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On generalisation of Brown's conjecture

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Abstract

Let P be the complex polynomial of the form $P(z) = z\prod_{j=1}^{n-1}(z - z_j)$, with $|z_j| \geq 1$, $1 \leq j \leq n-1$. Then according to famous Brown's Conjecture $p'(z) \neq 0$, for $|z| < \frac{1}{n}$. This conjecture was proved by Aziz and Zarger [1]. In this paper, we present some interesting generalisations of this conjecture and the results of several authors related to this conjecture.

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1. Introduction

If all the zeros of a polynomial lie in the disk $\mathbb{D} = \{z : |z - c| \leq r\}$, then according to Gauss-Lucas Theorem, every critical point of $p(z)$ lie in $\mathbb{D}$. B. Sendov conjectured that, if all the zeros of $p(z)$ lie in $|z| \leq 1$, then for any zero z_0 of $p(z)$, the disk $|z - z_0| \leq 1$ contains at least one zero of $p'(z)$ [5]. This conjecture has attracted much attention for the last six decades and a broad overview can be found in [7]. In connection with this conjecture, Brown [3] posed the following problem.

Let Q_n denote the set of all complex polynomials of the form $p(z) = z\prod_{j=1}^{n-1}(z - z_j)$ where $|z_j| \geq 1$ for $1 \leq j \leq n-1$. Find the best constant C_n such that $p'(z)$ does not vanish in $|z| < C_n$.

Brown himself conjectured that $C_n = \frac{1}{n}$. However, conjecture was verified by Aziz and Zarger [1] by proving the following more general result.

Theorem 1.1. *If all the zeros of polynomial $Q(z) = \prod_{j=1}^{n-1}(z - z_j)$ lie in $|z| \geq 1$ and $P(z) = z^m Q(z)$, then $P'(z)$ has $(m-1)$ fold zero at origin and remaining $(n-m)$ zeros lie in $|z| \geq \frac{m}{n}$.*

Remark 1.2. *For $m = 1$, it reduces to conjecture of Brown.*

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Zarger and Manzoor [2] have proved following result for s^{th} derivative of $P(z) = z^m \prod_{j=1}^{n-m}(z - z_j)$ with $|z_j| \geq 1$, $1 \leq j \leq n - m$.

Theorem 1.3. *Let $P(z) = z^m \prod_{j=1}^{n-m}(z - z_j)$ be a polynomial of degree n with $|z_j| \geq 1$, $1 \leq j \leq n - m$, then for $1 \leq s \leq m$ the polynomial $P^{(s)}(z)$, the s^{th} derivative of $P(z)$ does not vanish in*

$$0 < |z| < \frac{m(m-1)...(m-s+1)}{n(n-1)...(n-s+1)}$$

N. A Rather and F. Ahmad [6] have proved following results:

Theorem 1.4. *Let $p(z) = (z - a)\prod_{k=1}^{n-1}(z - z_k)$ with $|a| \leq 1$, be a polynomial of degree n with $|a| \leq 1$, and $|z_k| \geq 1$ for $1 \leq k \leq n - 1$, then $p'(z)$ does not vanish in the region*

$$\left| z - \left(\frac{n-1}{n}\right) a \right| < \frac{1}{n}.$$

The result is best possible as is shown by the polynomial

$$p(z) = (z - a)(z - e^{i\alpha})^{n-1}, 0 \leq \alpha < 2\pi.$$

Theorem 1.5. *Let $p(z) = (z - a)^m \prod_{k=1}^{n-m}(z - z_k)$ be a polynomial of degree n with $|a| \leq 1$, and $|z_k| \geq 1$ for $1 \leq k \leq n - m$, then $p'(z)$ has $(m - 1)$ fold zero at $z = a$ and remaining $(n - m)$ zeros of $p'(z)$ lie in the region*

$$\left| z - \left(\frac{n-m}{n}\right) a \right| \geq \frac{m}{n}.$$

The result is best possible as is shown by the polynomial

$$p(z) = (z - a)^m(z - e^{i\alpha})^{n-m}, 0 \leq \alpha < 2\pi.$$

In this paper, we first prove a result which generalises the result of Theorem 1.1 , Theorem 1.5 and includes Brown's Conjecture as a special case.

For the proof of our results, we shall use the following result, which is Walsh's Concidence Theorem [5].

