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Fixed point theorems of non-commuting mappings in $b-$multiplicative metric spaces

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Abstract

In this paper, we discuss some common fixed point theorems for compatible mappings of type (E) and $R-$weakly commuting mappings of type (P) of a complete $b-$multiplicative metric space along with some examples. As an application, we establish an existence and uniqueness theorem for a solution of a system of multiplicative integral equations. In the last section, we introduce the concept of $R-$multiplicative metric space by giving some examples and at the end of the section, we give an open question.

Keywords: compatible mappings of type (E), $R-$weakly commuting mappings of type (P), multiplicative metric space, $b-$multiplicative metric space, common fixed point, compatible mapping, $R-$metric space, orthogonal set.
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1. Introduction

The concept of multiplicative calculus, in which the role of addition and subtraction are replaced by multiplication and division, was not the interest of researchers for a long time, even though it was defined by Grossman and Katz [8] in the period from 1967 till 1970 (published a book called Non-Newtonian Calculus in 1972), and Stanley [20] published a paper 'A multiplicative calculus' in 1999. But in 2008, Bashirov et al. [3] draw the attention of researchers especially in the field of analysis by highlighting various properties like multiplicative derivatives, multiplicative integrals, etc. They also highlighted its application to various topics like Newtonian calculus, semi-groups of

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linear operators, multiplicative spaces, multiplicative differential equations, multiplicative calculus of variation, etc. For more applications, we refer ([4], [7]) and references therein.

In 2017, Ali et al. [1] introduced the concept of $b-$multiplicative metric space as a generalization of multiplicative metric space for fixed point results on multiplicative contraction mapping with an application in Fredholm multiplicative integral equation.

In this paper, we discuss some common fixed point theorems for compatible mappings of type (E) and $R-$weakly commuting mappings of type (P) of complete $b-$multiplicative metric spaces. Also, we study the existence and uniqueness of solution of a system of multiplicative integral equations. In the last section, we introduce the concept of $R-$multiplicative metric space by giving some examples and at the end of the section, we give an open question.

2. Preliminaries

Before going to our main work, we recall some definitions, properties, and lemmas that will be used in this paper.

Definition 2.1. *[3] Let $U \neq \emptyset$ be a set. A mapping $d : U \times U \to [1, +\infty)$ such that*

(i) $d(u, v) = 1$ *if and only if* $u = v$,
(ii) $d(u, v) = d(v, u),\ \forall u, v \in U$,
(iii) $d(u, v) \leq d(u, w).\ d(w, v),\ \forall u, v, w \in U$,

is called a multiplicative metric and (U, d) is called a multiplicative metric space.

Example 2.2. *Define a mapping $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by*

$$d(u, v) = e^{\max\left\{|u_1 - v_1|, |u_2 - v_2|\right\}},$$

where $u = (u_1, u_2),\ v = (v_1, v_2) \in \mathbb{R}^2$. Then, d is a multiplicative metric on $\mathbb{R}^2$.

Definition 2.3. *[1] Let $U \neq \emptyset$ be a set and $s \geq 1$ be a given real number. A mapping $d_b : U \times U \to [1, +\infty)$ such that*

(i) $d_b(u, v) = 1$ *if and only if* $u = v$,
(ii) $d_b(u, v) = d_b(v, u),\ \forall u, v \in U$,
(iii) $d_b(u, v) \leq d_b(u, w)^s.\ d_b(w, v)^s,\ \forall u, v, w \in U$,

is called a $b-$multiplicative metric and (U, d_b) is called $b-$multiplicative metric space.

Remark 2.4. *Every multiplicative metric space is $b-$multiplicative metric space, but the converse is not true.*

We give the following examples to illustrate the above remark.

Example 2.5. *Define a mapping*

$$d_b : \mathbb{R}^+ \cup \{0\} \times \mathbb{R}^+ \cup \{0\} \to [1, +\infty) \quad by \quad d_b(u, v) = e^{(u-v)^2},$$

for all $u, v \in \mathbb{R}^+ \cup \{0\}$. Then, d_b is a $b-$multiplicative metric on $\mathbb{R}^+ \cup \{0\}$ with $s = 2$. Note that d_b is not a multiplicative metric on $\mathbb{R}^+ \cup \{0\}$.

