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The numerical solution of bioheat equation based on shifted Legendre polynomial

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Abstract

The goal of this study is to expand the usage of a collection method based on shifted Legendre polynomials in matrix form to approximate the derivative to obtain numerical solutions for the unsteady state one-dimensional bioheat equation. The proposed methodology is used to two examples to illustrate its utility and accuracy. The numerical results shown that the techniques used are effective as well as gives high accuracy and good convergence

Keywords: collocation method, bioheat equation, Legendre polynomials, accuracy.

1. Introduction

Ordinary and partial differential equations (ODEs and PDEs) have a wide range of applications in physics, biology, chemistry, and engineering, in addition to their contribution to the study of mathematical analysis and their value in economics and sciences. The physical, geometric, and mathematical relationships and laws that bind the variables. Biological issues can be found in differential equations in a variety of ways, and they play a significant role in the development of differential equations. In addition, there are linear and nonlinear phenomena that are translated into differential equations, such as the bioheat model. As a result, ODEs and PDEs have emerged as one of the most intriguing subjects for many researchers.

Continuous or piecewise polynomials are incredibly useful mathematical tools as they are precisely

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defined, calculated rapidly on a modern computer system and can represent a great variety of functions. They can be differentiated and integrated without difficulty, and can be put together to form spline curves that can approximate any function to any accuracy desired [11]. The studies that have been achieved for solving various types of the differential equations in the last half century are still an active research area to develop, apply and combination some better polynomials in the matrix form for finding approximation solutions for these equations.

Yalçibas and et al. [17] developed a new Hermite collocation method to find approximate solutions for the generalized pantograph equations dependent on Hermite polynomials. The numerical results show that the accuracy and efficiency of the method improves when N is increased. Bhrawy and Alof [5] employed shifted Jacobi polynomials for solving nonlinear Lane-Emden type equation by using a shifted Jacobi−Gauss collocation spectral method The results show that the method is simple and accurate by selecting few collocation points. Tohidi and et al. [15] utilized a collocation method based on Bernoulli polynomials for solving the generalized pantograph equation, the results shows that the method is a powerful mathematical tool for finding the numerical solutions of a generalized pantograph equation.

Heydari and et al. [10] presented a numerical method for approximating solutions of the telegraph type equations by combining Chebyshev wavelets with their operational matrices of differentiation, in the proposed method a small number of grid points guarantees the necessary accuracy. Alshbool and et al. [3] introduced an approximate solution depending on collocation method and Bernstein polynomials for numerical solution of a singular nonlinear differential equations with the mixed conditions. The accuracy and efficiency of this technique are dependent on the size of the set of Bernstein polynomials, which that gives excellent agreement, is found between the exact and approximate solutions. Bahşi and Yalçibas [4] found a numerical scheme to solve the telegraph equation by using Fibonacci polynomials, the results show that the method yields either the exact solution or a high accuracy approximate solution for the telegraph equations. Gürbüz and Sezer [9] improved a matrix method based on collocation points and Laguerre polynomials to obtain the numerical approximations of the one dimensional nonlinear Klein-Gordon equations, the results demonstrated the accuracy and the effectiveness of the method. Khan and Ali [12] obtained an approximate solution for delay differential equation and stochastic delay differential equation based on Legendre spectral-collocation method, the result show that the presented method is high efficiency and accuracy.
In recent years, the Legendre polynomials in the matrix form has been used to find the approximate solutions of various types of differential equations [1, 2, 8, 12, 14, 16, 18]. The basic motivation of this work is to add a new application of collocation method based on the shifted Legendre polynomials for solving unsteady state one dimensional bioheat equation.

