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Common fixed point of generalized weakly contractive mappings on orthogonal modular spaces with applications

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Abstract

In this paper, we provide certain conditions under which guarantee the existence of a common fixed point for weakly contractive mappings defined on orthogonal modular spaces. Also, Banach fixed point theorem on an orthogonal modular space without completeness is obtained. To prove much stronger and more applicable results, some strong assumptions such as the convexity and the Fatou property of a modular are relaxed.

Keywords: Orthogonal set, Orthogonal modular space, Common fixed point
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1. Introduction

The Banach contraction mapping theorem [8] popularly known as Banach contraction mapping principle is a rewarding result in fixed point theory. It has widespread applications in both pure and applied mathematics. This celebrated principle has been generalized by several authors. In one approach, fixed point theory in modular spaces has received a lot of attention after being proposed as a generalization of normed spaces [36, 37, 38, 40, 41]. A growing literature on fixed point theorems in Modular spaces deals with rigorous formulations and proofs of many interesting problems which are applicable in a wide variety of settings, including Quantum Mechanics, Machine Learning and etc. The study of this theory in the context of modular function spaces was initiated by Khamsi et al. in [22] by using some constructive techniques for single-valued mappings. This work has been

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widely cited as the inspiration for a variety of fixed point work along with [23, 24]. This line of work was extended by several works in a variety of ways. Recently, Kutabi and Latif [29] studied fixed points of multivalued maps in modular function spaces. Then Lael et al. generalized their work in modular spaces by relaxing convexity and boundedness [31]. In one successful approach, Kuaket and Kumam [27] and Mongkolkeha and Kumam [33, 34, 35], considered and proved some fixed point and common fixed point results for generalized contraction mappings in modular spaces. Also, Kumam [28] obtained some fixed point theorems for non-expansive mappings in convex modular spaces. Then, Khamsi et al. explored the existence of fixed points of nonexpansive mapping and asymptotic pointwise nonexpansive mapping in modular function spaces in [5, 25], respectively. Further, on the basis of their result, in [3], an analog of DeMarr's common fixed point theorem for a family of symmetric Banach operator pairs in modular vector spaces is proved. Furthermore, authors in [2] study the existence and uniqueness of common fixed point results in partially ordered modular function spaces (see [45, 46]). Almost all researchers focus on key properties of modulars, convexity and Fatou property.

In another approach, Rhoades [44] introduced the concept of φ-contractive mappings. Afterwards, some researchers introduced a few φ- and $\psi-\varphi$-weakly contractive conditions and discussed the existence of fixed and common fixed point for these mappings [1, 42]. In particular, Abbas et al.[6] presented several common fixed point results of generalized weak contractive mappings in partially ordered b-metric spaces, and in [2], they showed the existence and uniqueness of common fixed point results in partially ordered modular function spaces. Very recently, Lael et al. [30, 32] proved some fixed point theorems for multivalued mappings with suitable φ-contraction on modular spaces.

Although fixed point theory is shown to be successful in challenging problems and has contributed significantly to many real-world problems, various fixed point theorems strongly are proved under strong assumptions. In particular, in modular spaces, some of these assumptions can lead to having some induced norms. So, some assumptions often do not hold in practice or can lead to their reformulations as a particular problem in a normed vector spaces. A recent trend of research has been dedicated to studying the fundamentals of fixed point theorems and relaxing their assumptions with the ambition of pushing the boundaries of fixed point theory in modular spaces (see [4, 10, 22]). But, you can hardy find literatures which relaxed the convexity and the Fatou property of the modular in fixed point theory [31].

Recently, Eshaghi et al. [16] introduced the notion of orthogonal set and then gave an extension of Banach's fixed point theorem. They proved, by means of an example, that their main theorem is a real generalization of Banach's fixed point theorem. The main result of [16] is the following theorem.

Theorem 1.1. *[16] Let $(X, \perp, d)$ be an O-complete metric space (not necessarily complete metric space) and $0 < k < 1$. Let $f : X \to X$ be $\perp$-continuous, $\perp$- contraction with Lipschitz constant k and $\perp$-preserving. Then, f has a unique fixed point $z \in X$. Also, f is a Picard operator, that is, $lim f^n(x) = z$ for all $x \in X$.*

After that in [43], orthogonal modular space is defined and a new generalized modular version of the Meir–Keeler fixed point theorem endowed with an orthogonal relation is presented. The famous Nguyen Van Dung, in [48], has talked about the importance of results which are proved on orthogonal set and showed that many existence results on fixed points in orthogonal-complete metric spaces can be proved by using the corresponding existence results in complete metric spaces. For more details about orthogonal space, we refer the reader to [16, 7, 19, 12].

Our main concern in this paper is to prove common fixed point theorems involving generalized weakly contractive conditions in orthogonal modular spaces by relaxing strong assumptions on the modular such as continuity, convexity and Fatou property. In a bird-eyes view, the paper starts with Section 2 which is a brief introduction to modular spaces and orthogonal modular spaces along with the required concepts. Section 3 includes Banach fixed point Theorem on an $SO-$complete space along with two main common fixed point theorems. Finally, in Section 4, as an application an integral equation is solved by using proved theorems.

2. Preliminaries

This section will serve as an introduction to some fundamental concepts of modular spaces and orthogonal sets. A detailed introduction in modular space can be found, for example, in the textbooks [26, 36].

A pair (X, ϱ) is called a modular space, where X is a real linear space and ϱ is a real valued functional on X which satisfies the conditions:

1. $\varrho(\mathfrak{x}) = 0$ if and only if $\mathfrak{x} = 0$,
2. $\varrho(-\mathfrak{x}) = \varrho(\mathfrak{x})$,
3. $\varrho(\alpha\mathfrak{x} + \beta\mathfrak{y}) \leq \varrho(\mathfrak{x}) + \varrho(\mathfrak{y})$, for any nonnegative real numbers α, β with $\alpha + \beta = 1$.

The functional ϱ is called a modular on X.
If (3) replaces by $\varrho(\alpha\mathfrak{x}+\beta\mathfrak{y}) \leq \alpha\varrho(\mathfrak{x})+\beta\varrho(\mathfrak{y})$, for any $\alpha, \beta \in \mathbb{R}^+$ with $\alpha+\beta = 1$ the modular is called a **convex** modular. Interestingly, it is shown that a modular induces a vector space $X_\varrho = \{\mathfrak{x} \in X : \alpha \to 0 \text{ implies } \varrho(\alpha\mathfrak{x}) \to 0\}$ which is called a modular linear space. For a modular space (X, ϱ), the function w_ϱ which is said growth function [9] is defined on $[0, \infty)$ as follows:

$$w_\varrho(t) = \inf\{w : \varrho(t\mathfrak{x}) \leq w\varrho(\mathfrak{x}) : \mathfrak{x} \in X,\ 0 < \varrho(\mathfrak{x})\}.$$

It is shown that a modular ϱ implies that

$$\|\mathfrak{x}\|_\varrho = \inf\{a > 0 : \varrho(\frac{\mathfrak{x}}{a}) \leq 1\},$$

defines an F-norm on X_ϱ. Specifically, if ϱ is convex, $\|\cdot\|_\varrho$ is a norm and it is frequently called the Luxembourg norm [17]. Note that a modular space determined by a function modular ϱ will be called a modular function space and will be denoted by L_ϱ. Then, it is not difficult to show that $\|\cdot\|_\varrho$ is an F-norm induced by ϱ. More importantly, $(L_\varrho, \|\cdot\|_\varrho)$ is a complete space. There are many arguably important special instances of well known spaces in which these properties are fulfilled [39, 40, 41, 47]. Furthermore, Musielak and Orlicz in [37, 40, 41] naturally provide the first definitions of the following key concepts in a modular space (X, ϱ):

D1. A sequence $(\mathfrak{x}_n)$ in $B \subseteq X$ is said to be convergent to a limit point $\mathfrak{x} \in B$ if $\lim \varrho(\mathfrak{x}_n - \mathfrak{x}) = 0$. It is easy to show that the limit point of a convergent sequence is unique.

D2. A closed subset $B \subseteq X$ is meant that it contains the limit of all its convergent sequences.

D3. A sequence $(\mathfrak{x}_n)$ in $B \subseteq X$ is said to be Cauchy if $\lim \varrho(\mathfrak{x}_m - \mathfrak{x}_n) = 0$ as $m, n \to \infty$. It is easy to show when (X, ϱ) satisfying $w_\varrho(2) < \infty$, then every convergent sequence in (X, ϱ) is Cauchy.

D4. A subset B of X is said to be complete if each Cauchy sequence in B is convergent to a point of B. It is clear that every closed subset of a complete modular space is complete.

D5. Fatou property: ϱ has the Fatou property, if $\varrho(\mathfrak{x}) \leq \liminf \varrho(\mathfrak{x}_n)$ whenever the sequence $(\mathfrak{x}_n)$ is convergent to $\mathfrak{x}$.

D6. bounded subsets: A subset $B \subseteq X_\varrho$ is called bounded if $\sup_{\mathfrak{x},\mathfrak{y}\in B} \varrho(\mathfrak{x}-\mathfrak{y}) < \infty$.

Now, we recall some definitions on orthogonal set and orthogonal modular space (for more details see [11, 13, 14, 15]).

Definition 2.1. Let $X \neq \emptyset$ and $\perp \subset X \times X$ be a binary relation. If $\perp$ satisfies the following condition

$$\exists \mathfrak{x}_0 \in X; ((\forall \mathfrak{x} \in X; \mathfrak{x} \perp \mathfrak{x}_0) \text{ or } (\forall \mathfrak{x} \in X; \mathfrak{x}_0 \perp \mathfrak{x}));$$

then X with $\perp$, is called an orthogonal set (briefly O-set). We denote this O-set by $(X, \perp)$.

Definition 2.2. Let $(X, \perp)$ be an O-set and (X, ϱ) be a modular space, then $(X, \varrho, \perp)$ is called an **orthogonal modular** space.

