H)$ is not subset of $\mathscr{L}_1(f^{-1}(V_H))$. Thus f is not a rough continuous mapping.

Definition 4.4. *Let $\mathscr{G}_m^1 = (G^1, Cl^1_{G_m}, \mathscr{F}^1_{G_m})$ and $\mathscr{G}_m^2 = (G^2, Cl^2_{G_m}, \mathscr{F}^2_{G_m})$ be two G_m-cl-approx-spaces. Then a mapping $f : \mathscr{G}_m^1 \to \mathscr{G}_m^2$ is said to be continuous if the inverse image of each open graph in G^2 is open in G^1.*

Example 4.5. *Let $\mathscr{G}_m^1 = (G^1, Cl^1_{G_m}, \mathscr{F}^1_{G_m})$ and $\mathscr{G}_m^2 = (G^2, Cl^2_{G_m}, \mathscr{F}^2_{G_m})$ be two G_m-cl-approx-spaces:*
$G^1 = (V_{G^1}, E_{G^1}); V_{G^1} = \{\varpi_1, \varpi_2, \varpi_3\}, E_{G^1} = \{(\varpi_1, \varpi_2), (\varpi_1, \varpi_3), (\varpi_2, \varpi_3)\}$,
$G^2 = (V_{G^2}, E_{G^2}); V_{G^2} = \{u_1, u_2, u_3\}, E_{G^2} = \{(u_2, u_3), (u_3, u_2)\}$.

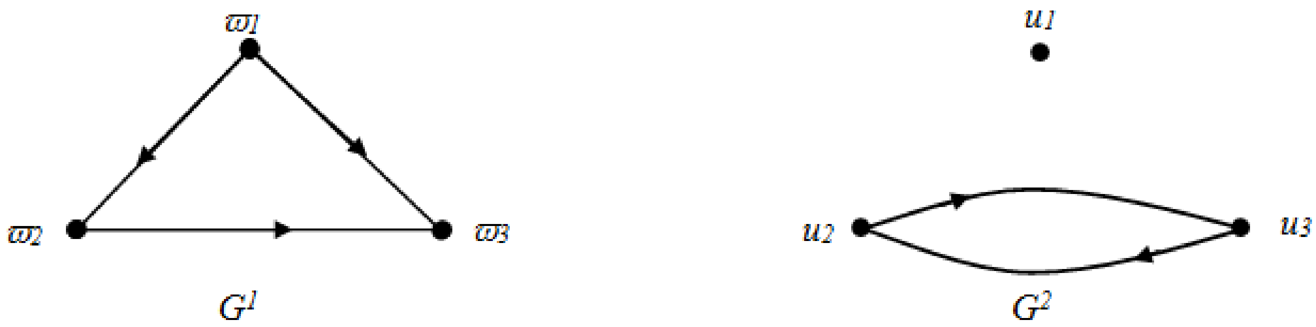

Figure 5: Graphs G^1 and G^2 in Exam. 4.5.

$\mathscr{F}^1_{G_1} = \{V_{G^1}, \phi, \{\varpi_3\}, \{\varpi_2, \varpi_3\}\}, \mathscr{T}^1_{G_1} = \{V_{G^1}, \phi, \{\varpi_1\}, \{\varpi_1, \varpi_2\}\}$ and
$\mathscr{F}^2_{G_1} = \{V_{G^2}, \phi, \{u_1\}, \{u_2, u_3\}\}, \mathscr{T}^2_{G_1} = \{V_{G^2}, \phi, \{u_1\}, \{u_2, u_3\}\}$.

Define a mapping $f : \mathscr{G}_m^1 \to \mathscr{G}_m^2$ such that $f(\varpi_1) = u_2, f(\varpi_2) = f(\varpi_3) = u_3$.
Then f is continuous, since $f^{-1}(H) \in \mathscr{T}^1_{G_1}$ for all $H \in \mathscr{T}^2_{G_1}$.

Example 4.6. *In example 4.5, a mapping f is not continuous, since*
$H = (V_H, E_H); V_H = \{u_2, u_3\}, E_H = \{(u_2, u_3), (u_3, u_2)\}$ *is an open subgraph in G^2 but*
$f^{-1}(H) = (V_{f^{-1}(H)}, E_{f^{-1}(H)}); V_{f^{-1}(H)} = \{\varpi_2, \varpi_3\}, E_{f^{-1}(H)} = \{(\varpi_2, \varpi_3)\}$ *is not open in G^1.*

Theorem 4.7. *Let $\mathscr{G}_m^1 = (G^1, Cl^1_{G_m}, \mathscr{F}^1_{G_m})$ and $\mathscr{G}_m^2 = (G^2, Cl^2_{G_m}, \mathscr{F}^2_{G_m})$ be two G_m-cl-approx-space's. Then $f : \mathscr{G}_m^1 \to \mathscr{G}_m^2$ is a rough continuous mapping if and only if f is continuous.*
Proof . *($\Rightarrow$) Let f be a rough continuous mapping and $H \subseteq G^2$ be an open graph, hence $\mathscr{L}_2(V_H) = Int_{G_m}(V_H) = V_H$). Thus $f^{-1}(V_H) = f^{-1}(\mathscr{L}_2(V_H) \subseteq \mathscr{L}_1(f^{-1}(V_H))$, since f is a rough continuous mapping.*
But $\mathscr{L}_1(f^{-1}(V_H)) = Int_{G_m}(f^{-1}(V_H))$. Then $f^{-1}(V_H) \subseteq Int_{G_m}(f^{-1}(V_H))$ and hence $f^{-1}(V_H) = Int_{G_m}(f^{-1}(V_H))$. Thus $f^{-1}(V_H)$ is an open graph in G^1.
Therefore f is continuous mapping.
($\Leftarrow$) Let f be a continuous mapping and $H \subseteq G^2$. Then $f^{-1}(\mathscr{L}_2(V_H) \subseteq f^{-1}(V_H)$, since $\mathscr{L}_2(V_H) \subseteq V_H$. Thus
$$\mathscr{L}_1(f^{-1}(\mathscr{L}_2(V_H))) \subseteq \mathscr{L}_1(f^{-1}(V_H)) \qquad (4.1.1)$$
But $f^{-1}(\mathscr{L}_2(V_H) = f^{-1}(Int_{G_m}(V_H)) \in \mathscr{T}^1_{G_1}$, since $Int_{G_m}(V_H) \in \mathscr{T}^2_{G_m}$ and f is continuous. Hence $f^{-1}(\mathscr{L}_2(V_H)) \subseteq Int_{G_m}(f^{-1}(\mathscr{L}_2(V_H)) = \mathscr{L}_1(f^{-1}(L_2(V_H)))$ and then from (4.1.1) we get $f^{-1}(\mathscr{L}_2(V_H)) \subseteq \mathscr{L}_1(f^{-1}(L_2(V_H))) \subseteq \mathscr{L}_1(f^{-1}(V_H))$.
Therefore f is a rough continuous mapping. □

5. Near Rough Continuous Mappings in G_m-Closure Approximation Spaces

Near rough (written as j-rough) continuous mappings represent different levels of continuity ; $j = r, s, p, \gamma, \alpha, \beta$. In this section we present thee concepts of j-rough continuous mappings between two G_m-cl-appro-spaces.