2. Governing equation

Pennes unsteady state one dimensional bioheat model is implemented to study the heat transfer in skin tissue[13]

$$\rho c\frac{\partial T(x,t)}{\partial t} - K\frac{\partial^2 T(x,t)}{\partial x^2} + W_b c_b(T(x,t) - T_a) = Q_{ext} + Q_{met}, \quad ,0 \leq x \leq b, t > 0 \qquad (1)$$

where $\rho, c, K, T, t, x, T_a, W_b = \rho_b \omega_b, Q_e xt$ and $Q_m et$ represents density, specific heat, thermal conductivity, temperature, time, distance, artillery temperature ,blood perfusion rate, metabolic heat generation in skin tissue and external heat source in skin tissue respectively. The units of the symbols

Symbol	T_a	ρ and ρ_b	c and c_b	k	ω_b	Q_{met}
Unit	°C	kg/m^3	J/kg°C	W/m°C	$m^3/s/m^3$	W/m^3
Value	37	1000	4000	0.5	0.0005	420

Table 1: The unit of the symbols expressed in the Pennes equation

expressed in this equation are listed in Table 1. The Pennes bioheat transfer equation appeared in the pioneering work of Pennes. To the best of our knowledge, in the general form the exact solution of this equation does not exist especially for the Pennes equation that the terms are changed during the domain. Therefore the numerical methods are needed to solve this equation. In the test Cases 1 and 2. with initial and boundary conditions

$$T(x,0) = T_c \tag{2}$$

$$-K\frac{\partial T}{\partial x}|_{x=0} = q_0 \tag{3}$$

$$-K\frac{\partial T}{\partial x}|_{x=b} = 0 \tag{4}$$

where, q_0 is the heat flux on the skin surface.

3. Shifted Legendre Polynomials

The $nth-$order Legendre polynomials which are orthogonal in the interval $[-1,1]$ are defined as

$$P_{n+1}(x) = \frac{2n+1}{n+1}P_n(x) - \frac{n}{n+1}P_{n-1}(x), \quad ,n = 1,2,\cdots \tag{5}$$

with $L_0(x) = 1$, $L_1(x) = x$.
In order to use these polynomials on the interval $[0,1]$, one can apply the change of variables $x = 2t-1$ in the above relation. Therefore, the shifted Legendre polynomials are constructed as follows $L_n(t) = P_n(2t-1), t \in [0,1]$. The analytic form of the shifted Legendre polynomial $L_n(t)$ of degree n is given by

$$L_n(t) = (-1)^{n+1}\sum_{i=0}^{n}\frac{(n+i)!}{(i!)^2(n-i)!}t^i, \quad n \in \mathbb{N} \tag{6}$$

where $L_n(0) = (-1)^n, L_n(0) = 1$, The orthogonal property of shifted Legendre polynomials is given by

$$\int_0^1 L_n(t)L_m(t)dt = \begin{cases} 0 & n \neq m \\ \frac{1}{2n+1} & n = m \end{cases} \tag{7}$$

A function $T(x,y)$ may be expressed in term of shifted Legendre polynomials as:

$$T(x,t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij}L_i(x)L_j(t)$$

In practice, we consider the $(n+1)$ and $(m+1)-$terms shifted Legendre polynomial with respect to x, t so that

$$\begin{aligned} T(x,t) &= \sum_{i=0}^{n}\sum_{j=0}^{m} a_{ij} L_i(x) L_j(t) \\ &= \phi(x)' A \phi(t) \qquad (8) \end{aligned}$$

where,

$$a_{ij} = (2i+1)(2j+1)\int_0^1\int_0^1 T(x,t) L_i(x) L_j(t)\, dx dt \qquad (9)$$

where the shifted Legendre coefficient matrix A when $n = m$, the shifted Legendre vector $\phi(x)$ and $\phi(t)$ are given by

$$A = \{a_{ij}\}_{i,j=0}^{n,n}, \quad \phi(x) = [L_0(x), L_1(x), \cdots, L_n(x)]', \quad \phi(t) = [L_0(t), L_1(t), \cdots, L_n(t)]'$$

respectively.