Definition 2.3. A sequence $(\mathfrak{x}_n)$, $n \in \mathbb{N}$ is called a orthogonal sequence (briefly, O-sequence) if

$$((\forall n; \mathfrak{x}_n \perp \mathfrak{x}_{n+1}) \text{ or } (\forall n; \mathfrak{x}_{n+1} \perp \mathfrak{x}_n)).$$

Also it is called an strongly orthogonal sequence (briefly, SO-sequence) if

$$((\forall n, k; \mathfrak{x}_n \perp \mathfrak{x}_{n+k}) \text{ or } (\forall n, k; \mathfrak{x}_{n+k} \perp \mathfrak{x}_n)).$$

It is clear that every orthogonal modular space is a modular space, so all definitions, **D1**–**D6**, can be defined for $(X, \varrho, \perp)$, similarly. And if a sequence $(\mathfrak{x}_n)$ is O-sequence (SO-sequence), then **D1**–**D3** in $(X, \varrho, \perp)$ are called convergent O-sequence (convergent SO-sequence), O-closed set (SO-closed set), Cauchy O-sequence (Cauchy SO-sequence), respectively.

Definition 2.4. Let $(X, \varrho, \perp)$ be an orthogonal modular space:

a. Then X is said to be O-complete (SO-complete) if every Cauchy O-sequence (Cauchy SO-sequence) is convergent. Clearly, every O-complete is SO-complete. If X is SO-complete then it is not necessary to be O-complete.

b. Let $B \subset X$. A mapping $f : B \to B$ is called:

(i) Orthogonal preserving mapping if $\mathfrak{x} \perp \mathfrak{y}$ implies $f(\mathfrak{x}) \perp f(\mathfrak{y})$.

(ii) O-continuous (SO-continuous) at $\mathfrak{x} \in B$ if $f\mathfrak{x}_n \to f\mathfrak{x}$, for each O-sequence (SO-sequence) $\mathfrak{x}_n \in B$ which $\mathfrak{x}_n \to \mathfrak{x}$. Also, f is O-continuous (SO-continuous) on B if f is O-continuous (SO-continuous) in each $\mathfrak{x} \in B$.

3. Main Results

In this section, we prove orthogonal modular version of Banach fixed point theorem and existence and uniqueness of common fixed point for generalized weakly contractive mappings (i.e. satisfying inequality (3.1) or (3.17)) in complete modular space. Also to support our main results, we give an example. We shall consider the contractive conditions are constructed via auxiliary functions defined with the families Ψ, Φ, respectively:

$$\Psi = \{\psi : [0, \infty) \to [0, \infty) \text{ is an increasing and continuous function}\},$$

and

$\Phi = \{\varphi : [0,\infty) \to [0,\infty)$ is an increasing and continuous function and $\varphi(t) = 0$ implies $t = 0\}$.

The following lemmas are handy tools that will be used in the sequel.

Lemma 3.1. *[31] Every sequence $(\mathfrak{x}_n)$ in a modular space (X,ϱ) is a Cauchy sequence if there exists $k \in [0,1)$ such that*

$$\varrho(\mathfrak{x}_n - \mathfrak{x}_{n+1}) \le k\varrho(\mathfrak{x}_{n-1} - \mathfrak{x}_n).$$

Lemma 3.2. *If X is a modular space and $(\mathfrak{x}_n)$ is a convergent sequence to $\mathfrak{x}$ in X, then*

$$\frac{1}{w_\varrho(2)}\varrho(\mathfrak{x}-\mathfrak{y}) \le \liminf \varrho(\mathfrak{x}_n - \mathfrak{y}) \le \limsup \varrho(\mathfrak{x}_n - \mathfrak{y}) \le w_\varrho(2)\varrho(\mathfrak{x}-\mathfrak{y}),$$

for every $\mathfrak{y} \in X$.

Proof . If we apply twice the modular's triangle inequality, we get for every $n \in \mathbb{N}$

$$\frac{1}{w_\varrho(2)}\varrho(\mathfrak{x}-\mathfrak{y}) - \varrho(\mathfrak{x}_n - \mathfrak{x}) \le \varrho(\mathfrak{x}_n - \mathfrak{y}) \le w_\varrho(2)\varrho(\mathfrak{x}-\mathfrak{y}) + w_\varrho(2)\varrho(\mathfrak{x}_n - \mathfrak{x}).$$

If we take lim inf on the left-hand side inequality and lim sup on the right-hand side inequality, we obtain the desired property. □

Theorem 3.3 is an orthogonal modoular version of Banach fixed point theorem, as we know the triangle inequality in a modoular space is weaker than the triangle inequality in a normed space, so we can say that Theorem3.3 is a generalization of Theorem 1.1 in some aspects. Also, it is a generalization of [Theorem 3 .3 in [31]] for single valued mapping.

Theorem 3.3. *Let $(X,\varrho,\perp)$ be SO-complete, and $f : X \to X$ be SO-continuous and orthogonal preserving mapping. Suppose that for every $\mathfrak{x},\mathfrak{y} \in X$ that $\mathfrak{x} \perp \mathfrak{y}$, we have*

$$\varrho(f\mathfrak{x} - f\mathfrak{y}) \le k\varrho(\mathfrak{x} - \mathfrak{y}),$$

where $k \in [0,1)$. Then f has a unique fixed point.

Proof . Since X is an O-set, then without loss of generality, there exists $\mathfrak{x}_0 \in X$ such that,

$$\mathfrak{x}_0 \perp \mathfrak{x} \quad \forall \mathfrak{x} \in X.$$

We define the sequence $\mathfrak{x}_n = f\mathfrak{x}_{n-1}$ for all $n \in \mathbb{N}$. Since for all $k \in \mathbb{N}$, $\mathfrak{x}_0 \perp \mathfrak{x}_k$ and f is orthogonal preserving, so $\mathfrak{x}_1 \perp \mathfrak{x}_{k+1}$, by induction, we have

$$\mathfrak{x}_n \perp \mathfrak{x}_{n+k}, \quad \forall n,k \in \mathbb{N}.$$

By contraction, we have that $\varrho(\mathfrak{x}_{n+1} - \mathfrak{x}_n) \le k\varrho(\mathfrak{x}_n - \mathfrak{x}_{n-1})$. Lemma 3.1 implies that the SO-sequence (x_n) is Cauchy. This implies that there exists $\mathfrak{z} \in X$ such that $\mathfrak{x}_n \to \mathfrak{z}$ as $n \to \infty$, since X is SO-complete space. Since f is SO-continuous, so $\mathfrak{x}_{n+1} = f\mathfrak{x}_n \to f\mathfrak{z}$. Thus $f\mathfrak{z} = \mathfrak{z}$. Now, we show that $\mathfrak{z}$ is a unique fixed point. By contradiction, we suppose that there is $\mathfrak{z} \neq \mathfrak{z}'$

such that $f\mathfrak{z} = \mathfrak{z}$ and $f\mathfrak{z}' = \mathfrak{z}'$. Since $\mathfrak{x}_0 \perp \mathfrak{z}$ and $\mathfrak{x}_0 \perp \mathfrak{z}'$, and f is orthogonal preserving, so by n-times using the orthogonality of f, we will have $\mathfrak{x}_n \perp \mathfrak{z}$ and $\mathfrak{x}_n \perp \mathfrak{z}'$. By using twice the contraction's definition, we have

$$\varrho(\frac{\mathfrak{z}-\mathfrak{z}'}{2}) = \varrho(\frac{f\mathfrak{z}-f\mathfrak{z}'}{2}) \leq \varrho(f\mathfrak{z} - f\mathfrak{x}_{\mathfrak{n}}) + \varrho(f\mathfrak{z}' - f\mathfrak{x}_{\mathfrak{n}}) \leq k[\varrho(\mathfrak{z}-\mathfrak{x}_n) + \varrho(\mathfrak{z}-\mathfrak{x}_n)].$$

as $n \to \infty$, $\mathfrak{z} = \mathfrak{z}'$. □

We would like to highlight that while the convexity of ϱ is required for the modular version of Banach fixed point Theorem which is proved in both papers [10] and [29], we showed in Theorem 3.3, that it can be removed. Also, completeness of the modular space ϱ is replaced with a weaker assumption SO-complete. So Theorem 3.3 generalized Banach fixed point theorems in many aspects.

Definition 3.4. *[18]* Let f and g be two self-mappings on a nonempty set X. If $c = f\mathfrak{x} = g\mathfrak{x}$, for some $\mathfrak{x} \in X$, then $\mathfrak{x}$ is said to be the coincidence point of f and g, and c is called the point of coincidence of f and g. Let $C(f, g)$ denote the set of all coincidence points of f and g. If for some $\mathfrak{x} \in X$, $\mathfrak{x} = f\mathfrak{x} = g\mathfrak{x}$, then $\mathfrak{x}$, is called a **common** fixed point of f and g.

Definition 3.5. *[18]* Let f and g be two self-mappings defined on a nonempty set X. Then f and g is said to be **weakly compatible** if they commute at every coincidence point, that is, $f\mathfrak{x} = g\mathfrak{x}$ implies $fg\mathfrak{x} = gf\mathfrak{x}$, for every $\mathfrak{x} \in C(f, g)$.

In the following, we suppose that $(X, \varrho, \perp)$ is complete to ease the notation let us now denote $X = (X, \varrho, \perp)$. And the mappings $f, g : X \to X$ satisfying $f(X) = g(X)$, $g(X)$ is an closed subset of X, g is one-to-one and $g^{-1} \circ f$ is an orthogonal preserving map. Also, we suppose that $\psi \in \Psi$ and $\varphi \in \Phi$.