Definition 5.1. *Let $\mathscr{G}^1_m = (G^1, Cl^1_{G_m}, \mathscr{F}^1_{G_m})$ and $\mathscr{G}^2_m = (G^2, Cl^2_{G_m}, \mathscr{F}^2_{G_m})$ be two G_m-cl-approx-spaces. A mapping $f : \mathscr{G}^1_m \to \mathscr{G}^2_m$ is said to be near rough (written as j-rough) continuous for all $j = r, s, p, \gamma, \alpha, \beta$ if $f^{-1}(\mathscr{L}_2(V_H) \subseteq \mathscr{L}^j_1(f^{-1}(V_H))$ for every subgraph H in G^2_m.*

Example 5.2. *Let $\mathscr{G}^1_m = (G^1, Cl^1_{G_m}, \mathscr{F}^1_{G_m})$ and $\mathscr{G}^2_m = (G^2, Cl^2_{G_m}, \mathscr{F}^2_{G_m})$ be two G_m-cl-approx-spaces where:*
$G^1 = (V_{G^1}, E_{G^1}); V_{G^1} = \{ \varpi_1, \varpi_2, \varpi_3, \varpi_4 \}, E_{G^1} = \{ (\varpi_2, \varpi_3), (\varpi_3, \varpi_4), (\varpi_4, \varpi_2) \}$,
$G^2 = (V_{G^2}, E_{G^2}); V_{G^2} = \{ u_1, u_2, u_3, u_4 \}, E_{G^2} = \{ (u_1, u_2), (u_1, u_3), (u_2, u_1), (u_2, u_3), (u_4, u_3) \}$.

Figure 6: Graphs G^1 and G^2 in Exam. 5.2.

$\mathscr{F}^1_{G_1} = \{ V_{G^1}, \phi, \{ \varpi_1 \}, \{ \varpi_2, \varpi_3, \varpi_4 \} \}$, $\mathscr{T}^1_{G_1} = \{ V_{G^1}, \phi, \{ \varpi_1 \}, \{ \varpi_2, \varpi_3, \varpi_4 \} \}$ and
$\mathscr{F}^2_{G_1} = \{ V_{G^2}, \phi, \{ u_3 \}, \{ u_3, u_4 \}, \{ u_1, u_2, u_3 \} \}$, $\mathscr{T}^2_{G_1} = \{ V_{G^2}, \phi, \{ u_4 \}, \{ u_1, u_2 \}, \{ u_1, u_2, u_3 \} \}$.

Hence $p - O_{G_2}(G^1) =$ the power set of vertices of G^1.
Define a mapping $f : \mathscr{G}^1_m \to \mathscr{G}^2_m$ such that $f(\varpi_1) = u_1, f(\varpi_2) = u_2, f(\varpi_3) = u_3, f(\varpi_4) = u_4$.
Thus f is p-rough continuous since $f^{-1}(\mathscr{L}_2(V_H) \subseteq \mathscr{L}^p_1(f^{-1}(V_H))$ for every subgraph H in $\mathscr{G}^2_1$ as illustrated in Table (3)..

Example 5.3. *In example (5.1), we get $s - O_{G2}(G^1) = \{ V_{G^1}, \phi, \{ \varpi_1 \}, \{ \varpi_2, \varpi_3, \varpi_4 \} \}$.
Let $H = (V_H, E_H); V_H = \{ u_1, u_2 \}, E_H = \{ (u_1, u_2), (u_2, u_1) \}$ be a subgraph of G^2. Then $f^{-1}(\mathscr{L}_2(V_H)) = f^{-1}(\{ u_1, u_2 \}) = \{ \varpi_1, \varpi_2 \}$, but $\mathscr{L}^s_1(f^{-1}(V_H)) = \mathscr{L}^s_1(\{ \varpi_1, \varpi_2 \}) = \{ \varpi_1 \}$. Hence there exists a subgrap H of G^2 such that $f^{-1}(\mathscr{L}_2(V_H)$ is not subset of $\mathscr{L}^s_1(f^{-1}(V_H))$. Thus f is not a s-rough continuous mapping.*

Proposition 5.4. *The implication between rough continuity and j-rough continuity for all $j = r, s, p, \gamma, \alpha, \beta$ are given by the following diagram.*

r-rough continuous
$\Downarrow$
α-rough continuous $\Rightarrow$ s-rough continuous

Table 3: $\mathscr{L}_2(V_H)$, $f^{-1}(V_H)$, $\mathscr{L}_1^p(f^{-1}(V_H))$ and $f^{-1}(\mathscr{L}_2(V_H)$ for every subgraph H in $\mathscr{G}_1^2$, where H, G^2, $\mathscr{L}_1$, $\mathscr{L}_2$ and f are given in Example 5.2.

