Int. J. Nonlinear Anal. Appl. 12 (2021) No. 2, 843-848
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.5141

A topology on a ring part of IS-algebra

Fatema F. Kareem[a], Reyadh. D. Ali[b]

[a]Department of Mathematics, College of Education for Pure Science,Ibn Al-Haithem, University of Baghdad, Baghdad, Iraq
[b]Department of Mathematics, College of Education for Pure Science,University of Karbala, Karbala, Iraq

(Communicated by Madjid Eshaghi Gordji)

Abstract

We give a topology on a ring part of IS-algebra, by define prime ideals of a commutative ring part of IS-algebra and study some of its properties.

Keywords: IS-algebra; adjoin ring part; prime ideal.
2010 MSC: 54A05, 06F35, 03E72

1. Introduction

In1966 [2], the notion of BCI-algebra was introduced by Iseki and Imai. A new class of algebra related to BCI-algebra was introduced by Jun and Hong [4], called a BCI-semigroup. After that, Jun et al. [5] renamed the BCI-semigroup as the IS-algebra and studied further properties of this algebra. In [10] the authors gave the ring part and adjoin ring part of IS-algebra. The concept of the topology was applied to a lot of algebraic structures by several authors; see [1, 6, 8, 7, 9]. This paper is intended to implement the new notion of adjoin ring part of IS-algebra and discusses some of their properties to study a topology on this structure.

2. Preliminaries

Definition 2.1. *[2, 3] Algebra $(L, *, 0)$ of (2, 0) is a BCI-algebra, if,$\forall$ $m, n, s \in L$*

$$(BCI_1)((m * n) * (m * s)) \leq (s * n)$$

$$(BCI_2)(m * (m * n)) \leq n$$

$$(BCI_3)m \leq m$$

(BCI_4) $m \leq n$ and $n \leq m$ imply $m = n$.

*Corresponding author
Email addresses: fa_sa20072000@yahoo.com (Fatema F. Kareem), eyadhdelphi@gmail.com (Reyadh. D. Ali)

Received: March 2021 *Accepted:* May 2021

In a BCI -algebra $(L, *, 0)$, the following properties are satisfied:

$$(BCI_{1\backslash}) : m * 0 = m$$
$$(BCI_{2\backslash}) : (m * n) * s = (m * s) * n$$
$$(BCI_{3\backslash}) : m * (m * (m * n)) = m * n$$
$$(BCI_{4\backslash}) : 0 * (0 * m) = m$$
$$(BCI_{5\backslash}) : 0 * (m * n) = (0 * m) * (0 * n).$$

Definition 2.2. *[3] Let $(L, *, 0)$ be a BCI-algebra and $\emptyset = I \subseteq L$, I is called an ideal of L if it satisfies the following conditions:*

(i) $0 \in I$,
(ii) $x * y \in I$ and $y \in I$ imply $x \in I$ (here x, $y \in L$).

Definition 2.3. *[4] An IS-algebra $(L, *, \bullet, 0)$ is $L \neq \phi$ with two binary operations $*, \bullet$ and constant 0 such that, $\forall$ $m, n, s \in L$*

(I) $(L, *, 0)$ is a BCI-algebra,
(II) $(L, \bullet)$ is a semigroup,
(III) $m \bullet (n * s) = (m \bullet n) * (m \bullet s)$ and $(m * n) \bullet s = (m \bullet s) * (n \bullet s)$.

Example 2.4. *[4] If$L = \{0 , e , f , g , h\}$is a set with the two operations $*$ and $\circ$ given by :*

$*$	0	e	f	g	h
0	0	e	0	0	h
e	e	0	e	e	0
f	f	f	0	0	0
g	g	g	g	0	0
h	h	h	h	h	0

$\circ$	0	e	f	g	h
0	0	0	0	0	0
e	0	0	0	0	0
f	0	0	0	0	f
g	0	0	0	f	g
h	0	e	f	g	h

Then $(L, *, \circ, 0)$ is an IS-algebra (by routine calculations).
In $(L, *, \circ, 0)$, we have $v \circ 0 = 0 \circ v = 0$, for any $v \in L$.