Theorem 1.6. *(Walsh's Concidence Theorem): Let $G(z_1, z_2, ...z_n)$ is a symmetric $n -$ linear form of total degree n in $(z_1, z_2, ..., z_n)$ and let C be a circular region containing the n points $\alpha_1, \alpha_2, ..., \alpha_n$, then there exists at least one point $\alpha \in C$ such that*

$$G(\alpha_1, \alpha_2, ...\alpha_n) = G(\alpha, \alpha, ..., \alpha).$$

2. Main Results

Theorem 2.1. *If all the zeros of a polynomial $P(z)$ of degree n lie in $|z| \geq 1$, then for every real $\lambda > 0$, the polynomial $\lambda P(z) + (z-a)P'(z)$, $|a| \leq 1$ has no zero in*

$$\left| z - \frac{na}{n+\lambda} \right| < \frac{\lambda}{n+\lambda}.$$

Proof . Let $z_1, z_2, ..., z_n$ be the zeros of $P(z)$, so that $|z_j| \geq 1$, $1 \leq j \leq n$. Let w be any zero of $\lambda P(z) + (z-a)P'(z)$, then

$$\lambda P(w) + (z-a)P'(w) = 0. \tag{1}$$

This is an equation which is linear and symmetric in $z_1, z_2, ..., z_n$. Hence by Theorem 1.6, for the circular region $C = \{z : |z| \geq 1\}$, there exists $\alpha \in C$ such that

$$P(z) = (z-\alpha)^n.$$

Thus equation (1) reduces to

$$\lambda(w-\alpha)^n + n(w-a)(w-\alpha)^{n-1} = 0.$$

Equivalently

$$(w-\alpha)^{n-1}\left(\lambda(w-\alpha) + n(w-a)\right) = 0.$$

This gives $w = \alpha$ and $w = \frac{\lambda\alpha}{n+\lambda} + \frac{na}{n+\lambda}$.

Now, If $w = \alpha$, then using the fact that $|a| \leq 1$, we have

$$\begin{aligned} \left| w - \frac{na}{n+\lambda} \right| &= \left| \alpha - \frac{na}{n+\lambda} \right| \\ &\geq |\alpha| - \frac{n}{n+\lambda}|a| \\ &\geq 1 - \frac{n}{n+\lambda} \\ &= \frac{\lambda}{n+\lambda}. \end{aligned}$$

That is

$$\left| w - \frac{na}{n+\lambda} \right| \geq \frac{\lambda}{n+\lambda}.$$

Again, if $w = \frac{\lambda\alpha}{n+\lambda} + \frac{na}{n+\lambda}$, then

$$\left| w - \frac{na}{n+\lambda} \right| = \frac{|\lambda||\alpha|}{|n+\lambda|} \geq \frac{\lambda}{n+\lambda}.$$

This gives

$$\left| w - \frac{na}{n+\lambda} \right| \geq \frac{\lambda}{n+\lambda}.$$

Since w is any zero of $\lambda P(z) + (z-a)P'(z)$, it follows that every zero of $\lambda P(z) + (z-a)P'(z)$ lies in

$$\left| z - \frac{na}{n+\lambda} \right| \geq \frac{\lambda}{n+\lambda}.$$

This completes the proof of the Theorem. □

Remark 2.2. *If all the zeros of polynomial* $Q(z) = \prod_{j=1}^{n-m}(z - z_j)$ *lie in* $|z| \geq 1$ *and* $P(z) = (z-a)^m Q(z)$, $|a| \leq 1$, *then*

$$P'(z) = (z-a)^{m-1}\left(mQ(z) + (z-a)Q'(z)\right).$$

Applying Theorem 2.1 with $\lambda = m$ *to the polynomial* $Q(z)$, *which is of degree* $n-m$, *we immediately obtain Theorem 1.5 .*

Remark 2.3. *For* $a = 0$ *and* $\lambda = m$ *in Remark 2, we obtain Theorem 1.1.*

Remark 2.4. *Theorem 2.1 also includes validity of the Brown's Conjecture as a special case when* $a = 0$, $\lambda = 1$ *and* $P(z)$ *is polynomial of degree* $n-1$.

Next, we shall prove following result which describes the regions which contains the zeros of higher derivatives of the polynomial $P(z) = (z-a)^m \prod_{j=1}^{n-m}(z - z_j)$ with $|a| \leq 1$ and $|z_j| \geq 1$.