Theorem 3.6. *If there are functions ψ and φ such that for mappings f and g, we have*

$$\psi(w_\varrho^2(2)[\varrho(f\mathfrak{x} - f\mathfrak{y})]^2) \leq \psi(N(\mathfrak{x},\mathfrak{y})) - \varphi(M(\mathfrak{x},\mathfrak{y})), \qquad \forall \mathfrak{x} \perp \mathfrak{y}, \tag{3.1}$$

where

$$\begin{aligned} N(\mathfrak{x},\mathfrak{y}) \;=\; & \max\{[\varrho(f\mathfrak{x}-g\mathfrak{x})]^2, [\varrho(g\mathfrak{x}-g\mathfrak{y})]^2, [\varrho(f\mathfrak{y}-g\mathfrak{y})]^2, \varrho(f\mathfrak{x}-g\mathfrak{y})\varrho(f\mathfrak{x}-g\mathfrak{x}), \\ & \varrho(f\mathfrak{x}-g\mathfrak{x})\varrho(f\mathfrak{x}-f\mathfrak{y}), \varrho(f\mathfrak{x}-g\mathfrak{x})\varrho(g\mathfrak{x}-g\mathfrak{y})\}, \end{aligned}$$

and

$$M(\mathfrak{x},\mathfrak{y}) = \max\{[\varrho(f\mathfrak{y}-g\mathfrak{y})]^2, [\varrho(f\mathfrak{x}-g\mathfrak{y})]^2, [\varrho(g\mathfrak{x}-g\mathfrak{y})]^2, \frac{[\varrho(f\mathfrak{x}-g\mathfrak{x})]^2[1+[\varrho(g\mathfrak{x}-g\mathfrak{y})]^2]}{1+[\varrho(f\mathfrak{x}-g\mathfrak{y})]^2}.$$

Then f and g have a unique coincidence point in X. Moreover, f and g have a unique common fixed point provided that f and g are weakly compatible.

Proof . Since X is an O-set, there exists $\mathfrak{x}_0 \in X$ such that without loss of generality

$$\mathfrak{x}_0 \perp \mathfrak{x} \quad \forall \mathfrak{x} \in X. \tag{3.2}$$

As $fX \subseteq gX$, there exists $\mathfrak{x}_1 \in X$ such that $f\mathfrak{x}_0 = g\mathfrak{x}_1$. Now, we define the sequence $(\mathfrak{x}_n)$ and $(\mathfrak{y}_n)$ in X by $f\mathfrak{x}_n = g\mathfrak{x}_{n+1}$ for all $n \in \mathbb{N}$, i.e. $\mathfrak{x}_{n+1} = (g^{-1} \circ f)\mathfrak{x}_n$, and $\mathfrak{y}_n = f\mathfrak{x}_n = g\mathfrak{x}_{n+1}$. If $\mathfrak{y}_n = \mathfrak{y}_{n+1}$ for some $n \in \mathbb{N}$, then we have $\mathfrak{y}_n = \mathfrak{y}_{n+1} = f\mathfrak{x}_{n+1} = g\mathfrak{x}_{n+1}$ and f and g have a point of coincidence. Without loss of generality, we assume that $\mathfrak{y}_n \neq \mathfrak{y}_{n+1}$ for all $n \in \mathbb{N}$.

By (3.2), for each $k \in \mathbb{N}$, we have $\mathfrak{x}_0 \perp \mathfrak{x}_k$, and since $g^{-1} \circ f$ is an orthogonal preserving map, these imply that $g^{-1} \circ f\mathfrak{x}_0 \perp g^{-1} \circ f\mathfrak{x}_k$ i.e $\mathfrak{x}_1 \perp \mathfrak{x}_{k+1}$. By induction, for each $n, k \in \mathbb{N}$, we will have

$$\mathfrak{x}_n \perp \mathfrak{x}_{n+k}. \tag{3.3}$$

Now, according (3.3) and (3.1) with $\mathfrak{x} = \mathfrak{x}_n$ and $\mathfrak{y} = \mathfrak{x}_{n+1}$, we obtain

$$\psi(w_\varrho^2(2)[\varrho(\mathfrak{y}_n - \mathfrak{y}_{n+1})]^2) = \psi(w_\varrho^2(2)[\varrho(f\mathfrak{x}_n - f\mathfrak{x}_{n+1})]^2) \leq \psi(N(\mathfrak{x}_n, \mathfrak{x}_{n+1})) - \varphi(M(\mathfrak{x}_n, \mathfrak{x}_{n+1})), \tag{3.4}$$

where

$$\begin{aligned} N(\mathfrak{x}_n, \mathfrak{x}_{n+1}) &= \max\{[\varrho(\mathfrak{y}_n - \mathfrak{y}_{n-1)}]^2, [\varrho(\mathfrak{y}_{n-1} - \mathfrak{y}_n)]^2, [\varrho(\mathfrak{y}_{n+1} - \mathfrak{y}_n)]^2, \\ &\quad \varrho(\mathfrak{y}_n - \mathfrak{y}_n)\varrho(\mathfrak{y}_n - \mathfrak{y}_{n-1}), \varrho(\mathfrak{y}_n - \mathfrak{y}_{n-1})\varrho(\mathfrak{y}_n - \mathfrak{y}_{n+1}), [\varrho(\mathfrak{y}_n - \mathfrak{y}_{n-1})]^2]\}, \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} M(\mathfrak{x}_n, \mathfrak{x}_{n+1}) &= \max\{[\varrho(\mathfrak{y}_{n+1} - \mathfrak{y}_n)]^2, [\varrho(\mathfrak{y}_n - \mathfrak{y}_n)]^2, [\varrho(\mathfrak{y}_{n-1} - \mathfrak{y}_n)]^2, \\ &\quad \frac{[\varrho(\mathfrak{y}_n - \mathfrak{y}_{n-1})]^2[1 + [\varrho(\mathfrak{y}_{n-1} - \mathfrak{y}_n)]^2]}{1 + [\varrho(\mathfrak{y}_n - \mathfrak{y}_n)]^2}\}. \end{aligned} \tag{3.6}$$

If $\varrho(\mathfrak{y}_n - \mathfrak{y}_{n+1}) \geq \varrho(\mathfrak{y}_n - \mathfrak{y}_{n-1}) > 0$ for some $n \in \mathbb{N}$, in view of (3.5) and (3.6), we have

$$N(\mathfrak{x}_n, \mathfrak{x}_{n+1}) = [\varrho(\mathfrak{y}_n - \mathfrak{y}_{n+1})]^2$$

and

$$M(\mathfrak{x}_n, \mathfrak{x}_{n+1}) \geq \max\{[\varrho(\mathfrak{y}_n - \mathfrak{y}_{n+1}]^2, [\varrho(\mathfrak{y}_n - \mathfrak{y}_{n-1})]^2\} = [\varrho(\mathfrak{y}_n - \mathfrak{y}_{n+1})]^2.$$

It follows from inequality (3.4) and the above inequalities,

$$\begin{aligned} \psi([\varrho(\mathfrak{y}_n - \mathfrak{y}_{n+1})]^2) &\leq \psi(w_\varrho^2(2)[\varrho(\mathfrak{y}_n - \mathfrak{y}_{n+1})]^2) \\ &\leq \psi(N(\mathfrak{x}_n, \mathfrak{x}_{n+1})) - \varphi(M(\mathfrak{x}_n, \mathfrak{x}_{n+1})) \\ &\leq \psi([\varrho(\mathfrak{y}_n - \mathfrak{y}_{n+1})]^2) - \varphi([\varrho(\mathfrak{y}_n - \mathfrak{y}_{n+1})]^2), \end{aligned} \tag{3.7}$$

which implies $\varphi([\varrho(\mathfrak{y}_n - \mathfrak{y}_{n+1})]^2) = 0$, that is, $\mathfrak{y}_n = \mathfrak{y}_{n+1}$, a contradiction. Hence, $\varrho(\mathfrak{y}_n - \mathfrak{y}_{n+1}) < \varrho(\mathfrak{y}_n - \mathfrak{y}_{n-1})$ and $(\varrho(\mathfrak{y}_n - \mathfrak{y}_{n+1}))$ is a non-increasing sequence and so there exists $r \geq 0$ such that $\lim \varrho(\mathfrak{y}_n - \mathfrak{y}_{n+1}) = r$. By virtue of (3.5) and (3.6) again, we have $N(\mathfrak{x}_n, \mathfrak{x}_{n+1}) = [\varrho(\mathfrak{y}_n - \mathfrak{y}_{n-1})]^2$, and $M(\mathfrak{x}_n, \mathfrak{x}_{n+1}) = [\varrho(\mathfrak{y}_n - \mathfrak{y}_{n-1})]^2$. It follows that

$$\begin{aligned} \psi([\varrho(\mathfrak{y}_n - \mathfrak{y}_{n+1})]^2) &\leq \psi(N(\mathfrak{x}_n, \mathfrak{x}_{n+1})) - \varphi(M(\mathfrak{x}_n, \mathfrak{x}_{n+1})) \\ &\leq \psi([\varrho(\mathfrak{y}_n - \mathfrak{y}_{n-1})]^2) - \varphi([\varrho(\mathfrak{y}_n - \mathfrak{y}_{n-1})]^2). \end{aligned}$$