Lemma 2.5. *[4] Let $(L, *, \bullet, 0)$ be an IS-algebra. Then for any $v, w, r \in L$,, we have: $v \leq w$ implies $v \bullet r \leq w \bullet r$ and $r \bullet v \leq r \bullet w$.*

Definition 2.6. *[5] If $I \neq \phi$is a subset of an IS-algebra $(L, *, \bullet, 0)$. Then $I$is called an ideal of $(L, *, \bullet, 0)$, if*

(I_1) $v * w \in I$ and $w \in I$, thenv $\in I$, $\forall v, w \in L$
(I_2) for any $v \in L$andr $\in I$, we have $v \bullet r \in I, r \bullet v \in I$.

Definition 2.7. *[10]If $(L, *, \bullet, 0)$ is an IS-algebra, then $K(L) = \{v \in L | 0 * v = v\}$is said to be ring part of L.*

Theorem 2.8. *[10] In IS-algebra$(L, *, \bullet, 0)$:*

1. *$K(L)$is a subalgebra of $(L, *, 0)$,*
2. *$(K(L), *, \bullet)$is a maximal ring .*
3. *$K(L)$is an ideal of $(L, *, \bullet, 0) \Leftrightarrow K(L)$is an ideal of BCI-algebra$(L, *, 0)$.*

3. A Topology on prime ideals of adjoin ring part

In this section, we study the prime spectrum$spec(N)$ of a ring part of an IS-algebra $(L, *, \bullet, 0)$. It turns out $spec(N)$ is T_0 and T_1-space. Moreover $f : spec(N) \to spec(K)$ is a continuous map.

Definition 3.1. *For every nonempty subset B of L, we define $N(L) = \{v \in L | b * (b * (b \bullet v)) \leq b \bullet v, \forall b \in B\}$, which will be called adjoin ring part of $L.N(L)$ in usual will be written $N$for short.*

Theorem 3.2. *In IS-algebra $(L, *, \bullet, 0)$:*

(a) $N$is a subalgebra of $(L, *, \bullet, 0)$
(b) If $m + n = m * (0 * n)$, then $(N, +, \bullet)$is a ring and $m + n = n + m, (m + n) + s = m + (n + s)$.

Proof .

(a) Since $0 \in N$, so $N \neq \phi$. For any $v, w \in N$, we get $d*(d*(d\bullet(m*n))) = (d*0)*(d*(d\bullet(m*n)) \leq (d \bullet (m * n)) * 0 = d \bullet (m * n)$ .... by $BCI_{1'}$, that is $m * n \in N$. In addition, we get $d * (d * (d \bullet (m \bullet n))) = (d * 0) * (d * (d \bullet (m \bullet n)) \leq (d \bullet (m \bullet n)) * 0 = d \bullet (m \bullet n)$ by... $BCI_{1'}$, that is $m \bullet n \in N$. Similarly we get $n \bullet m \in N$.

(b) $(N, *, 0)$ is BCI-algebra by (a), $\forall m, n, s \in N$, we get $d * (d * (d \bullet (m + n))) = (d * 0) * (d * (d \bullet (m * (0 * n))) \leq (d \bullet (m * (0 * n) * 0 \leq d \bullet (m * (0 * n))$, then $(m * (0 * n)) \in N$. So $m + n \in N$. In addition, since $m+n = m*(0*n) = (0*(0*m))*(0*n) = (0*(0*n))*(0*m) = n*(0*m) = n + m$, and$m + (n + s) = m * (0 * (n + s)) = m * (0 * (n * (0 * s))) = (n * (0 * s)) * (0 * m) = (n * (0 * m)) * (0 * s)(m * (0 * n) * (0 * s)) = (m + n) + s$.
Therefore " +" is associative and also commutative. Moreover,
$m + 0 = 0 + m = 0 * (0 * m) = m$ and$m + (0 * m) = (0 * m) + m = (0 * m) * (0 * m) = 0$, hence $0 * m$ is the inverse of m. Thus $(N, +)$is an abelian group.
Also, since N is closed about $\bullet$on IS-algebra $(L, *, \bullet, 0)$, so
$m\bullet(n+s) = m\bullet(n*(0*s)) = m\bullet n*(m\bullet(0*s)) = m\bullet n*(m\bullet 0*m\bullet s) = m\bullet n*(0*m\bullet s) = m \bullet n + m \bullet s$ in same reason $(m + n) \bullet s = m \bullet s + n \bullet s$, hence $(N, +, \bullet)$ is a ring.