Theorem 2.5. *Let* $P(z) = (z-a)^m \prod_{j=1}^{n-m}(z - z_j)$ *be a polynomial of degree* n, *with* $|a| \leq 1$ *and* $|z_j| \geq 1$, $1 \leq j \leq n-m$. *Then for* $1 \leq s \leq m$, *the polynomial* $P^{(s)}(z)$, s^{th} *derivative of* $P(z)$ *has* $(m-s)$ *fold zero at* $z = a$ *and remaining* $(n-m)$ *zeros lie in*

$$\left|z - \left(1 - \frac{m(m-1)...(m-s+1)}{n(n-1)...(n-s+1)}\right)a\right| \geq \frac{m(m-1)...(m-s+1)}{n(n-1)...(n-s+1)}.$$

Proof . Let $P(z) = (z-a)^m Q(z)$, where $Q(z) = \prod_{j=1}^{n-m}(z - z_j)$. Then

$$P'(z) = (z-a)^{m-1}\left(mQ(z) + (z-a)Q'(z)\right).$$

Applying Theorem 2.1 with $\lambda = m$ to the polynomial $Q(z)$ which is of degree $(n-m)$, it follows that $P'(z)$ has $(m-1)$ fold zero at $z = a$ and remaining $(n-m)$ zeros lie in

$$\left|z - \left(\frac{n-m}{n}\right)a\right| \geq \frac{m}{n}.$$

That is $P'(z) = (z-a)^{m-1}R(z)$, where $R(z) = (z-a)Q'(z) + mQ(z)$ has $(m-1)$ fold zero at $z = a$ and remaining $(n-m)$ zeros lie in

$$\left|z - \left(\frac{n-m}{n}\right)a\right| \geq \frac{m}{n}.$$

We can write

$$P''(z) = (z-a)^{m-2}T(z),$$

where $T(z) = (z-a)R'(z) + (m-1)R(z)$.

Consider the polynomial

$$S(z) = p'\left(\frac{m}{n}z + \frac{n-m}{n}a\right) \tag{2}$$

or

$$S(z) = \left(\frac{m}{n}\right)^{m-1}(z-a)^{m-1}R\left(\frac{m}{n}z + \frac{n-m}{n}a\right),$$

then $S(z)$ is a polynomial of degree $n-1$ with $(m-1)$ fold zero at $z=a$ and remaining $(n-m)$ zeros lie in $|z| \geq 1$. Thus,

$$S'(z) = \left(\frac{m}{n}\right)^{m-2} (z-a)^{m-2} \left[(z-a)\frac{m}{n}R'\left(\frac{m}{n}z + \frac{n-m}{n}a\right) + (m-1)R\left(\frac{m}{n}z + \frac{n-m}{n}a\right)\right].$$

Again applying Theorem 2.1 with $\lambda = m-1$ to the polynomial $R\left(\frac{m}{n}z + \frac{n-m}{n}a\right)$ which is of degree $(n-m)$, it follows that $S'(z)$ has $(m-2)$ fold zero at $z=a$ and remaining $(n-m)$ zero lie in

$$\left|z - \left(\frac{n-m}{n-1}\right)a\right| \geq \frac{m-1}{n-1}.$$

Replacing z by $\frac{n}{m}z + \left(\frac{m-n}{m}\right)a$ in (2) we obtain $P''(z)$ has $(m-2)$ fold zero at $z=a$ and remaining $(n-m)$ zeros lie in

$$\left|z - \left(1 - \frac{m(m-1)}{n(n-1)}\right)a\right| \geq \frac{m(m-1)}{n(n-1)}.$$

Proceeding similarly as above and by repeated application of Theorem 2.1, we get for any positive integer s such that $1 \leq s \leq m$, the polynomial $P^{(s)}(z)$, s^{th} derivative has $(m-s)$ zeros at $z=a$ and remaining zeros lie in

$$\left|z - \left(1 - \frac{m(m-1)...(m-s+1)}{n(n-1)...(n-s+1)}\right)a\right| \geq \frac{m(m-1)...(m-s+1)}{n(n-1)...(n-s+1)}.$$

□

Remark 2.6. *For $s=1$, $a=0$ and $m=1$, we get Brown's conjecture.*

Remark 2.7. *For $a=0$, we get Theorem 1.3 which is the result due to Zarger and Manzoor.*

For $a=0$ and $s=m$, following result proved in [8] immediately follows:

Corollary 2.8. *If $p(z) = z^m \prod_{k=1}^{n-m}(z-z_k)$ be a polynomial of degree n with $|z_k| \geq 1$ for $1 \leq k \leq n-m$, then the polynomial $p^{(m)}(z)$, $m \geq 1$ does not vanish in $|z| < \frac{m!}{n(n-1)...(n-m+1)}$.*

For $s=m$, we obtain following result proved in [4].

Corollary 2.9. *If $p(z) = (z-a)^m \prod_{k=1}^{n-m}(z-z_k)$ be a polynomial of degree n with $|a| \leq 1$ and $|z_k| \geq 1$ for $1 \leq k \leq n-m$, then the polynomial $p^{(m)}(z)$, $m \geq 1$ has all its zeros in the region*

$$\left|z - \left(1 - \frac{m!}{n(n-1)...(n-m+1)}\right)a\right| \geq \frac{m!}{n(n-1)...(n-m+1)}.$$

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