4. The Two Dimensional Shifted Legendre Operational Matrix of Differentiation

The shifted Legendre polynomials have an interesting property (a relation between shifted Legendre polynomials and their derivatives) which was used in many papers for solving different types of problems [2, 16]. This relation is as follows

$$\frac{d}{dx}\phi(x) = \frac{d}{dx}[L_0(x), L_1(x), \cdots, L_n(x)]$$

can be denoted in the matrix form by

$$\phi^{(1)}(x) = D_x \phi(x) \qquad (10)$$

where, if n is even and odd we get

$$D_x = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 6 & \cdots & 0 & 0 & 0 & 0 \\ 2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 2 & 0 & \cdots & 4n-6 & 0 & 0 & 0 \\ 0 & 6 & \cdots & 0 & 4n-2 & 0 & 0 \end{pmatrix}_{(n+1)\times(n+1)}$$

$$D_x = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 6 & \cdots & 0 & 0 & 0 & 0 \\ 2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 6 & \cdots & 4n-6 & 0 & 0 & 0 \\ 2 & 0 & \cdots & 0 & 4n-2 & 0 & 0 \end{pmatrix}_{(n+1)\times(n+1)}$$

respectively.
Accordingly, the $k-th$ derivative with respect to x of $\phi(x)$ can be obtained by

$$\left.\begin{array}{rcl} \phi^{(2)}(x) &=& (D_x)^2\phi(x) \\ \phi^{(3)}(x) &=& (D_x)^3\phi(x) \\ \vdots && \\ \phi^{(k)}(x) &=& (D_x)^k\phi(x) \end{array}\right\} \qquad (11)$$

In the same manner we can conclude that

$$\phi^{(k)}(t) = (D_t)^k\phi(t)$$

5. Method of the Solution

We will structure the numerical solution methodology of Equation (1), under the given conditions, in the series form of Equation (8) or in the matrix form $T(x,t) = \phi(x)'A\phi(t)$. Can be approximate the first , second spatial derivatives and first thermal derivative as follows:

$$\left.\begin{array}{rcl} \frac{\partial T(x,t)}{\partial x} &=& \phi'(x)D_x'A\phi(t) \\ \frac{\partial^2 T(x,t)}{\partial x^2} &=& \phi'(x)(D_x')^2A\phi(t) \\ \frac{\partial T(x,t)}{\partial t} &=& \phi'(x)AD_t\phi(t) \end{array}\right\} \qquad (12)$$

The right side of equation (1) is given as $g(x,y) = Q_{ext} + Q_{met} + W_bc_bT_a$, we can approximate as

$$\begin{array}{rcl} G(x,t) &=& \sum_{i=0}^{n}\sum_{j=0}^{n} g_{ij}L_i(x)L_j(t) \\ &=& \phi(x)'G\phi(t) \qquad (13) \end{array}$$

where,

$$G = \{g_{ij}\}_{i,j=0}^{n,n}.$$

Substituting the equations (8), (12) and (13) into the bioheat equation (1) and simplifying the result, we have the matrix equation

$$\rho c\phi'(x)AD_t\phi(t) - K\phi'(x)(D_x')^2A\phi(t) + W_bc_b\phi'(x)A\phi(t) = \phi'(x)Q_{ext}\phi(t) + \phi'(x)Q_{met}\phi(t) + \phi'(x)W_bc_bT_a\phi(t) \qquad (14)$$

with initial condition,

$$\begin{array}{rcl} T(x,0) &=& T_c \\ A\phi(0) &=& F, \qquad F = [f_0 \;\; f_1, \cdots, f_n]', \qquad f_j = (2j+1)\int_0^1 T(x,0)L_j(x)dx \end{array}$$

and boundary conditions,

$$\begin{array}{rcl} -K\phi'(0)D_x'A &=& K', \qquad K' = [k_0 \;\; k_1, \cdots, k_n], \qquad k_j = (2j+1)\int_0^1 T_x(0,t)L_j(t)dt \\ -K\phi'(b)D_x'A &=& H', \qquad H' = [h_0 \;\; h_1, \cdots, h_n], \qquad h_j = (2j+1)\int_0^1 T_x(b,t)L_j(t)dt \end{array}$$

and,

$$G = Q_1 + Q_2 + Q_3$$

where,

$$Q_1 = \{(q_{ext})_{ij}\}_{i,j=0}^{n,n}, Q_2 = \{(q_{met})_{ij}\}_{i,j=0}^{n,n} \quad and\ Q_3 = \{(\omega_b \rho_b c_b T_a)_{ij}\}_{i,j=0}^{n,n}$$

which can be compute it by depending on (9).