Now suppose that $r > 0$. By taking the limit as $n \to +\infty$ in (3.7), we have $\psi(r^2) \leq \psi(r^2) - \varphi(r^2)$ a contradiction. This yields that

$$\lim \varrho(\mathfrak{y}_n - \mathfrak{y}_{n+1}) = r = 0. \tag{3.8}$$

Now we shall prove that $(\mathfrak{y}_n)$ is a Cauchy sequence in X. Suppose on the contrary that, $\lim_{n,m\to+\infty} \varrho(\mathfrak{y}_n - \mathfrak{y}_m) \neq 0$. It follows that there exists $\epsilon > 0$ for which one can find sequences $(\mathfrak{y}_{m_k})$ and $(\mathfrak{y}_{n_k})$ of $(\mathfrak{y}_n)$ satisfying n_k is the smallest index for which $n_k > m_k > k$, $\epsilon \leq \varrho(\mathfrak{y}_{m_k} - \mathfrak{y}_{n_k})$ and $\varrho(\mathfrak{y}_{m_k} - \mathfrak{y}_{n_k-1}) < \epsilon$. In view of the triangle inequality in modular space, we get

$$\begin{aligned} \epsilon^2 &\leq [\varrho(\mathfrak{y}_{m_k} - \mathfrak{y}_{n_k})]^2 \\ &\leq [w_\varrho(2)\varrho(\mathfrak{y}_{m_k} - \mathfrak{y}_{n_k-1}) + w_\varrho(2)\varrho(\mathfrak{y}_{n_k-1} - \mathfrak{y}_{n_k})]^2 \\ &= w_\varrho^2(2)[\varrho(\mathfrak{y}_{m_k} - \mathfrak{y}_{n_k-1})]^2 \\ &+ w_\varrho^2(2)[\varrho(\mathfrak{y}_{n_k-1} - \mathfrak{y}_{n_k})]^2 + 2w_\varrho^2(2)\varrho(\mathfrak{y}_{m_k} - \mathfrak{y}_{n_k-1})\varrho(\mathfrak{y}_{n_k-1} - \mathfrak{y}_{n_k}) \\ &\leq w_\varrho^2(2)\epsilon^2 + w_\varrho^2(2)[\varrho(\mathfrak{y}_{n_k-1} - \mathfrak{y}_{n_k})]^2 + 2w_\varrho^2(2)\varrho(\mathfrak{y}_{m_k} - \mathfrak{y}_{n_k-1})\varrho(\mathfrak{y}_{n_k-1} - \mathfrak{y}_{n_k}) \end{aligned}$$

Using equality (3.8) and taking the upper limit as $k \to \infty$ in the above inequality, we obtain

$$\epsilon^2 \leq \limsup_k [\varrho(\mathfrak{y}_{m_k} - \mathfrak{y}_{n_k})]^2 \leq w_\varrho^2(2)\epsilon^2.$$

As the same arguments, we deduce the following results:

$$\begin{aligned} \epsilon^2 &\leq [\varrho(\mathfrak{y}_{m_k} - \mathfrak{y}_{n_k})]^2 \\ &\leq [w_\varrho(2)\varrho(\mathfrak{y}_{m_k} - \mathfrak{y}_{n_k-1}) + w_\varrho(2)\varrho(\mathfrak{y}_{n_k-1} - \mathfrak{y}_{n_k})]^2 \\ &= w_\varrho^2(2)[\varrho(\mathfrak{y}_{m_k} - \mathfrak{y}_{n_k-1})]^2 + w_\varrho^2(2)[\varrho(\mathfrak{y}_{n_k-1} - \mathfrak{y}_{n_k})]^2 \\ &\quad +2w_\varrho^2(2)\varrho(\mathfrak{y}_{m_k} - \mathfrak{y}_{n_k-1})\varrho(\mathfrak{y}_{n_k-1} - \mathfrak{y}_{n_k}) \end{aligned} \tag{3.9}$$

$$\begin{aligned} \epsilon^2 &\leq [\varrho(\mathfrak{y}_{m_k} - \mathfrak{y}_{n_k})]^2 \\ &\leq [w_\varrho(2)\varrho(\mathfrak{y}_{m_k} - \mathfrak{y}_{m_k-1}) + w_\varrho(2)\varrho(\mathfrak{y}_{m_k-1} - \mathfrak{y}_{n_k})]^2 \\ &= w_\varrho^2(2)[\varrho(\mathfrak{y}_{m_k} - \mathfrak{y}_{m_k-1})]^2 + w_\varrho^2(2)[\varrho(\mathfrak{y}_{m_k-1} - \mathfrak{y}_{n_k})]^2 \\ &+ 2w_\varrho^2(2)\varrho(\mathfrak{y}_{m_k} - \mathfrak{y}_{m_k-1})\varrho(\mathfrak{y}_{m_k-1} - \mathfrak{y}_{n_k}) \end{aligned} \tag{3.10}$$

and

$$\begin{aligned}[\varrho(\mathfrak{y}_{m_k-1}-\mathfrak{y}_{n_k})]^2 &\leq [w_\varrho(2)\varrho(\mathfrak{y}_{m_k-1}-\mathfrak{y}_{m_k})+w_\varrho(2)\varrho(\mathfrak{y}_{m_k}-\mathfrak{y}_{n_k})]^2\\ &= w_\varrho^2(2)[\varrho(\mathfrak{y}_{m_k-1}-\mathfrak{y}_{m_k})]^2+w_\varrho^2(2)[\varrho(\mathfrak{y}_{m_k}-\mathfrak{y}_{n_k})]^2\\ &+ 2w_\varrho^2(2)\varrho(\mathfrak{y}_{m_k-1}-\mathfrak{y}_{m_k})\varrho(\mathfrak{y}_{m_k}-\mathfrak{y}_{n_k}) \qquad (3.11)\end{aligned}$$

In view of (3.9), we have

$$\frac{\epsilon^2}{w_\varrho^2(2)} \leq \limsup_{k\to\infty}[\varrho(\mathfrak{y}_{m_k}-\mathfrak{y}_{n_k-1})]^2 \leq \epsilon^2.$$

Using (3.10) and (3.11), we obtain

$$\frac{\epsilon^2}{w_\varrho^2(2)} \leq \limsup_{k}[\varrho(\mathfrak{y}_{m_k-1}-\mathfrak{y}_{n_k})]^2 \leq w_\varrho^4(2)\epsilon^2$$

Similarly, we deduce that

$$\begin{aligned}[\varrho(\mathfrak{y}_{m_k-1}-\mathfrak{y}_{n_k-1})]^2 &\leq [w_\varrho(2)\varrho(\mathfrak{y}_{m_k-1}-\mathfrak{y}_{m_k})+w_\varrho(2)\varrho(\mathfrak{y}_{m_k}-\mathfrak{y}_{n_k-1})]^2\\ &= w_\varrho^2(2)[\varrho(\mathfrak{y}_{m_k-1}-\mathfrak{y}_{m_k})]^2+w_\varrho^2(2)[\varrho(\mathfrak{y}_{m_k}-\mathfrak{y}_{n_k-1})]^2\\ &+ 2w_\varrho^2(2)\varrho(\mathfrak{y}_{m_k-1}-\mathfrak{y}_{m_k})\varrho(\mathfrak{y}_{m_k}-\mathfrak{y}_{n_k-1}),\end{aligned}$$

and

$$\begin{aligned}[\varrho(\mathfrak{y}_{m_k}-\mathfrak{y}_{n_k})]^2 &\leq [w_\varrho(2)\varrho(\mathfrak{y}_{m_k}-\mathfrak{y}_{m_k-1})+w_\varrho(2)\varrho(\mathfrak{y}_{m_k-1}-\mathfrak{y}_{n_k})]^2\\ &= w_\varrho^2(2)[\varrho(\mathfrak{y}_{m_k}-\mathfrak{y}_{m_k-1})]^2+w_\varrho^2(2)[\varrho(\mathfrak{y}_{m_k-1}-\mathfrak{y}_{n_k})]^2\\ &+ 2w_\varrho^2(2)\varrho(\mathfrak{y}_{m_k}-\mathfrak{y}_{m_k-1})\varrho(\mathfrak{y}_{m_k-1}-\mathfrak{y}_{n_k})\\ &\leq w_\varrho^2(2)[\varrho(\mathfrak{y}_{m_k}-\mathfrak{y}_{m_k-1})]^2+w_\varrho^2(2)[w_\varrho(2)\varrho(\mathfrak{y}_{m_k-1}-\mathfrak{y}_{n_k-1})\\ &+ w_\varrho(2)\varrho(\mathfrak{y}_{n_k-1}-\mathfrak{y}_{n_k})]^2\\ &+ 2w_\varrho^2(2)\varrho(\mathfrak{y}_{m_k}-\mathfrak{y}_{m_k-1})[w_\varrho(2)\varrho(\mathfrak{y}_{m_k-1}-\mathfrak{y}_{n_k-1})\\ &+ w_\varrho(2)\varrho(\mathfrak{y}_{n_k-1}-\mathfrak{y}_{n_k})].\end{aligned}$$

It follows that

$$\frac{\epsilon^2}{w_\varrho^4(2)} \leq \limsup_{k}[\varrho(\mathfrak{y}_{m_k-1}-\mathfrak{y}_{n_k-1})]^2 \leq w_\varrho^2(2)\epsilon^2.$$

Through the definition of N, we have

$$\begin{aligned}N(\mathfrak{x}_{m_k},\mathfrak{x}_{n_k}) &= \max\{[\varrho(\mathfrak{y}_{m_k}-\mathfrak{y}_{m_k-1})]^2,[\varrho(\mathfrak{y}_{m_k-1}-\mathfrak{y}_{n_k-1})]^2,\\ &[\varrho(\mathfrak{y}_{n_k}-\mathfrak{y}_{n_k-1})]^2,\\ &\varrho(\mathfrak{y}_{m_k}-\mathfrak{y}_{n_k-1})\varrho(\mathfrak{y}_{m_k}-\mathfrak{y}_{m_k-1}),\varrho(\mathfrak{y}_{m_k}-\mathfrak{y}_{m_k-1})\varrho(\mathfrak{y}_{m_k}-\mathfrak{y}_{n_k}),\\ &\varrho(\mathfrak{y}_{m_k}-\mathfrak{y}_{m_k-1})\varrho(\mathfrak{y}_{m_k-1}-\mathfrak{y}_{n_k-1})\}\end{aligned}$$

which yields that

$$\limsup_k N(\mathfrak{x}_{m_k},\mathfrak{x}_{n_k}) \leq \; max\{0, w_\varrho^2(2)\epsilon^2, 0, 0, 0, 0\} = \epsilon^2 w_\varrho^2(2). \tag{3.12}$$