$H \subseteq G^2$	$\mathscr{L}_2(V_H)$	$f^{-1}(V_H)$	$\mathscr{L}_1^p f^{-1}(V_H)$	$f^{-1}(\mathscr{L}_2((V_H)$
V_{G^2}	V_{G^2}	V_{G^1}	V_{G^1}	V_{G^1}
ϕ	ϕ	ϕ	ϕ	ϕ
$\{u_1\}$	ϕ	$\{\varpi_1\}$	$\{\varpi_1\}$	ϕ
$\{u_2\}$	ϕ	$\{\varpi_2\}$	$\{\varpi_2\}$	ϕ
$\{u_3\}$	ϕ	$\{\varpi_3\}$	$\{\varpi_3\}$	ϕ
$\{u_4\}$	$\{u_4\}$	ϖ_4	$\{\varpi_4\}$	$\{\varpi_4\}$
$\{u_1, u_2\}$	$\{u_1, u_2\}$	$\{\varpi_1, \varpi_2\}$	$\{\varpi_1, \varpi_2\}$	$\{\varpi_1, \varpi_2\}$
$\{u_1, u_3\}$	$\{\phi\}$	$\{\varpi_1, \varpi_3\}$	$\{\varpi_1, \varpi_3\}$	$\{\phi\}$
$\{u_1, u_4\}$	$\{u_4\}$	$\{\varpi_1, \varpi_4\}$	$\{\varpi_1, \varpi_4\}$	$\{\varpi_4\}$
$\{u_2, u_3\}$	$\{\phi\}$	$\{\varpi_2, \varpi_3\}$	$\{\varpi_2, \varpi_3\}$	$\{\phi\}$
$\{u_2, u_4\}$	$\{u_4\}$	$\{\varpi_2, \varpi_4\}$	$\{\varpi_2, \varpi_4\}$	$\{\varpi_4\}$
$\{u_3, u_4\}$	$\{u_4\}$	$\{\varpi_3, \varpi_4\}$	$\{\varpi_3, \varpi_4\}$	$\{\varpi_4\}$
$\{u_1, u_2, u_3\}$	$\{u_1, u_2\}$	$\{\varpi_1, \varpi_2, \varpi_3\}$	$\{\varpi_1, \varpi_2, \varpi_3\}$	$\{\varpi_1, \varpi_2\}$
$\{u_1, u_2, u_4\}$	$\{u_1, u_2, u_4\}$	$\{\varpi_1, \varpi_2, \varpi_4\}$	$\{\varpi_1, \varpi_2, \varpi_4\}$	$\{\varpi_1, \varpi_2, \varpi_4\}$
$\{u_1, u_3, u_4\}$	$\{u_4\}$	$\{\varpi_1, \varpi_3, \varpi_4\}$	$\{\varpi_1, \varpi_3, \varpi_4\}$	$\{\varpi_4\}$
$\{u_2, u_3, u_4\}$	$\{u_4\}$	$\{\varpi_2, \varpi_3, \varpi_4\}$	$\{\varpi_2, \varpi_3, \varpi_4\}$	$\{\varpi_4\}$

$\Downarrow$ $\Downarrow$

p-rough continuous $\Rightarrow$ *γ-rough continuous* $\Rightarrow$ *β-rough continuous.*

Proof . *By using Proposition 2.7 and Proposition 3.5, the proof if obvious.* □

The implication of the other side of Proposition 5.4 is not true as illustrated by the example:

Example 5.5. *In example (5.1), we get* $\alpha - O_{G_2}(G^1) = \{V_{G^1}, \phi, \{\varpi_1\}, \{\varpi_2, \varpi_3, \varpi_4\}\}$.
Let $H = (V_H, E_H); V_H = \{u_4\}, E_H = \phi$, *be a subgraph of* G^2. *Then* $f^{-1}(\mathscr{L}_2(V_H)) = f^{-1}(\{u_4\}) = \{\varpi_4\}$, *but* $\mathscr{L}_1^\alpha(f^{-1}(V_H)) = \mathscr{L}_1^\alpha(\{\varpi_4\}) = \phi$. *Hence there exists a subgrap* H *of* G^2 *such that* $f^{-1}(\mathscr{L}_2(V_H)$ *is not subset of* $\mathscr{L}_1^\alpha(f^{-1}(V_H))$. *Thus* f *is not a* α*-rough continuous mapping, but it is p-rough continuous.*

6. Rough Separation Axioms in G_m-Closure Approximation Spaces

The main goal of this section is to give one of the G_m-topological applications that represented by the concept of rough separations axioms. Two different objects of the universe can be belong to the same category and then they are indiscernible in view of the available information which making the imprecise and uncertainty about data. The main purpose of separation axioms is to make vertices and graphs of spaces topologically distinguishable that is a very useful in the information systems to extracting the given data.

Definition 6.1. *If* $\mathscr{G}_m = (G, Cl_{G_m}, \mathscr{F}_{G_m})$ *is a* G_m*-cl-approx-space. Then* $\mathscr{G}_m$ *is said to be a rough* $\mathscr{G}_{m0}$ *space (written as* $\mathscr{G}_{m0}$*-space), if for every two distinct vertices* $\varpi, u \in G$, *find a subgraph* $H \subseteq G$ *; either* $\varpi \in \mathscr{L}(V_H), u \in V_G - \mathscr{L}(V_H)$ *or* $u \in \mathscr{L}(V_H), \varpi \in V_G - \mathscr{L}(V_H)$.

Theorem 6.2. *If $\mathscr{G}_m = (G, Cl_{G_m}, \mathscr{F}_{G_m})$ is a G_m-cl-approx-space. Then $\mathscr{G}_m$ is $\mathscr{G}_{m0}$-space if and only if $\mathscr{U}(\{\varpi\}) \neq \mathscr{U}(\{u\})$ for every two distinct vertices $\varpi, u \in G$.*