6. Applications and Numerical Results

We offer some numerical results acquired using the algorithms provided in the preceding part in this part. The collocation method is used to solve the Pennes bioheat equation based on Legendre polynomials in two instances in this section. Cases 1 and 2 are offered to demonstrate the collocation method's capacity to achieve high accuracy. The solution produced using this method is compared to the exact solution in these problems. As mentioned before, the exact solution for this equation does not exist in the general form so we assume unrealistic Q_{ext} to deriving the exact solution in Cases 1 and 2.

Case 1 Consider the bioheat equation (1) has the following exact solution [6]:

$$T(x,t) = \exp(-t)\cos(\pi x) + \tanh(x) + 37 \tag{15}$$

we can get the scours function Q_{ext} as follows:

$$Q_{ext}(x,t) = (-3998000 + \frac{1}{2}\pi^2)\exp(-t)\cos(\pi x) + \frac{\tanh(x)}{\cosh^2(x)} + 2000\tanh(x) - 420. \tag{16}$$

Case 2 Consider (1) has the following exact solution[7] :

$$T(x,t) = t^{\frac{3}{2}} x^2 (\frac{3}{2} - x) + 37 \tag{17}$$

also, we can get Q_{ext} as follows:

$$Q_{ext}(x,t) = (6000000\sqrt{t}x^2 - t^{\frac{3}{2}} + 2000 t^{\frac{3}{2}} x^2)(\frac{3}{2} - x) + 2xt^{\frac{3}{2}} - 420. \tag{18}$$

We take $n = 4$,

$$\begin{aligned} T(x,t) &= \sum_{i=0}^{4}\sum_{j=0}^{4} a_{ij} L_i(x) L_j(t) \\ A(\rho c D_t + \omega_b \rho_b c_b I) - K(D'_x)^2 A &= G \end{aligned}$$

where,

$$D_x = D_t = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 \\ 2 & 0 & 10 & 0 & 0 \\ 0 & 6 & 0 & 14 & 0 \end{pmatrix},$$

$$G = \begin{pmatrix} 7.4868e+04 & 0 & 0 & 0 & 0 \\ 3.0735e+06 & -1.5114e+06 & 2.5011e+05 & -2.4913e+04 & 1.7750e+03 \\ -1.1024e+02 & 0 & 0 & 0 & 0 \\ -5.6836e+05 & 2.7955e+05 & -4.6262e+04 & 4.6080+03 & -3.2831e+02 \\ 4.2458 & 0 & 0 & 0 & 0 \end{pmatrix}$$

so,

$$A\begin{pmatrix} 2000 & 0 & 0 & 0 & 0 \\ 8000000 & 2000 & 0 & 0 & 0 \\ 0 & 24000000 & 2000 & 0 & 0 \\ 8000000 & 0 & 4000000 & 2000 & 0 \\ 0 & 24000000 & 0 & 56000000 & 2000 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -6 & 0 & -20 \\ 0 & 0 & 0 & -30 & 0 \\ 0 & 0 & 0 & 0 & -70 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} A = G$$

The initial condition obtain as $T(x,0) = T_c$ where $T_c = \cos(\pi x) + \tanh(x) + 37$.

Then, $A\phi(0) = F$,

$$\begin{aligned} f_i &= (2i+1)\int_0^1 T_c L(x)dx \\ &= [3.7434e+01 \quad -8.2999e-01 \quad -5.5052e-02 \quad 2.1989e-01 \quad 2.1181e-03]' \end{aligned}$$

At the boundary conditions , when $x = 0$ get $q_0 = 1$ then $-k\phi'(0)D_x'A = H'$ where,

$$h_j = (2j+1)\int_0^1 q_0 L(t)dt = [1 \;\; 0 \;\; 0 \;\; 0 \;\; 0],\ and\ \phi(0) = [1 \quad -1 \quad 1 \quad -1 \quad 1]'.$$