Also,

$$\begin{aligned} M(\mathfrak{x}_{m_k},\mathfrak{x}_{n_k}) &= \max\{[\varrho(\mathfrak{y}_{n_k}-\mathfrak{y}_{n_k-1})]^2, [\varrho(\mathfrak{y}_{m_k}-\mathfrak{y}_{n_k-1})]^2, [\varrho(\mathfrak{y}_{m_k-1}-\mathfrak{y}_{n_k-1})]^2, \\ &\quad \frac{[\varrho(\mathfrak{y}_{m_k}-\mathfrak{y}_{m_k-1})]^2[1+[\varrho(\mathfrak{y}_{m_k-1}-\mathfrak{y}_{n_k-1}]^2]}{1+[\varrho(\mathfrak{y}_{m_k}-\mathfrak{y}_{n_k-1})]^2}\} \end{aligned}$$

It is easy to show that

$$\liminf_k M(\mathfrak{x}_{m_k},\mathfrak{x}_{n_k}) \geq \max\{0, \frac{\epsilon^2}{w_\varrho^2(2)}, \frac{\epsilon^2}{w_\varrho^4(2)}, 0\} \geq \frac{\epsilon^2}{w_\varrho^4(2)}. \tag{3.13}$$

Again, according (3.1) and (3.3), with $\mathfrak{x}=\mathfrak{x}_{m_k}$ and $\mathfrak{y}=\mathfrak{x}_{n_k}$, since $\mathfrak{x}_{m_k}\perp\mathfrak{x}_{n_k}$, we get

$$\psi([\varrho(\mathfrak{y}_{m_k}-\mathfrak{y}_{n_k})]^2) \leq \psi(w_\varrho^2(2)[\varrho(\mathfrak{y}_{m_k}-\mathfrak{y}_{n_k})]^2) \leq \psi(N(\mathfrak{x}_{m_k},\mathfrak{x}_{n_k})) - \varphi(M(\mathfrak{x}_{m_k},\mathfrak{x}_{n_k})).$$

In light of (3.12), one can obtain

$$\begin{aligned} \psi(w_\varrho^2(2)\epsilon^2) &\leq \psi(w_\varrho^2(2)\limsup_k[\varrho(f\mathfrak{x}_{m_k}-f\mathfrak{x}_{n_k})]^2) \\ &\leq \psi(\limsup_k N(\mathfrak{x}_{m_k},\mathfrak{x}_{n_k})) - \varphi(\liminf_k M(\mathfrak{x}_{m_k},\mathfrak{x}_{n_k})) \\ &\leq \psi(w_\varrho^2(2)\epsilon^2) - \varphi(\liminf_k M(\mathfrak{x}_{m_k},\mathfrak{x}_{n_k})) \end{aligned}$$

which implies that

$$\liminf_k M(\mathfrak{x}_{m_k},\mathfrak{x}_{n_k}) = 0$$

a contradiction to (3.13). It follows that $(\mathfrak{y}_n)$ is a Cauchy sequence in X. Since X is complete modular space, there exists $\mathfrak{u}\in X$ such that

$$\lim \varrho(\mathfrak{y}_n-\mathfrak{u}) = \lim \varrho(f\mathfrak{x}_n-\mathfrak{u}) = \lim \varrho(g\mathfrak{x}_{n+1}-\mathfrak{u}) = 0. \tag{3.14}$$

Furthermore, we have $\mathfrak{u}\in g(X)$ since $g(X)$ is closed. It follows that one can choose a $\mathfrak{z}\in X$ such that $\mathfrak{u}=g\mathfrak{z}$, and one can write (3.14) as

$$\lim \varrho(\mathfrak{y}_n-g\mathfrak{z}) = \lim \varrho(f\mathfrak{x}_n-g\mathfrak{z}) = \lim \varrho(g\mathfrak{x}_{n+1}-g\mathfrak{z}) = 0.$$

Now, we prove that $\mathfrak{x}_n\perp\mathfrak{z}$. Since $f(X)=g(X)$, we have $g^{-1}\circ f(X)=X$, by continuing this way, we will have $(g^{-1}\circ f\circ\ldots g^{-1}\circ f)(X)=X$. Therefore there is $\mathfrak{z}_1$ such that $(g^{-1}\circ f\circ\ldots g^{-1}\circ f)\mathfrak{z}_1=\mathfrak{z}$. Since $\mathfrak{x}_0\perp\mathfrak{z}_1$ and $g^{-1}\circ f$ is orthogonal preserving map, we have $\mathfrak{x}_n\perp\mathfrak{z}$. From (3.1) with taking $\mathfrak{x}=\mathfrak{x}_{n_k}$ and $\mathfrak{y}=\mathfrak{z}$, we get

$$\psi(w_\varrho^2(2)[\varrho(\mathfrak{y}_{n_k} - f\mathfrak{z})]^2) = \psi(w_\varrho^2(2)[\varrho(f\mathfrak{x}_{n_k} - f\mathfrak{z})]^2) \leq \psi(N(\mathfrak{x}_{n_k}, \mathfrak{z})) - \varphi(M(\mathfrak{x}_{n_k}, \mathfrak{z})), \tag{3.15}$$

where

$$\begin{aligned} N(\mathfrak{x}_{n_k}, \mathfrak{z}) &= \max\{[\varrho(\mathfrak{y}_{n_k} - \mathfrak{y}_{n_k-1})]^2, [\varrho(\mathfrak{y}_{n_k-1} - g\mathfrak{z})]^2, [\varrho(f\mathfrak{z} - g\mathfrak{z})]^2, \\ &\quad \varrho(\mathfrak{y}_{n_k} - g\mathfrak{z})\varrho(\mathfrak{y}_{n_k} - \mathfrak{y}_{n_k-1}), \varrho(\mathfrak{y}_{n_k} - \mathfrak{y}_{n_k-1})\varrho(\mathfrak{y}_{n_k} - f\mathfrak{z}), \\ &\quad \varrho(\mathfrak{y}_{n_k} - \mathfrak{y}_{n_k-1})\varrho(\mathfrak{y}_{n_k-1} - g\mathfrak{z})\}, \end{aligned}$$

and

$$\begin{aligned} M(\mathfrak{x}_{n_k}, \mathfrak{z}) &= \max\{[\varrho(f\mathfrak{z} - g\mathfrak{z})]^2, [\varrho(\mathfrak{y}_{n_k} - g\mathfrak{z})]^2, \\ &\quad [\varrho(\mathfrak{y}_{n_k-1} - g\mathfrak{z})]^2, \frac{[\varrho(\mathfrak{y}_{n_k} - \mathfrak{y}_{n_k-1})]^2[1 + [\varrho(\mathfrak{y}_{n_k-1} - g\mathfrak{z})]^2]}{1 + [\varrho(\mathfrak{y}_{n_k} - g\mathfrak{z})]^2}\}. \end{aligned}$$

And then we obtain

$$\limsup_k N(\mathfrak{x}_{n_k}, \mathfrak{z}) = \max\{0, 0, [\varrho(g\mathfrak{z} - f\mathfrak{z})]^2, 0, 0, 0\} = [\varrho(g\mathfrak{z} - f\mathfrak{z})]^2,$$

and

$$\liminf_k M(\mathfrak{x}_{n_k}, \mathfrak{z}) = \max\{[\varrho(g\mathfrak{z} - f\mathfrak{z})]^2, 0, 0, 0\} = [\varrho(g\mathfrak{z} - f\mathfrak{z})]^2.$$

Taking the upper limit as $k \to \infty$ in (3.15), by Lemma 3.2, we will have

$$\begin{aligned} \psi([\varrho(g\mathfrak{z} - f\mathfrak{z})]^2) &= \psi(w_\varrho^2(2)\frac{1}{w_\varrho^2(2)}[\varrho(g\mathfrak{z} - f\mathfrak{z})]^2) \\ &\leq \psi(w_\varrho^2(2)[\limsup_k \varrho(f\mathfrak{x}_{n_k} - f\mathfrak{z})]^2) \\ &\leq \psi(\limsup_k N(\mathfrak{x}_{n_k}, \mathfrak{z})) - \varphi(\liminf_k M(\mathfrak{x}_{n_k}, \mathfrak{z})) \\ &= \psi([\varrho(g\mathfrak{z} - f\mathfrak{z})]^2) - \varphi([\varrho(g\mathfrak{z} - f\mathfrak{z})]^2) \end{aligned}$$

which implies that $\varphi([\varrho(f\mathfrak{z} - g\mathfrak{z})]^2) = 0$. It follows that $\varrho(f\mathfrak{z} - g\mathfrak{z}) = 0$. That is, $f\mathfrak{z} = g\mathfrak{z}$. Therefore, $\mathfrak{u} = f\mathfrak{z} = g\mathfrak{z}$ is a point of coincidence for f and g. We also conclude that the point of coincidence is unique. Assume on the contrary that, there exist $\mathfrak{c}, \mathfrak{c}' \in C(f, g)$ i.e. $f\mathfrak{z} = g\mathfrak{z} = \mathfrak{c}$, and $f\mathfrak{z}' = g\mathfrak{z}' = \mathfrak{c}'$ but $\mathfrak{c} \neq \mathfrak{c}'$. Since $\mathfrak{x}_0 \perp \mathfrak{z}'$, $g^{-1} \circ f$ is orthogonal preserving map and $g^{-1} \circ f\mathfrak{z}' = \mathfrak{z}'$, so $\mathfrak{x}_n \perp \mathfrak{z}'$. Using (3.1) with $\mathfrak{x} = \mathfrak{x}_n$ and $\mathfrak{y} = \mathfrak{z}'$, implies that

$$\psi(w_\varrho^2(2)[\varrho(f\mathfrak{x}_n - f\mathfrak{z}')]^2) \leq \psi(N(\mathfrak{x}_n, \mathfrak{z}')) - \varphi(M(\mathfrak{x}_n, \mathfrak{z}')). \tag{3.16}$$

By Lemma 3.2, we obtain

$$\lim N(\mathfrak{x}_n, \mathfrak{z}') \leq \lim[\varrho(f\mathfrak{x}_n - f\mathfrak{z}')]^2,$$

and

$$\lim M(\mathfrak{x}_n, \mathfrak{z}') \leq \lim[\varrho(f\mathfrak{x}_n - f\mathfrak{z}']^2.$$