Proof . *(⇒) If $\mathscr{G}_m = (G, Cl_{G_m}, \mathscr{F}_{G_m})$ is a $\mathscr{G}_{m0}$-space. Then for every two distinct vertices $\varpi, u \in G$ find a subgraph $H \subseteq G$; $\varpi \in \mathscr{L}(V_H) = Int_{G_m}(V_H)$ and $u \notin \mathscr{L}(V_H) = Int_{G_m}(V_H)$. Thus $\varpi \notin [Int_{G_m}(V_H)]^c, u \in [Int_{G_m}(V_H)]^c$ and $[Int_{G_m}(V_H)]^c$ is closed graph. Hence $\varpi \notin \cap\{K; V_K \subseteq [Int_{G_m}(V_H)]^c, K \in \mathscr{F}_{G_m}, \{u\} \subseteq H\} = Cl_{G_m}(\{u\}) = \mathscr{U}(\{u\})$. Thus $\varpi \notin \mathscr{U}(\{u\})$, but $\varpi \in Cl_{G_m}(\{\varpi\}) = \mathscr{U}(\{\varpi\})$. Therefore $\mathscr{U}(\{\varpi\}) \neq \mathscr{U}(\{u\})$.*
(⇐) Let $\mathscr{U}(\{\varpi\}) \neq \mathscr{U}(\{u\})$ for every two distinct vertices $\varpi, u \in G$. Then find $w \in G$; $w \in \mathscr{U}(\{\varpi\})$ and $w \notin \mathscr{U}(\{u\})$. Now, suppose thus $\varpi \in \mathscr{U}(\{u\})$, then $\mathscr{U}(\{\varpi\}) \subseteq \mathscr{U}(\{u\})$, since $\varpi \in \mathscr{U}(\{\varpi\})$ and thus $w \in \mathscr{U}(\{u\})$, which is a contradiction. Hence $\varpi \notin \mathscr{U}(\{u\})$ and then $\varpi \in [\mathscr{U}(\{u\})]^c = \mathscr{L}(\{u\}^c)$. But $u \notin [\mathscr{U}(\{u\})]^c = \mathscr{L}(\{u\}^c)$, hence for every two distinct vertices $\varpi, u \in G$ find a subgraph $H = \{u\}^c$ of G ; $\varpi \in \mathscr{L}(V_H) = \mathscr{L}(\{u\}^c)$ and $u \notin \mathscr{L}(V_H) = \mathscr{L}(\{u\}^c)$ (i.e. $u \in V_G - \mathscr{L}(V_H)$). Therefore $\mathscr{G}_m = (G, Cl_{G_m}, \mathscr{F}_{G_m})$ is a $\mathscr{G}_{m0}$-space. □

Definition 6.3. *If $\mathscr{G}_m = (G, Cl_{G_m}, \mathscr{F}_{G_m})$ is a G_m-cl-approx-space. Then $\mathscr{G}_m$ is said to be a rough $\mathscr{G}_{m1}$ space (written as $\mathscr{G}_{m1}$-space), if for every two distinct vertices $\varpi, u \in G$, find two subgraph H and K of G ; $\varpi \in \mathscr{L}(V_H), u \notin \mathscr{L}(V_H)$ and $u \in \mathscr{L}(V_K), \varpi \notin \mathscr{L}(V_K)$.*

Theorem 6.4. *If $\mathscr{G}_m = (G, Cl_{G_m}, \mathscr{F}_{G_m})$ is a G_m-cl-approx-space. Then $\mathscr{G}_m$ is a $\mathscr{G}_{m1}$-space if and only if $\{\varpi\} = \mathscr{U}(\{\varpi\})$ for every $\varpi \in G$.*

Proof . *(⇒) If $\mathscr{G}_m = (G, Cl_{G_m}, \mathscr{F}_{G_m})$ is a $\mathscr{G}_{m1}$-space. Then for every two distinct vertices $\varpi, u \in G$ there exists two subgraph H and K of G such that $\varpi \in \mathscr{L}(V_H), u \notin \mathscr{L}(V_H)$ and $u \in \mathscr{L}(V_K), \varpi \notin \mathscr{L}(V_K)$. Clearly $\varpi \in \{u\}^c$ and $u \in \{\varpi\}^c$, thus for all $u \in \{\varpi\}^c$ find a subgraph K_u of G ; $u \in \mathscr{L}(V_{K_u}) = Int_{G_m}(V_{K_u}) \subseteq \{\varpi\}^c$ and thus $\{\varpi\}^c = \cup_{u\in\{\varpi\}^c}\mathscr{L}(V_{K_u}) = \cup_{u\in\{\varpi\}^c} Int_{G_m}(V_{K_u})$. Hence $\{\varpi\}^c \in \mathscr{T}_{G_m}$, that is $\{\varpi\} \in \mathscr{F}_{G_m}$. Thus $\{\varpi\} = Cl_{G_m}(\{\varpi\}) = \mathscr{U}(\{\varpi\})$. Therefore $\{\varpi\} = \mathscr{U}(\{\varpi\})$ for every $\varpi \in G$.*
(⇐) Let $\{\varpi\} = \mathscr{U}(\{\varpi\})$ for every $\varpi \in G$. Then $\{\varpi\} \in \mathscr{F}_{G_m}$ and $\{\varpi\}^c \in \mathscr{T}_{G_m}$ for every $\varpi \in G$. Thus for every two distinct vertices $\varpi, u \in G$, we get $\varpi \in \{u\}^c, u \notin \{u\}^c$ and $u \in \{\varpi\}^c, \varpi \notin \{\varpi\}^c$ such that $\{\varpi\}^c, \{u\}^c \in \mathscr{T}_{G_m}$. Since $\mathscr{L}(\{\varpi\}^c) = \{\varpi\}^c$ and $\mathscr{L}(\{u\}^c) = \{u\}^c$, then for every two distinct vertices $\varpi, u \in G$ there exists two subgraphs $H = \{u\}^c$ and $K = \{\varpi\}^c$ of G such that $\varpi \in \mathscr{L}(V_H), u \notin \mathscr{L}(V_K)$ and $u \in \mathscr{L}(V_K), \varpi \notin \mathscr{L}(V_K)$. Therefore $\mathscr{G}_m = (G, Cl_{G_m}, \mathscr{F}_{G_m})$ is a $\mathscr{G}_{m1}$-space. □

Theorem 6.5. *If $\mathscr{G}_m = (G, Cl_{G_m}, \mathscr{F}_{G_m})$ is a G_m-cl-approx-space. If $\mathscr{G}_m$ is a $\mathscr{G}_{m1}$-space and $\{v\} = \mathscr{L}(\{v\})$ for all $\varpi \in G$, then for all subgraph H of G, H is an exact graph and $\mathscr{T}_{G_m}$ is the discrete topology.*

Proof . *If $\mathscr{G}_m = (G, Cl_{G_m}, \mathscr{F}_{G_m})$ is a $\mathscr{G}_{m1}$-space. Then by Theorem (6.2), for all $\varpi \in G, \{\varpi\} = \mathscr{U}(\{\varpi\}) = Cl_{G_m}(\{\varpi\})$, hence*

$$\{\varpi\} \in \mathscr{F}_{G_m} and \{\varpi\}^c \in \mathscr{T}_{G_m} \qquad (6.5.1)$$

But it is given that for all $v \in G, \{\varpi\} = \mathscr{L}(\{\varpi\}) = Int_{G_m}(\{\varpi\})$. Hence

$$\{\varpi\} \in \mathscr{T}_{G_m} \text{ and } \{\varpi\}^c \in \mathscr{F}_{G_m} \qquad (6.5.2)$$