At $x = b$ then $-k\phi'(b)D_x'A = R'$ where,

$$\begin{aligned} r_j &= (2j+1)\int_0^1 \frac{\partial}{\partial x}T(b,t)L(t)dt \\ &= [4.1997e-01 \quad 1.1962e-16 \quad -1.9796e-17 \quad 1.9718e-18 \quad -1.4049e-19] \end{aligned}$$

Now, by substituting the columns $\phi(0)$ and f_j in fourth columns of temporal derivative and the matrix G respectively also substituting the row $-k\phi'(0)D_x'$ and column h_j in third rows of spatial derivative and the matrix G respectively. Finally, substituting the row $-k\phi'(b)D_x'$ and column r_j in fourth rows of spatial derivative and the matrix G respectively.

Then solve the resulting system by using Sylvester equation to find A .

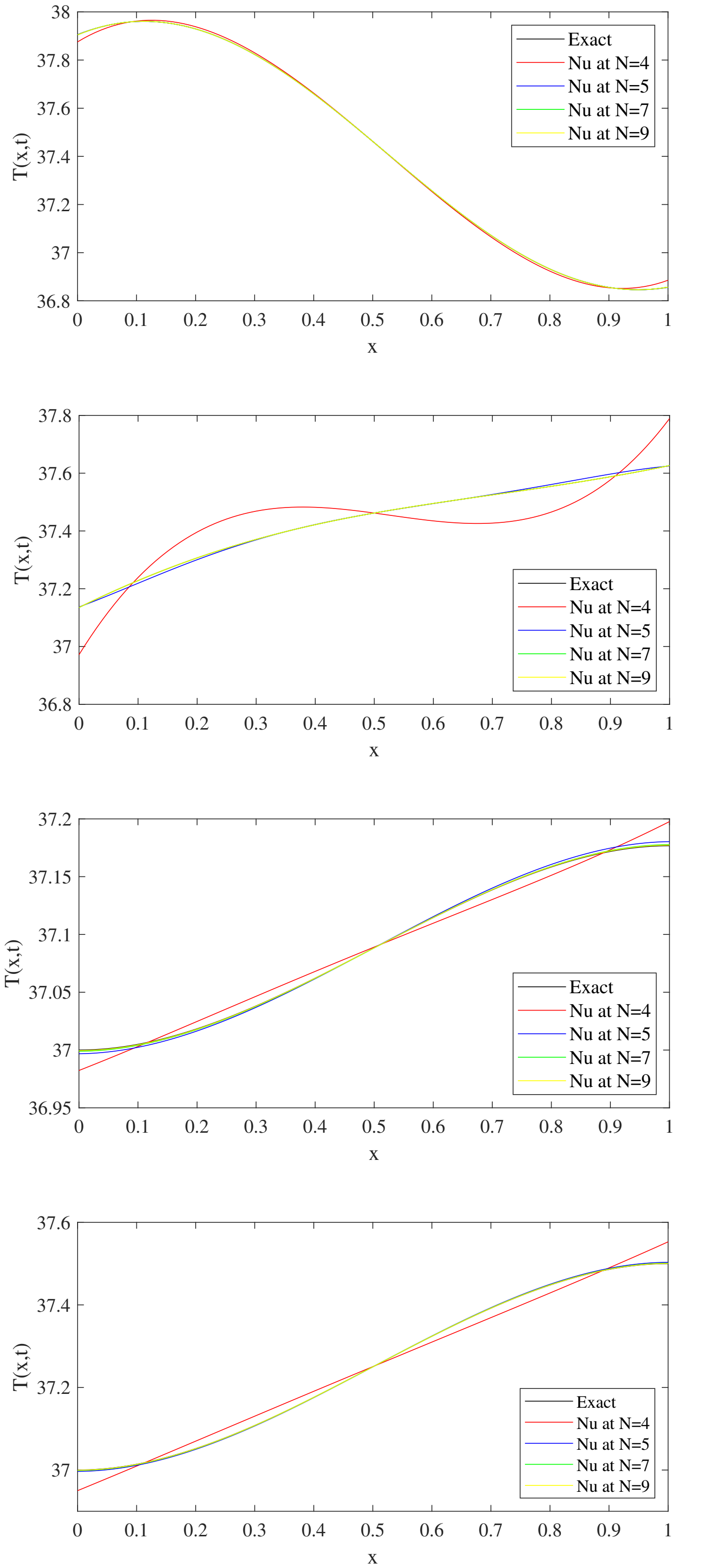
In cases 1 and 2 we consider the following data $b = 0.1$ and $b = 1$. Tables 2-3 show the errors obtained from solving the bioheat equation by using shifted Legendre polynomial at $x \in [0, b]$ for different values of $n = 4, 5, 7$ and 9. Figures 1-4 clarify a comparison between exact solution and numerical solutions of the present cases. The results show that the numerical solutions have a high accuracy and good convergence when increasing order the numerical solutions .