As $n \to +\infty$, and replacing the last inequality in (3.16), we will have

$$\begin{aligned}\psi(w_\varrho^2(2)\lim[\varrho(f\mathfrak{x}_n - f\mathfrak{z}']^2) &\leq \psi(\lim[\varrho(f\mathfrak{x}_n - f\mathfrak{z}']^2) - \varphi(\lim[\varrho(f\mathfrak{x}_n - f\mathfrak{z}']^2) \\ &\leq \psi(\lim[\varrho(f\mathfrak{x}_n - f\mathfrak{z}']^2).\end{aligned}$$

This implies that $\lim \varrho(f\mathfrak{x}_n - f\mathfrak{z}') = 0$. Again, by Lemma 3.2, we have

$$\frac{1}{w_\varrho(2)}\varrho(f\mathfrak{z} - f\mathfrak{z}') \leq \lim \varrho(f\mathfrak{x}_n - f\mathfrak{z}') = 0.$$

From the above inequality, we get $f\mathfrak{z} = f\mathfrak{z}'$. That is, the point of coincidence is unique. Considering the weak compatibility of f and g, for each coincidence point $\mathfrak{z}$, that $f\mathfrak{z} = g\mathfrak{z}$, we have $gf\mathfrak{z} = gg\mathfrak{z}$ i.e. $fg\mathfrak{z} = gg\mathfrak{z}$. This implies that $gg\mathfrak{z}$ is a point of coincidence. Since the point of coincidence is unique so $gg\mathfrak{z} = g\mathfrak{z}$. The mapping g is one-to- one, so $g\mathfrak{z} = \mathfrak{z}$. These imply that $\mathfrak{z}$ is a common fixed point of f and g. □

Example 3.7. Define $\varrho(\mathfrak{x}) = \mathfrak{x}^2$, $\mathfrak{x} \perp \mathfrak{y}$ if $\mathfrak{x} \leq \mathfrak{y}$, and $f, g : [0, \infty) \to [0, \infty)$ by

$$f\mathfrak{x} = \frac{\mathfrak{x}}{64}, \qquad g\mathfrak{x} = \frac{\mathfrak{x}}{2}.$$

The auxiliary functions $\psi, \varphi : [0, \infty) \to [0, \infty)$ are defined as

$$\psi(t) = \frac{5t}{4}, \quad \varphi(t) = \frac{48545t}{87846} \quad \forall t \in [0, \infty).$$

It is clearly that $fX = gX$, gX is closed, f, g are nondecreasing so they are orthogonal functions and in the same way $g^{-1} \circ f$ is orthogonal preserving. For all $\mathfrak{x}, \mathfrak{y} \in X$ such that $\mathfrak{x} \perp \mathfrak{y}$, we have

$$\psi(w_\varrho^2(2)[\varrho(f\mathfrak{x} - f\mathfrak{y})]^2) = \psi(4(\frac{\mathfrak{x}}{64} - \frac{\mathfrak{y}}{64})^4) = \frac{5}{4}.4(\frac{\mathfrak{x}}{64} - \frac{\mathfrak{y}}{64})^4 = \frac{5}{64^4}(\mathfrak{x} - \mathfrak{y})^4.$$

We have

$$\psi(N(\mathfrak{x}, \mathfrak{y})) \geq \psi([\varrho(g\mathfrak{x} - g\mathfrak{y})]^2) = \frac{5}{4}(\frac{\mathfrak{x}}{2} - \frac{\mathfrak{y}}{2})^4 = \frac{5}{64}(\mathfrak{x} - \mathfrak{y})^4;$$

and

$$\varphi(M(\mathfrak{x}, \mathfrak{y})) = \max\{(\frac{\mathfrak{y}}{64} - \frac{\mathfrak{y}}{2})^4, (\frac{\mathfrak{x}}{64} - \frac{\mathfrak{y}}{2})^4, (\frac{\mathfrak{x}}{2} - \frac{\mathfrak{y}}{2})^4, \frac{(\frac{\mathfrak{x}}{64} - \frac{\mathfrak{x}}{2})^4[1 + (\frac{\mathfrak{x}}{2} - \frac{\mathfrak{y}}{2})^4]}{1 + (\frac{\mathfrak{x}}{64} - \frac{\mathfrak{y}}{2})^4}\}$$

It follows that

$$\begin{aligned}\varphi(M(\mathfrak{x}, \mathfrak{y})) &\leq \varphi(2(\frac{33\mathfrak{x}}{64} - \frac{33\mathfrak{y}}{64})^4) \\ &= \frac{5.64^3 - 5}{2.33^4}.2.(\frac{33}{64})^4(\mathfrak{x} - \mathfrak{y})^4 \\ &= \frac{1310715}{644}(\mathfrak{x} - \mathfrak{y})^4\end{aligned}$$

It is easy to see that

$$\begin{aligned}\psi(w_\varrho^2(2)[\varrho(f\mathfrak{x}-f\mathfrak{y})]^2) &= \psi([\varrho(g\mathfrak{x}-g\mathfrak{y})]^2)-\varphi(2(\frac{33}{64})^4(\mathfrak{x}-\mathfrak{y})^4)\\ &\le \psi(N(\mathfrak{x},\mathfrak{y}))-\varphi(M(\mathfrak{x},\mathfrak{y}))\end{aligned}$$

Therefore, the conditions of Theorem 3.6 are satisfied. It is obviously that 0 is the common fixed point of f and g.

If in Theorem 3.6,we put $\psi(t)=t$ and $\varphi(t)=t$, we can get the following result.

Corollary 3.8. *Every mapping f which is satisfied:*

$$w_\varrho^2(2)[\varrho(f\mathfrak{x}-f\mathfrak{y})]^2 \le N(\mathfrak{x},\mathfrak{y})-M(\mathfrak{x},\mathfrak{y}), \quad \forall \mathfrak{x}\perp\mathfrak{y}.$$

has a fixed point.

Theorem 3.9. *If there are functions ψ and φ such that for mappings f and g, we have*

$$\psi(w_\varrho^2(2)[\varrho(f\mathfrak{x}-f\mathfrak{y})]^2) \le \psi(\mathcal{N}(\mathfrak{x},\mathfrak{y}))-\varphi(\mathcal{M}(\mathfrak{x},\mathfrak{y})), \quad \forall\ \mathfrak{x}\perp\mathfrak{y} \tag{3.17}$$

where

$$\begin{aligned}\mathcal{N}(\mathfrak{x},\mathfrak{y}) &= \max\{\varrho(f\mathfrak{x}-f\mathfrak{y})\varrho(g\mathfrak{x}-g\mathfrak{y}),\varrho(f\mathfrak{y}-g\mathfrak{y})\varrho(f\mathfrak{x}-g\mathfrak{y}),\\ &\quad [\varrho(g\mathfrak{x}-g\mathfrak{y})]^2,\frac{[\varrho(f\mathfrak{y}-g\mathfrak{y})]^2+[\varrho(f\mathfrak{x}-g\mathfrak{y})]^2}{w_\varrho^2(2)}\},\end{aligned}$$

and

$$\begin{aligned}\mathcal{M}(\mathfrak{x},\mathfrak{y}) &= \max\{[\varrho(f\mathfrak{y}-g\mathfrak{y})]^2,[\varrho(f\mathfrak{x}-g\mathfrak{y})]^2,[\varrho(g\mathfrak{x}-g\mathfrak{y})]^2,\\ &\quad \frac{[\varrho(f\mathfrak{x}-g\mathfrak{x})]^2[1+[\varrho(g\mathfrak{x}-g\mathfrak{y})]^2]}{1+[\varrho(f\mathfrak{x}-g\mathfrak{y})]^2},\frac{[\varrho(g\mathfrak{x}-g\mathfrak{y})]^2[1+[\varrho(g\mathfrak{x}-g\mathfrak{y})]^2]}{1+[\varrho(f\mathfrak{x}-g\mathfrak{x})]^2}\}.\end{aligned}$$

have a unique coincidence point in X. Moreover, f and g have a unique common fixed point provided that f and g are weakly compatible.

Proof . As same as Theorem 3.6, we define the sequences $(\mathfrak{x}_n)$ and $(\mathfrak{y}_n)$ in X by $\mathfrak{y}_n=f\mathfrak{x}_n=g\mathfrak{x}_{n+1}$. It is easy to show that for all $n\in\mathbb{N}$ and $\mathfrak{x}\in X$, we will have $\mathfrak{x}_n\perp\mathfrak{x}$. We also suppose that $\mathfrak{y}_n\neq\mathfrak{y}_{n+1}$ for each $n\in\mathbb{N}$, it follows from (3.17) that

$$\psi(w_\varrho^2(2)[\varrho(\mathfrak{y}_n-\mathfrak{y}_{n+1})]^2)=\psi(w_\varrho^2(2)[(f\mathfrak{x}_n-f\mathfrak{x}_{n+1})]^2)\le\psi(\mathcal{N}(\mathfrak{x}_n,\mathfrak{x}_{n+1}))-\varphi(\mathcal{M}(\mathfrak{x}_n,\mathfrak{x}_{n+1})), \tag{3.18}$$

where

$$\begin{aligned}\mathcal{N}(\mathfrak{x}_n,\mathfrak{x}_{n+1}) &= \max\{\varrho(\mathfrak{y}_n-\mathfrak{y}_{n+1})\varrho(\mathfrak{y}_{n-1}-\mathfrak{y}_n),\varrho(\mathfrak{y}_{n+1}-\mathfrak{y}_n)\varrho(\mathfrak{y}_n-\mathfrak{y}_n),[\varrho(\mathfrak{y}_{n-1}-\mathfrak{y}_n)]^2,\\ &\quad \frac{[\varrho(\mathfrak{y}_{n+1}-\mathfrak{y}_n)]^2+[\varrho(\mathfrak{y}_n-\mathfrak{y}_n)]^2}{w_\varrho^2(2)}\}\end{aligned} \tag{3.19}$$

$$\begin{aligned}\mathcal{M}(\mathfrak{x}_n,\mathfrak{x}_{n+1}) &= \max\{[\varrho(\mathfrak{y}_{n+1}-\mathfrak{y}_n)]^2,[\varrho(\mathfrak{y}_n-\mathfrak{y}_n)]^2,[\varrho(\mathfrak{y}_{n-1}-\mathfrak{y}_n)]^2,\\ &\frac{[\varrho(\mathfrak{y}_n-\mathfrak{y}_{n-1})]^2[1+[\varrho(\mathfrak{y}_{n-1}-\mathfrak{y}_n)]^2]}{1+[\varrho(\mathfrak{y}_n-\mathfrak{y}_n)]^2},\\ &\frac{[\varrho(\mathfrak{y}_{n-1}-\mathfrak{y}_n)]^2[1+[\varrho(\mathfrak{y}_{n-1}-\mathfrak{y}_n)]^2]}{1+[\varrho(\mathfrak{y}_n-\mathfrak{y}_{n-1})]^2}\}.\end{aligned} \tag{3.20}$$