Thus for all $\varpi \in G, \{\varpi\}$ is exist graph (i.e. $\mathscr{L}(\{\varpi\}) = \{\varpi\} = \mathscr{U}(\{\varpi\})$).
Let H be any subgraph of G, then $V_H = \cup_{\varpi\in V_H}\{\varpi\}$ and $V_G - V_H = \cup_{u\in V_G - V_H}\{u\}$
Hence $V_H, V_G - V_H \in \mathscr{T}_{G_m}$ since by (6.5.3) V_H and $V_G - V_H$ are union of open graphs. By taking the complement of (6.5.3) we get
$V_G - V_H = \cap_{\varpi\in V_H}\{\varpi\}^c$ and $V_G - (V_G - V_H) = V_H = \cap_{u\in V_G - V_H}\{u\}^c$.
Then $V_G - V_H, V_H \in \mathscr{F}_{G_m}$ since $V_G - V_H, V_H$ are intersection of closed graphs. Therefore H is an exact graph for every subgraph H of G and thus $\mathscr{T}_{G_m}$ is the discrete topology. □

Definition 6.6. *If $\mathscr{G}_m = (G, Cl_{G_m}, \mathscr{F}_{G_m})$ is a G_m-cl-approx-space. Then $\mathscr{G}_m$ is said to be a rough $\mathscr{G}_{m2}$ space (written as $\mathscr{G}_{m2}$-space), if for every two distinct vertices $\varpi, u \in G$, there exists two subgraph H and K of G such that $\varpi \in \mathscr{L}(V_H), u \in \mathscr{L}(V_K)$ and $\mathscr{L}(V_H) \cap \mathscr{L}(V_K) = \phi$.*

Remark 6.7. *The implication between $\mathscr{G}_{m0}$-spaces, $\mathscr{G}_{m1}$-spaces and $\mathscr{G}_{m2}$-spaces are given in the following diagram.*

$$\mathscr{G}_{m2} - spaces \;\; \Rightarrow \;\; \mathscr{G}_{m1} - spaces \;\; \Rightarrow \;\; \mathscr{G}_{m0} - spaces.$$

In general, the converse of Remark 6.7 is not true as an example:

Example 6.8. *If $\mathscr{G}_m = (G, Cl_{G_m}, \mathscr{F}_{G_m})$ is a G_m-cl-approx-spaces;*
$G = (V_G, E_G); V_G = \{\varpi_1, \varpi_2, \varpi_3\}, E_G = \{(\varpi_2, \varpi_3)\}$,
$\mathscr{F}_{G_1} = \{V_G, \phi, \{\varpi_1\}, \{\varpi_3\}, \{\varpi_1, \varpi_3\}, \{\varpi_2, \varpi_3\}\}$,
$\mathscr{T}_{G_1} = \{V_G, \phi, \{\varpi_1\}, \{\varpi_2\}, \{\varpi_1, \varpi_2\}, \{\varpi_2, \varpi_3\}\}$

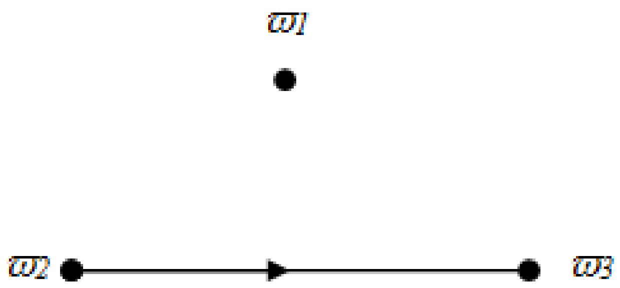

Figure 7: Graph G in Exam. 6.8.

7. Near Rough Separation Axioms in G_m-Closure Approximation Spaces

The concepts of near approximation have an important role in separation axioms. By using these concepts we can construct many several separation axioms. The following definition introduces some new separation axioms.

Definition 7.1. *If $\mathscr{G}_m = (G, Cl_{G_m}, \mathscr{F}_{G_m})$ is a G_m-cl-approx-space. Then for all $j = r, s, p, \gamma, \alpha, \beta$, $\mathscr{G}_m$ is said to be near rough $\mathscr{G}^j_{m0}$ space (written as $\mathscr{G}^j_{m0}$-space), if for every two distinct vertices $\varpi, u \in G$, find a subgraph $H \subseteq G$; either $\varpi \in \mathscr{L}^j(V_H), u \in V_G - \mathscr{L}^j(V_H)$ or $u \in \mathscr{L}^j(V_H), \varpi \in V_G - \mathscr{L}^j(V_H)$.*

Theorem 7.2. *If $\mathscr{G}_m = (G, Cl_{G_m}, \mathscr{F}_{G_m})$ is a G_m-cl-approx-space. Then $\mathscr{G}_m$ is a $\mathscr{G}^j_{m0}$-space for all $j = s, p, \gamma, \alpha, \beta$ iff $\mathscr{U}^j(\{\varpi\}) \neq \mathscr{U}^j(\{u\})$ for every two distinct vertices $\varpi, u \in G$.*
Proof . *For $j = \beta$: Now,*
($\Rightarrow$) If $\mathscr{G}_m = (G, Cl_{G_m}, \mathscr{F}_{G_m})$ is a $\mathscr{G}^j_{m0}$-space. Then for every two distinct vertices $\varpi, u \in G$ find a subgraph $H \subseteq G$; either $\in \mathscr{L}^\beta(V_H) = Int^\beta_{G_m}(V_H)$ and $u \notin L^\beta(V_H) = Int^\beta_{G_m}(V_H)$. Thus $\varpi \notin [Int^\beta_{G_m}(V_H)]^c, u \in [Int^\beta_{G_m}(V_H)]^c$ and $[Int^\beta_{G_m}(V_H)]^c \in \beta C_{G_m}(G)$. Hence $\varpi \notin \cap\{K; V_K \subseteq [Int^\beta_{G_m}(V_H)]^c, K \in \beta C_{G_m}(G), \{u\} \subseteq H\} = Cl^\beta_{G_m}(\{u\}) = \mathscr{U}^\beta(\{u\})$. Thus $v \notin \mathscr{U}^\beta(\{u\})$, but $\varpi \in Cl^\beta_{G_m}(\{\varpi\}) = \mathscr{U}^\beta(\{\varpi\})$. Therefore $\mathscr{U}^\beta(\{\varpi\}) \neq \mathscr{U}^\beta(\{u\})$.
($\Leftarrow$) Let $\mathscr{U}^\beta(\{\varpi\}) \neq \mathscr{U}^\beta(\{u\})$ for every two distinct vertices $\varpi, u \in G$. Then find $w \in G$;