□

Lemma 3.3. *N is an ideal of an IS-algebra $(L, *, \bullet, 0) \Leftrightarrow N$ is an ideal of a BCI-algebra $(L, *, 0)$*
Proof .*If N is an ideal of an IS-algebra $(L, *, \bullet, 0)$, then by definition above, N is an ideal of a BCI-algebra $(L, *, 0)$. Conversely, suppose N is an ideal of a BCI-algebra $(L, *, 0)$, $\forall v \in L, r \in N$ $d * (d * (d \bullet (r \bullet v))) = (d * 0) * (d * (d \bullet r) \bullet v)) \leq ((d \bullet r) \bullet v) * 0 = ((d \bullet r) \bullet v)$ .... by BCI_1, Therefore $r \bullet v \in N$, in same reasoning $v \bullet r \in N$, hence N is an ideal of an IS-algebra $(L, *, \bullet, 0)$.* □

Definition 3.4. *Let N be a ring part of IS-algebra $(L, *, \bullet, 0)$. A proper ideal J of N is called a prime if $cd \in J$ for elements c and d of N, either $c \in J$ or $d \in J$.*

Definition 3.5. *Let N be a ring part of IS-algebra $(L, *, \bullet, 0)$ and spec(N) be the collection of all prime ideals of N. Now for each ideal Y of N, we define the variety of Y by $V(Y) = \{J \in spec(N) | Y \subseteq J\}$,Therefore $V(N) = \phi$ and $V(\{0\}) = spec(N)$.*

Theorem 3.6. *Let $(L, *, \bullet, 0)$ be an IS-algebra and N be a ring part of L. If Y and $H$are two ideals of N,then*

$$H \subseteq Y \Rightarrow V(Y) \subseteq V(H). \tag{3.1}$$

$$V(Y) \cup V(H) \subseteq V(Y \cap H). \tag{3.2}$$

Proof .

(3.1) *If $O \in V(Y)$, then $Y \subseteq O$ and since $H \subseteq Y$, therefore $O \in V(H)$. It follows that $V(Y) \subseteq V(H)$.*

(3.2) *Let $O \in V(Y) \cup V(H)$, then $Y \subseteq O$or $H \subseteq O$. Hence $Y \cap H \subseteq O$, therefore $O \in V(Y \cap H)$. It follows that $V(Y) \cup V(H) \subseteq V(Y \cap H)$.*

□

Lemma 3.7. *Let $(N, +, \bullet)$ be a ring of IS-algebra $(L, *, \bullet, 0)$, For any $Y_i(i \in I)$ of an ideals of N. Then $\cap_{i \in I} V(Y_i) = V(\sum_{i \in I} Y_i)$.*
Proof . *Let $J \in \cap_{i \in I} V(Y_i)$, then$Y_i \subseteq J, \forall i \in I$, hence $\sum Y_i \subseteq J$. So $J \in V(\sum Y_i)$, It follows that $\cap_{i \in I} V(Y_i) \subseteq V(\sum_{i \in I} Y_i)$.*
Now, if $J \in V(\sum Y_i)$.So$\sum Y_i \subseteq J$and since $Y_i \subseteq \sum Y_i$, for $i \in I$, hence $Y_i \subseteq J$, then $J \in V(Y_i)$, for $i \in I$. It follows that $J \in \cap_{i \in I} V(Y_i)$. □

Definition 3.8. *Let N be a ring part of an IS-algebra$L$. Then a prime ideal$J$ of N is extraordinary if for any two ideals Y and $H$of N,$Y \cap H \subseteq J$ implies $Y \subseteq J$ or $H \subseteq J$.*

Theorem 3.9. *Let N be a ring part of an IS-algebra $(L, *, \bullet, 0)$. If every prime ideal of $N$is an extraordinary, then $V(Y) \cup V(H) = V(Y \cap H)$, for any two ideals Y and H of N.*
Proof . *By Theorem 3.6, $V(Y) \cup V(H) \subseteq V(Y \cap H)$.Now, let$J \in V(Y \cap H)$, then $Y \cap H \subseteq J$ and since J is extraordinary. Then $Y \subseteq J$ or $H \subseteq J \Rightarrow J \in V(Y)$ or $J \in V(H) \Rightarrow J \in V(Y) \cup V(H) \Rightarrow V(Y \cap H) \subseteq V(Y) \cup V(H)$. Hence $V(Y) \cup V(H) = V(Y \cap H)$.* □