If we assume that, for some $n\in\mathbb{N}$,

$$\varrho(\mathfrak{y}_n-\mathfrak{y}_{n+1})\geq\varrho(\mathfrak{y}_{n-1}-\mathfrak{y}_n)>0;$$

then from inequality (3.19) and (3.20), we get that

$$\mathcal{N}(\mathfrak{x}_n,\mathfrak{x}_{n+1})\leq[\varrho(\mathfrak{y}_{n+1}-\mathfrak{y}_n)]^2;$$

$$\mathcal{M}(\mathfrak{x}_n,\mathfrak{x}_{n+1})\geq[\varrho(\mathfrak{y}_{n+1}-\mathfrak{y}_n)]^2.$$

In view of (3.18), we have the following inequality,

$$\begin{aligned}\psi([\varrho(\mathfrak{y}_n-\mathfrak{y}_{n+1})]^2) &\leq \psi(w_\varrho^2(2)[\varrho(\mathfrak{y}_n-\mathfrak{y}_{n+1})]^2)\\ &\leq \psi(\mathcal{N}(\mathfrak{x}_n,\mathfrak{x}_{n+1}))-\varphi(\mathcal{M}(\mathfrak{x}_n,\mathfrak{x}_{n+1}))\\ &\leq \psi([\varrho(\mathfrak{y}_n-\mathfrak{y}_{n+1})]^2)-\varphi([\varrho(\mathfrak{y}_n-\mathfrak{y}_{n+1})]^2),\end{aligned}$$

which gives that $\varrho(\mathfrak{y}_n-\mathfrak{y}_{n+1})=0$, a contradiction to $\varrho(\mathfrak{y}_n,\mathfrak{y}_{n+1})>0$. It follows that

$$\varrho(\mathfrak{y}_n-\mathfrak{y}_{n+1})<\varrho(\mathfrak{y}_n-\mathfrak{y}_{n-1}).$$

Hence, the sequence $(\varrho(\mathfrak{y}_n-\mathfrak{y}_{n+1}))$ is non-increasing. Consequently, the limit of the sequence is a nonnegative number, say $r\geq 0$. That is, $\lim\varrho(\mathfrak{y}_n-\mathfrak{y}_{n+1})=r$. According to (3.19) and (3.20), we have

$$\mathcal{N}(\mathfrak{x}_n,\mathfrak{x}_{n+1})\leq[\varrho(\mathfrak{y}_n-\mathfrak{y}_{n-1})]^2;$$

and

$$\mathcal{M}(\mathfrak{x}_n,\mathfrak{x}_{n+1})\geq[\varrho(\mathfrak{y}_n-\mathfrak{y}_{n-1})]^2.$$

So,

$$\begin{aligned}\psi([\varrho(\mathfrak{y}_n-\mathfrak{y}_{n+1})]^2) &\leq \psi(\mathcal{N}(\mathfrak{x}_n,\mathfrak{x}_{n+1}))-\varphi(\mathcal{M}(\mathfrak{x}_n,\mathfrak{x}_{n+1}))\\ &\leq \psi([\varrho(\mathfrak{y}_n-\mathfrak{y}_{n-1})]^2)-\varphi([\varrho(\mathfrak{y}_n,\mathfrak{y}_{n-1})]^2).\end{aligned}$$

If $r>0$, then letting $n\to\infty$ in above inequality, we obtain that $\psi(r^2)\leq\psi(r^2)-\varphi(r^2)$ which implies that $r=0$, i.e. $\lim\varrho(\mathfrak{y}_n-\mathfrak{y}_{n+1})=0$. Now we prove that $(\mathfrak{y}_n)$ is a Cauchy sequence. If not, as the proof of Theorem 3.6, there exists $\epsilon>0$ for which one can find sequences $(\mathfrak{y}_{m_k})$ and $(\mathfrak{y}_{n_k})$ of $(\mathfrak{y}_n)$ so that n_k is the smallest index for which $n_k>m_k>k$, and the following inequalities hold:

$$\begin{aligned}\epsilon &\leq \limsup_k\varrho(\mathfrak{y}_{m_k}-\mathfrak{y}_{n_k})\leq w_\varrho(2)\epsilon,\\ \frac{\epsilon}{w_\varrho(2)} &\leq \limsup_k\varrho(\mathfrak{y}_{m_k}-\mathfrak{y}_{n_k-1})\leq\epsilon,\\ \frac{\epsilon}{w_\varrho(2)} &\leq \limsup_k\varrho(\mathfrak{y}_{m_k-1}-\mathfrak{y}_{n_k})\leq w_\varrho^2(2)\epsilon,\\ \frac{\epsilon}{w_\varrho^2(2)} &\leq \limsup_k\varrho(\mathfrak{y}_{m_k-1}-\mathfrak{y}_{n_k-1})\leq w_\varrho(2)\epsilon.\end{aligned} \tag{3.21}$$

We deduce the following equation according to the definitions of $\mathcal{N}(\mathfrak{x},\mathfrak{y})$ and $\mathcal{M}(\mathfrak{x},\mathfrak{y})$,

$$\begin{aligned}\mathcal{N}(\mathfrak{x}_{m_k},\mathfrak{x}_{n_k}) &= \max\{\varrho(\mathfrak{y}_{m_k}-\mathfrak{y}_{n_k})\varrho(\mathfrak{y}_{m_k-1}-\mathfrak{y}_{n_k-1}),\\ &\quad \varrho(\mathfrak{y}_{n_k}-\mathfrak{y}_{n_k-1})\varrho(\mathfrak{y}_{m_k}-\mathfrak{y}_{n_k-1}),[\varrho(\mathfrak{y}_{m_k-1}-\mathfrak{y}_{n_k-1})]^2,\\ &\quad \frac{[\varrho(\mathfrak{y}_{n_k}-\mathfrak{y}_{n_k-1})]^2+[\varrho(\mathfrak{y}_{m_k}-\mathfrak{y}_{n_k-1})]^2}{w_\varrho^2(2)}\},\end{aligned}$$

and

$$\begin{aligned}\mathcal{M}(\mathfrak{x}_{m_k},\mathfrak{x}_{n_k}) &= \max\{[\varrho(\mathfrak{y}_{n_k}-\mathfrak{y}_{n_k-1})]^2,[\varrho(\mathfrak{y}_{m_k}-\mathfrak{y}_{n_k-1})]^2,[\varrho(\mathfrak{y}_{m_k-1}-\mathfrak{y}_{n_k-1})]^2,\\ &\quad \frac{[\varrho(\mathfrak{y}_{m_k}-\mathfrak{y}_{m_k-1})]^2[1+[\varrho(\mathfrak{y}_{m_k-1}-\mathfrak{y}_{n_k-1})]^2]}{1+[\varrho(\mathfrak{y}_{m_k}-\mathfrak{y}_{n_k-1})]^2},\\ &\quad \frac{[\varrho(\mathfrak{y}_{m_k-1}-\mathfrak{y}_{n_k-1})]^2[1+[\varrho(\mathfrak{y}_{m_k-1}-\mathfrak{y}_{n_k-1})]^2]}{1+[\varrho(\mathfrak{y}_{m_k}-\mathfrak{y}_{m_k-1})]^2}\}.\end{aligned}$$

Using (3.21), one can obtain that

$$\limsup_k \mathcal{N}(\mathfrak{x}_{m_k},\mathfrak{x}_{n_k}) \le \max\{w_\varrho^2(2)\epsilon^2,0,w_\varrho^2(2)\epsilon^2,\frac{\epsilon^2}{w_\varrho^2(2)}\} = w_\varrho^2(2)\epsilon^2$$

and

$$\liminf_k \mathcal{M}(\mathfrak{x}_{m_k},\mathfrak{x}_{n_k}) \ge \max\{0,\frac{\epsilon^2}{w_\varrho^2(2)},\frac{\epsilon^2}{w_\varrho^4(2)},0,\frac{\epsilon^2}{w_\varrho^4(2)}(1+\frac{\epsilon^2}{w_\varrho^4(2)})\} \ge \frac{\epsilon^2}{w_\varphi^4(2)} \tag{3.22}$$

Taking $\mathfrak{x}=\mathfrak{x}_{m_k}$ and $\mathfrak{y}=\mathfrak{x}_{n_k}$ in (3.17), we get

$$\psi([\varrho(\mathfrak{y}_{m_k}-\mathfrak{y}_{n_k})]^2) \le \psi(w_\varrho^2(2)[\varrho(\mathfrak{y}_{m_k}-\mathfrak{y}_{n_k})]^2) \le \psi(\mathcal{N}(\mathfrak{x}_{m_k},\mathfrak{x}_{n_k}))-\varphi(\mathcal{M}(\mathfrak{x}_{m_k},\mathfrak{x}_{n_k}))$$