$w \in \mathscr{U}^{\beta}(\{\varpi\})$ and $w \notin \mathscr{U}^{\beta}(\{u\})$. Now, suppose that $\varpi \in \mathscr{U}^{\beta}(\{u\})$, then $\mathscr{U}^{\beta}(\{v\}) \subseteq \mathscr{U}^{\beta}(\{u\})$, since $\varpi \in \mathscr{U}^{\beta}(\{v\})$, and thus $w \in \mathscr{U}^{\beta}(\{u\})$, which is a contradiction. Hence $\varpi \notin \mathscr{U}^{\beta}(\{u\})$ and then $\varpi \in [\mathscr{U}^{\beta}(\{u\})]^c = \mathscr{L}^{\beta}(\{u\}^c)$. But $u \notin [\mathscr{U}^{\beta}(\{u\})]^c = \mathscr{L}^{\beta}(\{u\}^c)$, hence for every two distinct vertices $\varpi, u \in G$ find a subgraph $H = \{u\}^c$ of G ; $v \in \mathscr{L}^{\beta}(V_H) = \mathscr{L}^{\beta}(\{u\}^c)$ and $u \notin \mathscr{L}^{\beta}(V(H)) = \mathscr{L}^{\beta}(\{u\}^c)$. Therefore $\mathscr{G}_m = (G, Cl_{G_m}, \mathscr{F}_{G_m})$ is a $\mathscr{G}^{\beta}_{m0}$-space. □

The proofs of the other cases are similar

Definition 7.3. *Let $\mathscr{G}_m = (G, Cl_{G_m}, \mathscr{F}_{G_m})$ be a G_m-cl-approx-space. Then for all $j = r, s, p, \gamma, \alpha, \beta$, $\mathscr{G}_m$ is said to be near rough $\mathscr{G}^{j}_{m1}$ space (written as $\mathscr{G}^{j}_{m1}$-space), if for every two distinct vertices $\varpi, u \in G$, there exists two subgraph H and K of G such that $\varpi \in \mathscr{L}^{j}(V_H), u \notin \mathscr{L}^{j}(V_H)$ and $u \in \mathscr{L}^{j}(V_K), \varpi \notin \mathscr{L}^{j}(V_K)$.*

Theorem 7.4. *If $\mathscr{G}_m = (G, Cl_{G_m}, \mathscr{F}_{G_m})$ is a G_m-cl-approx-space. Then G_m is a $\mathscr{G}^{j}_{m1}$-space for all $j = s, p, \gamma, \alpha, \beta$ iff $\{\varpi\} = \mathscr{U}^{j}(\{v\})$ for every $\varpi \in G$.*

Proof . *For $j = \alpha$: Now,*

(⇒) If $\mathscr{G}_m = (G, Cl_{G_m}, \mathscr{F}_{G_m})$ is a $\mathscr{G}^{\alpha}_{m1}$-space. Then for every two distinct vertices $\varpi, u \in G$ find two subgraph H and K of G ; $\varpi \in \mathscr{L}^{\alpha}(V_H), u \notin \mathscr{L}^{\alpha}(V_H)$ and $u \in \mathscr{L}^{\alpha}(V_K), \varpi \notin \mathscr{L}^{\alpha}(V_K)$. Clearly $\alpha \in \{u\}^c$ and $u \in \{v\}^c$, thus for all $u \in \{\varpi\}^c$ find a subgraph K_u of G ; $u \in \mathscr{L}^{\alpha}(V_{K_u}) = Int^{\alpha}_{G_m}(V_{K_u}) \subseteq \{\varpi\}^c$ and thus $\{\varpi\}^c = \cup^{c}_{u\in\{\varpi\}}\mathscr{L}^{\alpha}(V_{K_u}) = \cup_{u\in\{\varpi\}^c} Int^{\alpha}_{G_m}(V_{K_u})$. Hence $\{\varpi\}^c$ is an α-open graph, that is $\{\varpi\}$ is an α-closed graph. Thus $\{\varpi\} = Cl^{\alpha}_{G_m}(\{\varpi\}) = \mathscr{U}^{\alpha}(\{\varpi\})$. Therefore $\{\varpi\} = \mathscr{U}^{\alpha}(\{\varpi\})$ for every $\varpi \in G$.

(⇐) Let $\{\varpi\} = \mathscr{U}^{\alpha}(\{v\})$ for every $\varpi \in G$. Then $\{\varpi\}$ is an α-closed graph and $\{\varpi\}^c$ is an α-open graph for all $\varpi \in G$. Thus for every two distinct vertices $\varpi, u \in G$, we get $\varpi \in \{u\}^c, u \notin \{u\}^c$ and $u \in \{\varpi\}^c, \varpi \notin \{\varpi\}^c$ such that $\{\varpi\}^c, \{u\}^c$ are α-open graphs. Since $\mathscr{L}^{\alpha}(\{\varpi\}^c) = \{\varpi\}^c$ and $\mathscr{L}^{\alpha}(\{u\}^c) = \{u\}^c$, then for every two distinct vertices $\varpi, u \in G$ there exists two subgraph $H = \{u\}^c$ and $K = \{\varpi\}^c$ such that $\varpi \in \mathscr{L}^{\alpha}(V(H)), u \notin \mathscr{L}^{\alpha}(V_K)$ and $u \in \mathscr{L}^{\alpha}(V_K), \varpi \notin \mathscr{L}^{\alpha}(V_K)$. Therefore $\mathscr{G}_m = (G, Cl_{G_m}, \mathscr{F}_{G_m})$ is a $\mathscr{G}^{\alpha}_{m1}$-space. □

The proofs of the other cases are similar

Theorem 7.5. *If $\mathscr{G}_m = (G, Cl_{G_m}, \mathscr{F}_{G_m})$ is a G_m-cl-approx-space. If G_m is a $\mathscr{G}^{j}_{m1}$-space and $\{\varpi\} = \mathscr{L}^{j}(\{\varpi\})$ for all $\varpi \in G$ and $j = s, p, \gamma, \alpha, \beta$, then for every subgraph H of G, H is a j-exact graph and the family of all j-open graphs is the discrete topology.*