By Definition 3.5, Lemma 3.7 and Theorem 3.9, it follows that the family $\{V(Y)\}_{Y \subseteq N}$of subsets of $spec(N)$ satisfies the axioms for closed sets in a topological space. The topological space $spec(N)$ is called the prime spectrum of Nalso the resulting topology is called the Zariski topology.

Example 3.10. *Let $(\mathrm{Z}_6, +, \bullet)$ be a ring part of IS-algebra $(\mathrm{Z}_6, -, \bullet, 0)$, then the set of all ideals of $\mathrm{Z}_6$are $\{\{\overline{0}\}, \{\overline{0}, \overline{3}\}, \{\overline{0}, \overline{2}, \overline{4}\}, \mathrm{Z}_6\}$. Now, the set$\{\{\overline{0}, \overline{3}\}, \{\overline{0}, \overline{2}, \overline{4}\}\}$ is all prime ideals of $\mathrm{Z}_6$and that is extraordinary, hence $Spec(\mathrm{Z}_6) = \{\{\overline{0}, \overline{3}\}, \{\overline{0}, \overline{2}, \overline{4}\}\}$. Therefore the topology on spectrum is $\tau = \{\phi, spec(\mathrm{Z}_6)\}$.*

Remark 3.11. *For any $Y \subseteq N$, we're denoting the complement of $V(Y)$ by $W(Y)$. So $W(Y) = \{J \in spec(N) | Y \not\subseteq J\}$, so the collection $\{W(Y)\}_{Y \subseteq N}$ is the collection of open sets of a topological space $Spec(N)$. By duality, we get the following:*

Proposition 3.12. *Let N be a ring part of IS-algebra$(L, *, \bullet, 0)$, then*

(i) $W(N) = Spec(N), W(\{0\}) = \phi$,
(ii) If $\{Y_i\}_{i \in I}$ is any family ideals of$N$, then $\cup_{i \in I} W(Y_i) = W(\cup_{i \in I} Y_i)$,
(iii) $W(Y_1 \cap Y_2) = W(Y_1) \cap W(Y_2)$, for some ideals $Y_1, Y_2 \subseteq N$
(iv) For any two ideals $Y, H \in N, Y \subseteq H \Rightarrow W(Y) \subseteq W(H)$.

Proof .*Clear.* □

Remark 3.13. *For any $b \in N$,we denote $V(\{b\})$ by $V(b)$ and $W(\{b\})$ by $W(b)$.So $V(b) = \{J \in spec(N) | b \in J\}$and $W(b) = \{J \in spec(N) | b \notin J\}$.*

Theorem 3.14. *If N is a ring part of IS-algebra $(L, *, \bullet, 0)$, the collection $\{W(b)\}_{b\in N}$ is a basis for the topology on $Spec(N)$.*

Proof .*If $Y \subseteq N$,$W(Y)$an open and $W(Y) \subseteq Spec(N)$, then by proposition 3.12, we get $W(Y) = W(\cup_{b\in Y}\{b\}) = \cup_{b\in Y}W(b)$. Hence, any open set of $Spec(N)$is the union of subsets from the collection $\{W(b)\}_{b\in N}$.* □

Theorem 3.15. *$Spec(N)$is a T_0 topological space.*

Proof .*Let$J$ and Q be any two distinct prime ideals in $Spec(N)$. Then either $J \not\subset Q$or $Q \not\subset J$.
If $J \not\subset Q \Rightarrow \exists\ b \in J \ni b \notin Q \Rightarrow Q \in W(b)$ and $J \notin W(b)$
$\Rightarrow\exists$an open set $W(b)$containing Q, but not J.
If $Q \not\subset J \Rightarrow \exists b \in Q \ni b \notin J \Rightarrow Q \notin W(b)andJ \in W(b)$.
$\Rightarrow\exists$an open set $W(b)$containing J, but not Q.
Hence $Spec(N)$is a T_0-space.* □