Table 2: Absolute error for case 1 with b=0.1

(x,t)	$n=4$	$n=5$	$n=7$	$n=9$
(0,0)	9.0494e-03	3.4101e-04	1.5974e-06	2.0436e-07
(0.01,0.01)	8.4316e-03	3.0690e-04	7.0173e-07	1.9844e-07
(0.02,0.02)	6.0203e-03	2.5357e-04	6.7483e-07	1.8422e-07
(0.03,0.03)	6.8141e-03	1.8702e-04	2.1923e-06	1.7001e-07
(0.04,0.04)	5.8355e-03	1.1248e-04	3.6038e-06	1.6049e-07
(0.05,0.05)	4.7578e-03	3.4474e-05	4.7406e-06	1.5774e-07
(0.06,0.06)	3.5920e-03	4.3110e-05	5.4988e-06	1.6199e-07
(0.07,0.07)	2.3490e-03	1.1701e-04	5.8269e-06	1.7229e-07
(0.08,0.08)	1.0398e-03	1.8454e-04	5.7159e-06	1.8697e-07
(0.09,0.09)	3.2461e-04	2.4352e-04	5.1889e-06	2.0408e-07
(0.1,0.1)	1.7332e-03	2.9226e-04	4.2936e-06	2.2161e-07

Table 3: Absolute error for case 2 with b=0.1

(x,t)	$n=4$	$n=5$	$n=7$	$n=9$
(0,0)	1.0567e-08	2.9547e-03	9.5384e-04	4.2242e-04
(0.01,0.01)	1.9610e-04	2.9713e-03	9.7892e-04	4.4318e-04
(0.02,0.02)	3.6351e-04	2.9763e-03	9.9188e-04	4.5100e-04
(0.03,0.03)	5.0033e-04	2.9707e-03	9.9485e-04	4.5149e-04
(0.04,0.04)	6.0537e-04	2.9554e-03	9.8978e-04	4.4435e-04
(0.05,0.05)	6.7804e-04	2.9317e-03	9.7829e-04	4.3372e-04
(0.06,0.06)	7.1824e-04	2.9005e-03	9.6177e-04	4.2070e-04
(0.07,0.07)	7.2629e-04	2.8627e-03	9.4137e-04	4.0646e-04
(0.08,0.08)	7.0284e-04	2.8192e-03	9.1803e-04	3.9186e-04
(0.09,0.09)	6.4889e-04	2.7706e-03	8.9254e-04	3.7747e-04
(0.1,0.1)	5.6571e-04	2.7176e-03	8.6551e-04	3.6371e-04



(a) (b) (c) (d)

Figure 1: Solutions of Legendre Polynomial for case 1 at $t = 0.1, 2$ in $(a) - (b)$ and for case 2 at $t = 0.5, 1$ in $(c) - (d)$ respectively.

7. Conclusions

To solve the bioheat equation successfully in this article, we used a shifted Legendre polynomial in matrix form. The numerical results reveal that by employing fewer grid points, the current methodology offers improved accuracy, good convergence, and reasonable stability, as well as a lower computational workload. The numerical results reveal that by employing fewer grid points, the current methodology offers improved accuracy, good convergence by increasing n and noting that the accuracy will be increasing when increasing the time t , and reasonable stability, as well as a lower computational workload .