Proof . *For $j = p$: Now, let $\mathscr{G}_m = (G, Cl_{G_m}, \mathscr{F}_{G_m})$ be a $\mathscr{G}^{P}_{m1}$-space. Then by Theorem 7.4, for all $\varpi \in G, \{\varpi\} = \mathscr{U}^{p}(\{\varpi\}) = Cl^{p}_{G_m}(\{\varpi\})$, hence*

$$\{\varpi\} \in p - C_{G_m}(G) \text{ and } \{\varpi\}^c \in p - O_{G_m}(G) \tag{7.1}$$

But it is given that for all $\varpi \in G, \{\varpi\} = \mathscr{L}^{p}(\{\varpi\}) = Int^{p}_{G_m}(\{\varpi\})$. Hence

$$\{\varpi\} \in p - O_{G_m}(G) \text{ and } \{\varpi\}^c \in p - C_{G_m}(G) \tag{7.2}$$

Thus for all $\varpi \in G, \{\varpi\}$ is a p-exist graph (i.e. $\mathscr{L}^{p}(\{\varpi\}) = \{\varpi\} = \mathscr{U}^{p}(\{\varpi\})$). Let H be any subgraph of G, then

$$V_H = \cup_{\varpi\in V_H}\{\varpi\} \text{ and } V(G) - V_H = \cup_{u\in V_G - V_H}\{u\} \tag{7.3}$$

Hence $V_H, V_G - V_H \in p - O_{G_m}(G)$ since by (7.3) V_H and $V - G - V_H$ are union of p-open graphs. By taking the complement of 7.3) we get

$V_G - V_H = \cap_{\varpi \in V_H}\{\,\varpi\,\}^c$ *and* $V_G - (V_G - V_H) = V_H = \cap_{u \in V_G - V_H}\{\,u\,\}^c$.
Then $V_G - V_H, V_H \in p - C_{G_m}(G)$ *since* $V_G - V_H, V_H$ *are intersection of p-closed graphs. Therefore* H *is a p-exact graph for every subgraph* H *of* G *and thus the family of all p-open graphs is the discrete topology.* □

The proofs of the other cases are similar.

Definition 7.6. *If* $\mathscr{G}_m = (G, Cl_{G_m}, \mathscr{F}_{G_m})$ *is a* G_m*-cl-approx-space. Then for all* $j = r, s, p, \gamma, \alpha, \beta$, $\mathscr{G}_m$ *is said to be a near rough* $\mathscr{G}_{m2}$ *space (written as* $\mathscr{G}^j_{m2}$*-space), if for every two distinct vertices* $\varpi, u \in G$*, there exists two subgraph* H *and* K *of* G *such that* $\varpi \in \mathscr{L}^j(V_H), u \in \mathscr{L}^j(V_K)$ *and* $\mathscr{L}^j(V_H) \cap \mathscr{L}^j(V_K) = \phi$.

Remark 7.7. *The implication between* $\mathscr{G}^j_{m0}$*-spaces,* $\mathscr{G}^j_{m1}$*-spaces and* $\mathscr{G}^j_{m2}$*-spaces for all* $j = r, s, p, \gamma, \alpha, \beta$ *are given in the diagram.*

$$\mathscr{G}^j_{m2}\text{-spaces} \;\Rightarrow\; \mathscr{G}^j_{m1}\text{-spaces} \;\Rightarrow\; \mathscr{G}^j_{m0}\text{-spaces.}$$

In general, the converse of Remark 7.7 is not true. Example 7.8 illustrated that the converse of Remark 7.7 is not true if $j = p$.

Example 7.8. *If* $\mathscr{G}_m = (G, Cl_{G_m}, \mathscr{F}_{G_m})$ *is a* G_m*-cl-approx-spaces in example 2.2.*
$\mathscr{F}_{G_1} = \{\,V_G, \phi, \{\,\varpi_3\,\}, \{\,\varpi_3, \varpi_4\,\}, \{\,\varpi_1, \varpi_2, \varpi_3\,\}\,\}$, $\mathscr{T}_{G_1} = \{\,V_G, \phi, \{\,\varpi_4\,\}, \{\,\varpi_1, \varpi_2\,\}, \{\,\varpi_1, \varpi_2, \varpi_4\,\}$ *and* $p - O_{G1}(G) = \{\,V_G, \phi, \{\,\varpi_1\,\}, \{\,\varpi_2\,\}, \{\,\varpi_4\,\}, \{\,\varpi_1, \varpi_2\,\}, \{\,\varpi_1, \varpi_4\,\}, \{\,\varpi_2, \varpi_4\,\}, \{\,\varpi_1, \varpi_2, \varpi_4\,\},$
$\{\,\varpi_1, \varpi_3, \varpi_4\,\}, \{\,\varpi_2, \varpi_3, \varpi_4\,\}\,\}$.
Hence $\mathscr{G}_m = (G, Cl_{G_m}, \mathscr{F}_{G_m})$ *is a* $\mathscr{G}^P_{m0}$*-space, but it is not a* $\mathscr{G}^P_{m1}$*-space.*

Theorem 7.9. *The implications between* $\mathscr{G}_{mi}$*-spaces and* $\mathscr{G}^j_{mi}$*-spaces for all* $i \in \{\,0, 1, 2\,\}$ *and* $j \in \{\,r, s, p, \gamma, \alpha, \beta\,\}$ *are given by the following diagram.*

$$\begin{array}{ccccc} \mathscr{G}_{m2}\text{-spaces} & \Rightarrow & \mathscr{G}_{m1}\text{-spaces} & \Rightarrow & \mathscr{G}_{m0}\text{-spaces.} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \mathscr{G}^j_{m2}\text{-spaces} & \Rightarrow & \mathscr{G}^j_{m1}\text{-spaces} & \Rightarrow & \mathscr{G}^j_{m0}\text{-spaces.} \end{array}$$