Theorem 3.16. *$Spec(N)$is a T_1 topological space.*

Proof .*If $Spec(N) = \phi \Rightarrow spec(N)$ is trivial space and so it is a T_1-space. Now, if $Spec(N) \neq \phi$, then there exist J prime ideal of $Spec(N)$,$V(J) = \{J\}$and so $\{J\}$is closed set in $Spec(N)$,i.e. $Spec(N)$is a T_1-space.* □

Proposition 3.17. *If$l : N \to K$is a homomorphism of two ring parts $N$and $K$of IS-algebra$(L, *, \bullet, 0)$, then $\forall$prime ideal of K, $l^{-1}(J) = \{b \in N/l(b) \in J\}$ is also a prime ideal of S.*

Proof .*For any $c, d \in J$such that$c \bullet d \in l^{-1}(J) \Rightarrow l(c \bullet d) \in J \Rightarrow l(c) \bullet l(d) \in J$(by homomorphism) $\Rightarrow l(c) \in J$or $l(d) \in J \Rightarrow c \in l^{-1}(J)ord \in l^{-1}(J)$.Hence $l^{-1}(J)$ is prime ideal.* □

Theorem 3.18. *If $l : N \to K$ is a homomorphism of two ring parts $N$and K of IS-algebra $(L, *, \bullet, 0)$, then $f : SpecK \to SpecN$ define by $f(J) = l^{-1}(J)$,$\forall J \in SpecK$ is continuous map.*

Proof .*For any $b \in N$, Let$W(b)$ be a basic open set in $Spec(N)$,then*

$$\begin{aligned} f^{-1}(W(b)) &= \{J \in SpecK/f(J) \in W(b)\} \\ &= \{J \in SpecK/l^{-1}(J) \in W(b)\} \\ &= \{J \in SpecK/b \notin l^{-1}(J)\} \\ &= \{J \in SpecK/l(b) \notin J\} \end{aligned}$$

which is open in $Spec(K)$. Hence $f$is a continuous map. □

4. Conclusion

We have studied the topology of a ring part of IS -algebra by using prime ideals of a commutative ring part of IS-algebra and discussed few results of this topology, for example, the prime spectrum $spec(N)$of a ring part of an IS-algebra and study some of its properties. Also, proved that $spec(N)$ is T_0 and T_1 -space. Furthermore, $f : spec(N) \to spec(K)$ it is a continuous mapping.

5. Open problems

The following are some open problems for future works:

1. Studying the theory of soft topological space on IS-algebra.
2. Introducing a compact and simply compact of a ring part of an IS-algebra.
3. Studying of soft simply path connected spaces and soft simply compact spaces.
4. Studying the filter of this structure.

References

[1] E. Eslami and F. Kh. Haghani, *Pure filters and stable topology on BL-algebras*, Kybernetika, 45 (2009) 491–506.

[2] Y. Imai and K. Iseki, *On axiom systems of propositional calculi*, XIV,Proc. Japan Acad. Ser A. Math. Sci. 42 (1966) 19–22.

[3] K. Iseki, *On BCI-algebras*, Math. Sem. Notes, 8 (1980) 125–130.

[4] Y. Jun and S. M. Hong, *BCI-semigroups*, Honam Math. J. 15(1) (1993) 59–64.

[5] Y. Jun, Roh E. H. and X. L. Xin, *I-ideals generated by a set in IS-algebras*, Bull Korean Math. Soc. 35 (1998) 615–624.

[6] F. Kareem, R. D. Ali and S. M. Mostafa, *Fuzzy topological spectrum of a KU-algebra*, Mater. Sci. Eng. (2019) 571 012015.

[7] S. M. Mostafa and F. F. Kareem, *A topology spectrum of a KU-algebra*, J. New Theo. 5 (2015) 78–91.

[8] T. Roudbari and N. Motahari, *A topology on BCK-modules via prime sub-BCK-modules*, J. Hyper Struct. 1 (2012) 24–30.

[9] K. Venkateswarlu and B. V. N. Murthy, *Spectrum of Boolean like semi ring*, Int. J. Math. Sci. Appl. 1 (2011).

[10] Y. Wenqi, *Two rings in IS-algebras*, Proc. Fifth Int. Conf. Number Theory and Smarandache Notions, (2009) 98–101.