References

[1] M. Abdelkawy, E. Ahmed and P. Sanchez, *A method based on Legendre Pseudo-Spectral Approximations for Solving Inverse Problems of Parabolic Types Equations*, Mathematical Sciences Letters An International Journal, 1 (2011) 81-90.

[2] F. Ahmad, A. Alomari, A. Bataineh, J. Sulaiman and I. Hashim, *On the approximate solutions of systems of ODEs by Legendre operational matrix of differentiation*, Italian Journal of Pure and Applied Mathematics, 36 (2016) 483-494.

[3] M. Alshbool , A. Bataineh , I. Hashim and O. Isik, *Approximate solutions of singular differential equations with estimation error by using Bernstein polynomials*, International Journal of Pure and Applied Mathematics, 1 (2015) 109-125.

[4] A. Bahşi and S. Yalçinbas, *A new algorithm for the numerical solution of telegraph equations by using Fibonacci polynomials*, Mathematics and Computational Applications, 15 (2016) 1-12.

[5] A. Bhrawy and A. Alof, *A Jacobi-Gauss collocation method for solving nonlinear Lane-Emden type equations*, Commun Nonlinear Sci Numer Simulat, 17 (2012) 62-70.

[6] M. Dehghan and M. Sabouri, *A spectral element method for solving the Pennes bioheat transfer equation by using triangular and quadrilateral elements*, Applied Mathematical Modelling, 36 (2012) 6031-6049.

[7] L. Ferrás, N. Ford, M. Morgado, J. Nobrega and M. Rebelo, *Fractional Pennes̕ bioheat equation: Theoretical and numerical studies*, Fractional Calculus and Applied Analysis, 4 (2015) 1080-1106.

[8] A. Güner and S. Yalçinbas, *Legendre collocation method for solving nonlinear differential equations*, Mathematical and Computational Applications, 3 (2013) 521-530.

[9] B. Gürbüz and M. Sezer, *Laguerre Polynomial Approach for Solving Nonlinear Klein-Gordon Equations*, Malaysian Journal of Mathematical Sciences, 2 (2017) 191-203.

[10] M. Heydari, M. Hooshmandasl and F. Ghaini, *A new approach of the Chebyshev wavelets method for partial differential equations with boundary conditions of the telegraph type*, Applied Mathematical Modelling, 38 (2014) 1597-1606.

[11] M. Idrees and P. Bracken, *Solutions of differential equations in a Bernstein polynomial basis*, Journal of Computational and Applied Mathematics, 205 (2007) 272-280.

[12] S. Khan and I. Ali, *Application of Legendre spectral-collocation method to delay differential and stochastic delay differential equation* , AIP Advances, 8 (2018) 1-10.

[13] H. Pennes, *Analysis of tissue and arterial blood temperature in the resting forearm*, Journal of applied physiology, 1 (1984) 93-122.

[14] A. Saadatmandi and M. Razzaghi, *A Tau method approach for the diffusion equation with nonlocal boundary conditions*, International Journal of Computer Mathematics, 11 (2004) 1427-1432.

[15] E.Tohidi, A. Bhrawy and K.Erfani, *A collocation method based on Bernoulli operational matrix for numerical solution of generalized pantograph equation*, Applied Mathematical Modelling, 6 (2013) 4283-4294.

[16] E. Tohidi, *Legendre approximation for solving linear HPDEs and comparison with Taylor and Bernoulli matrix methods*, Applied Mathematics, 3 (2012) 410-416.

[17] S. Yalçibas, M. Aynigül and M. Sezer, *A collocation method using Hermite polynomials for approximate solution of pantograph equations*, Journal of the Franklin Institute, 348 (2011) 1128-1139.

[18] S. Yousefi, *Legendre scaling function for solving generalized Emden-Fowler equations*, International Journal of Information and Systems Sciences, 2 (2007) 243-250.