Therefore, we have

$$\begin{aligned}\psi(w_\varrho^2(2)\epsilon^2) &\le \psi(w_\varrho^2(2)\limsup_k[\varrho(f\mathfrak{x}_{m_k}-f\mathfrak{x}_{n_k})]^2)\\ &\le \psi(\limsup_k(\mathcal{N}(\mathfrak{x}_{m_k},\mathfrak{x}_{n_k})))-\varphi(\liminf_k \mathcal{M}(\mathfrak{x}_{m_k},\mathfrak{x}_{n_k}))\\ &\le \psi(w_\varrho^2(2)\epsilon^2)-\varphi(\liminf_k \mathcal{M}(\mathfrak{x}_{m_k},\mathfrak{x}_{n_k})),\end{aligned}$$

and we concluded that $\liminf_k \mathcal{M}(\mathfrak{x}_{m_k},\mathfrak{x}_{n_k})=0$ which gives a contradiction to (3.22). Hence,

$$\lim_{n,m\to\infty}\varrho(\mathfrak{y}_n-\mathfrak{y}_m)=0.$$

The completeness of X ensures that there exists $u\in X$ such that

$$\lim\varrho(\mathfrak{y}_n-\mathfrak{u})=\lim\varrho(f\mathfrak{x}_n-\mathfrak{u})=\lim\varrho(g\mathfrak{x}_{n+1}-\mathfrak{u})=0.$$

In view of the hypothesis $g(X)$ is closed, we obtain that $\mathfrak{u}\in g(X)$. It follows that one can choose $\mathfrak{z}\in X$ such that $\mathfrak{u}=g\mathfrak{z}$, and we write the above equality as

$$\lim\varrho(\mathfrak{y}_n-g\mathfrak{z})=\lim\varrho(f\mathfrak{x}_n-g\mathfrak{z})=\lim\varrho(g\mathfrak{x}_{n+1}-g\mathfrak{z})=0$$

If $f\mathfrak{z} \neq g\mathfrak{z}$, putting $\mathfrak{x} = \mathfrak{x}_{n_k}$ and $\mathfrak{y} = \mathfrak{z}$ into contractive condition (3.17), we have

$$\psi(w_\varrho^2(2)[\varrho(f\mathfrak{x}_{n_k} - f\mathfrak{z})]^2) \leq \psi(\mathcal{N}(\mathfrak{x}_{n_k}, \mathfrak{z})) - \varphi(\mathcal{M}(\mathfrak{x}_{n_k}, \mathfrak{z})); \tag{3.23}$$

where

$$\begin{aligned}\mathcal{N}(\mathfrak{x}_{n_k}, \mathfrak{z}) \;=\; & \max\{\varrho(\mathfrak{y}_{n_k} - f\mathfrak{z})\varrho(\mathfrak{y}_{n_k-1} - g\mathfrak{z}), \varrho(f\mathfrak{z} - g\mathfrak{z})\varrho(\mathfrak{y}_{n_k} - g\mathfrak{z}), [\varrho(\mathfrak{y}_{n_k-1} - g\mathfrak{z})]^2,\\ & \frac{[\varrho(f\mathfrak{z} - g\mathfrak{z})]^2 + [\varrho(\mathfrak{y}_{n_k} - g\mathfrak{z})]^2}{w_\varrho^2(2)}\},\end{aligned}$$

and

$$\begin{aligned}\mathcal{M}(\mathfrak{x}_{n_k}, \mathfrak{z}) \;=\; & \max\{[\varrho(f\mathfrak{z} - g\mathfrak{z})]^2, [\varrho(\mathfrak{y}_{n_k} - g\mathfrak{z})]^2, [\varrho(\mathfrak{y}_{n_k-1} - g\mathfrak{z})]^2,\\ & \frac{[\varrho(\mathfrak{y}_{n_k} - \mathfrak{y}_{n_k-1})]^2[1 + [\varrho(\mathfrak{y}_{n_k-1} - g\mathfrak{z})]^2]}{1 + [\varrho(\mathfrak{y}_{n_k} - g\mathfrak{z})]^2},\\ & \frac{[\varrho(\mathfrak{y}_{n_k-1} - g\mathfrak{z})]^2[1 + [\varrho(\mathfrak{y}_{n_k-1} - g\mathfrak{z})]^2]}{1 + \varrho(\mathfrak{y}_{n_k} - \mathfrak{y}_{n_k-1})}\}.\end{aligned}$$

Consequently, we get

$$\limsup_k \mathcal{N}(\mathfrak{x}_{n_k}, \mathfrak{z}) = \max\{0, 0, 0, \frac{[\varrho(f\mathfrak{z} - g\mathfrak{z})]^2}{w_\varrho^2(2)}\} \leq [\varrho(f\mathfrak{z} - g\mathfrak{z})]^2,$$

and

$$\liminf_k \mathcal{M}(\mathfrak{x}_{n_k}, \mathfrak{z}) = \max\{[\varrho(f\mathfrak{z} - g\mathfrak{z})]^2, 0, 0, 0, 0\} = [\varrho(f\mathfrak{z} - g\mathfrak{z})]^2.$$

Taking the upper limit as $k \to \infty$ in (3.23), by Lemma 3.2, we have

$$\begin{aligned}\psi([\varrho(g\mathfrak{z} - f\mathfrak{z})]^2) \;&=\; \psi(w_\varrho^2(2)\frac{1}{w_\varrho^2(2)}[\varrho(g\mathfrak{z} - f\mathfrak{z})]^2)\\ &\leq\; (w_\varrho^2(2)\limsup_k[\varrho(f\mathfrak{x}_{n_k} - f\mathfrak{z})]^2)\\ &\leq\; \psi(\limsup_k \mathcal{N}(\mathfrak{x}_{n_k}, \mathfrak{z})) - \varphi(\liminf_k \mathcal{M}(\mathfrak{x}_{n_k}, \mathfrak{z}))\\ &\leq\; \psi([\varrho(g\mathfrak{z} - f\mathfrak{z})]^2) - \varphi([\varrho(g\mathfrak{z} - f\mathfrak{z})]^2),\end{aligned}$$

which implies that $\varrho(f\mathfrak{z} - g\mathfrak{z}) = 0$. That is, $\mathfrak{u} = f\mathfrak{z} = g\mathfrak{z}$ is a point of coincidence for f and g. Using the same technique of the proof of Theorem 3.6, it can be proved that $\mathfrak{z}$ is a unique common fixed point. This completes the proof. □

Note. If in Theorem 3.6 and 3.9, we let an additional assumption that f is an orthogonal preserving map, then they could be proved with weaker assumption, SO-complete modular space and SO-closed set $g(X)$.

4. Application

In this section, we will use Theorem 3.6 to show that there is a solution to the following integral equation:

$$x(t) = \int_0^t K(r, x(r))dr. \tag{4.1}$$

Let $X = C[0, T]$ be the set of real continuous functions defined on $[0, T]$ for $T > 0$. We consider the following orthogonal relation in X:

$$x \perp y \iff \forall t, s \in [0, T] : x(t)y(t) \geq 0.$$

Define the modular ϱ by

$$\varrho(x) = \max_{t\in[0,T]} |x(t)|^{\frac{m}{2}},$$

for all $x \in X$, where $m > 2$. It is evident that $(X, \varrho, \perp)$ is SO-complete. Consider the mapping $f : X \to X$ by

$$fx(t) = \int_0^t K(r, x(r))dr. \tag{4.2}$$

Theorem 4.1. *Consider the integral equations (4.1) and suppose that the following conditions hold:*

(i) $K : [0, T] \times \mathbb{R} \to \mathbb{R}^+$ *is continuous;*

(ii) there exists a continuous function $\nu : [0, T] \to [1, \infty)$ *such that we have*

$$|K(r, x(r)) - K(r, y(r))| \leq \nu(r)|x(r) - y(r)|, \quad \forall x \perp y.$$

(iii) there exists a constant $k \in (0, 1)$ *such that for all* $r \in [0, T]$

$$\sup_{t\in[0,T]} \int_0^t 1 + \nu(r)dr \leq k.$$

Then the integral equations 4.1 has a unique solution in X*.*

Proof . Let $x, y \in X$ and $x \perp y$, from conditions (i), (ii) and (iii),

$$\begin{aligned}
|fx(t) - fy(t)|^m &= |\int_0^t K(r, x(r))dr - \int_0^t K(r, y(r))dr|^m \\
&\leq (\int_0^t |K(r, x(r)) - K(r, y(r))|dr)^m \\
&\leq (\int_0^t \nu(r)(|x(r) - y(r)|)dr)^m \\
&\leq (\int_0^t 1 + \nu(r)dr)^m \varrho^2(x(t) - y(t)) \\
&\leq k\varrho^2(x(t) - y(t)).
\end{aligned} \tag{4.3}$$

Thus

$$\varrho(fx(t) - fy(t)) \leq \sqrt{k}\varrho(x(t) - y(t)).$$

By Theorem 3.3, the mapping f defined in (4.2) has a unique fixed point in X, which is the solution of the integral equation (4.2). □

Recently, fixed point theory on orthogonal metric space and orthogonal modoular space is discussed in [7, 11, 13, 14, 15, 16, 19, 12] and some fixed point theorems are generalized on these spaces. In our paper, we proved some new fixed point theorems on orthogonal modular space. In 2020, S. Khalehoghli, H. Rahimi and M. Eshaghi Gordji, introduce the notion of R-metric spaces which is a generalization of orthogonal metric space, and give a real generalization of Banach fixed point theorem and the Brouwer fixed point on R-metric space (see [20, 21]). Now, there are open questions whether:

1) Is it possible to generalize our main results on R-metric space?

2) could be our main results generalized to multivalued mappings f and g?

5. Conclusion

Although fixed point theorems in modular spaces have remarkably applied to a wide variety of mathematical problems, these theorems strongly depend on some assumptions which often do not hold in practice or can lead to their reformulations as particular problems in normed vector spaces. A recent trend of research has been dedicated to studying the fundamentals of fixed point theorems and relaxing their assumptions with the ambition of pushing the boundaries of fixed point theory in modular spaces further. In this paper, we focus on convexity, continuity and Fatou property of the orthogonal modular in common fixed point results taken from the literature for contractive mappings. To relax these three assumptions, we seek to use a new method to prove the convergent of the constructed sequence that its limit is the common fixed point of two self-mappings.

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