Proof . *Using Remark 6.7 (resp. Remark 7.7), it is clear that*
$\mathscr{G}_{mi}$*-spaces* $\Rightarrow \mathscr{G}_{mi}$*-spaces* $\Rightarrow \mathscr{G}_{mi}$*-spaces.*
(resp. $\mathscr{G}^j_{mi}$*-spaces* $\Rightarrow \mathscr{G}^j_{mi}$*-spaces* $\Rightarrow \mathscr{G}^j_{mi}$*-spaces. for all* $j \in \{\,s, p, \gamma, \alpha, \beta\,\}$*);*
Now, we shall prove that $\mathscr{G}_{m0}$*-spaces* $\Rightarrow \mathscr{G}^j_{m0}$*-spaces for all* $j \in \{\,s, p, \gamma, \alpha, \beta\,\}$*. Let* $\mathscr{G}_m = (G, Cl_{G_m}, \mathscr{F}_{G_m})$ *be a* $\mathscr{G}_{m0}$*-space. Then for every two distinct vertices* $\varpi, u \in G$ *find a subgraph* H *of* G *;* $\varpi \in \mathscr{L}(V_H) = Int_{Gm}(V_H)$ *and* $u \notin \mathscr{L}(V_H) = Int_{G_m}(V_H)$*. But* $Int_{Gm}(V_H) \in \mathscr{T}_{Gm}$*, then* $Int_{Gm}(V_H)$ *is a j-open graph for all* $j \in \{\,s, p, \gamma, \alpha, \beta\,\}$*. Hence* $Int_{G_m}(V_H) = Int^j_{Gm}(V_H) = \mathscr{L}^j(V_H)$*. Thus for every two distinct vertices* $\varpi, u \in G$ *find a subgraph* H *of* G *;* $\varpi \in \mathscr{L}^j(V_H)$ *and* $u \notin \mathscr{L}^j(V_H)$*. Therefore* $\mathscr{G}_m = (G, Cl_{G_m}, \mathscr{F}_{G_m})$ *is a* $\mathscr{G}^j_{m0}$*-space for all* $j \in \{\,s, p, \gamma, \alpha, \beta\,\}$*.*
Similarly we can show that
$\mathscr{G}^j_{mi}$*-spaces* $\Rightarrow \mathscr{G}^j_{mi}$*-spaces* $\Rightarrow \mathscr{G}^j_{mi}$*-spaces. for all* $j \in \{\,s, p, \gamma, \alpha, \beta\,\}$ □

In general, the converse of Theorem 7.9 is not true, as illustrated by the following example.

Example 7.10. *Using the same* G_m*-cl-approx-space* $\mathscr{G}_m = (G, Cl_{G_m}, \mathscr{F}_{G_m})$ *which is given in example (7.8), we get*
$\mathscr{F}_{G_1} = \{V_G, \phi, \{\varpi_3\}, \{\varpi_3, \varpi_4\}, \{\varpi_1, \varpi_2, \varpi_3\}\}$, $\mathscr{T}_{G_1} = \{V_G, \phi, \{\varpi_4\}, \{\varpi_1, \varpi_2\}, \{\varpi_1, \varpi_2, \varpi_4\}$ *and* $\beta - O_{G1}(G) = \{\,V_G, \phi, \{\,\varpi_1\,\}, \{\varpi_2\}, \{\varpi_4\}, \{\varpi_1, \varpi_2\}, \{\varpi_1, \varpi_3\}, \{\varpi_1, \varpi_4\}, \{\varpi_2, \varpi_3\,\}, \{\,\varpi_2, \varpi_4\,\}, \{\,\varpi_3, \varpi_4\,\},$
$\{\,\varpi_1, \varpi_2, \varpi_3\,\}, \{\,\varpi_1, \varpi_2, \varpi_4\,\}, \{\varpi_1, \varpi_3, \varpi_4\}, \{\varpi_2, \varpi_3, \varpi_4\}\}$.
Hence $\mathscr{G}_m = (G, Cl_{G_m}, \mathscr{F}_{G_m})$ *is a* $\mathscr{G}^\beta_{10}$*-space, but it is not a* $\mathscr{G}_{10}$*-space.*

Theorem 7.11. *The implications between $\mathscr{G}_{mi}^{j}$-spaces for all $i \in \{0,1,2\}$ and $j \in \{s,p,\gamma,\alpha,\beta\}$ are given by the following diagram.*

$$\begin{array}{ccccc} \mathscr{G}_{mi}^{\alpha}\text{-spaces} & \Rightarrow & \mathscr{G}_{mi}^{S}\text{-spaces.} & & \\ \Downarrow & & \Downarrow & & \\ \mathscr{G}_{mi}^{P}\text{-spaces} & \Rightarrow & \mathscr{G}_{mi}^{\gamma}\text{-spaces} & \Rightarrow & \mathscr{G}_{mi}^{\beta}\text{-spaces.} \end{array}$$

Proof . *Using Proposition 5.4, the proof is similar to Theorem 7.9.* □

In general, the converse of Theorem 7.11 is not true, as illustrated by the following example.

Example 7.12. *Using the same G_m-cl-approx-space $\mathscr{G}_m = (G, Cl_{G_m}, \mathscr{F}_{G_m})$ which is given in example 7.8, we get*
$s - O_{G_1}(G) = \{V_G, \phi, \{\varpi_4\}, \{\varpi_1, \varpi_2\}, \{\varpi_3, \varpi_4\}, \{\varpi_1, \varpi_2, \varpi_3\}, \{\varpi_1, \varpi_2, \varpi_4\}\}$,
$\beta - O_{G1}(G) = \{V_G, \phi, \{\varpi_1\}, \{\varpi_2\}, \{\varpi_4\}, \{\varpi_1, \varpi_2\}, \{\varpi_1, \varpi_3\}, \{\varpi_1, \varpi_4\}, \{\varpi_2, \varpi_3\}, \{\varpi_2, \varpi_4\}, \{\varpi_3, \varpi_4\}, \{\varpi_1, \varpi_2, \varpi_3\}, \{\varpi_1, \varpi_2, \varpi_4\}, \{\varpi_1, \varpi_3, \varpi_4\}, \{\varpi_2, \varpi_3, \varpi_4\}\}$.
Hence $\mathscr{G}_m = (G, Cl_{G_m}, \mathscr{F}_{G_m})$ is a $\mathscr{G}_{10}^{\beta}$-space, but it is not a $\mathscr{G}_{10}$-space.

8. Conclusions

The continuity in the G_m-cl-approx-spaces is useful, since it connected two different G_m-cl-approx-spaces and this helps to make a comparison between the G_m-cl-approx-spaces. Also, the separation axioms which introduce in this paper considered as tools for separate the vertices under certain condition.

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