Example 2.6. Let $U = L_p[0,1]$ be the space of all real functions $u(t),\ t \in [0,1]$ such that $\int_0^1 |u(t)|^p < \infty$ with $0 < p < 1$. Define $d_b : U \times U \to \mathbb{R}^+$ as

$$d_b(u,v) = e^{\left(\int_0^1 |u(t)-v(t)|^p dt\right)^{\frac{1}{p}}}.$$

It is obvious that the conditions (i) and (ii) of Definition 2.3 hold true. For condition (iii), we have

$$\begin{aligned}
\ln d_b(u,v) &= \Big(\int_0^1 |u(t)-v(t)|^p dt\Big)^{\frac{1}{p}} \\
&= \Big(\int_0^1 |\big(u(t)-w(t)\big)+\big(w(t)-v(t)\big)|^p dt\Big)^{\frac{1}{p}} \\
&\le 2\Big(\int_0^1 |u(t)-w(t)|^p dt + \int_0^1 |w(t)-v(t)|^p dt\Big)^{\frac{1}{p}}
\end{aligned}$$

Since $0 < p < 1$, so $1 < \frac{1}{p} < \infty$. By the property of convexity, we obtain

$$\begin{aligned}
\ln d_b(u,v) &\le 2^{\frac{1}{p}}\Big[\big(\int_0^1 |u(t)-w(t)|^p dt\big)^{\frac{1}{p}} + \big(\int_0^1 |w(t)-v(t)|^p dt\big)^{\frac{1}{p}}\Big] \\
d_b(u,v) &\le e^{2^{\frac{1}{p}}\Big[\big(\int_0^1 |u(t)-w(t)|^p dt\big)^{\frac{1}{p}} + \big(\int_0^1 |w(t)-v(t)|^p dt\big)^{\frac{1}{p}}\Big]} \\
&= e^{2^{\frac{1}{p}}\big(\int_0^1 |u(t)-w(t)|^p dt\big)^{\frac{1}{p}}} . e^{2^{\frac{1}{p}}\big(\int_0^1 |w(t)-v(t)|^p dt\big)^{\frac{1}{p}}} \\
&= d_b(u,w)^{2^{\frac{1}{p}}} . d_b(w,v)^{2^{\frac{1}{p}}}
\end{aligned}$$

Thus, $d_b(u,v)$ is a $b-$multiplicative metric space with $s = 2^{\frac{1}{p}}$.

Definition 2.7. *[1] Let (U, d_b) be a $b-$multiplicative metric space and $u \in U$. Then,*

(i) a sequence $\{u_n\}$ is $b-$multiplicative convergent to $u \in U$ if for each $\varepsilon > 1$, there exist some $n_0 \in \mathbb{N}$ such that $d_b(u_n, u) < \varepsilon$ for each $n \ge n_0$.

(ii) a sequence $\{u_n\}$ is $b-$multiplicative Cauchy, if for each $\varepsilon > 1$, there exists $n_0 \in \mathbb{N}$ such that $d_b(u_n, u_m) < \varepsilon$ for each $m, n \ge n_0$.

(iii) a $b-$multiplicative metric space is complete if every $b-$multiplicative Cauchy sequence in it is $b-$multiplicative convergent to some $u \in U$.

Lemma 2.8. *[1] Let (U, d_b) be a $b-$multiplicative metric space. If a sequence $\{u_n\}$ is a $b-$multiplicative convergent, then the limit is unique.*

In 1976, Jungck [11] used the notion of commuting mappings to prove the existence of a common fixed point theorem on a metric space. Sessa [18] introduced the notion of weakly commuting mappings, which is a generalization of commuting mappings. Recently in 2013, Gu et al. [9] introduced the notion of commuting and weakly commuting mappings in a multiplicative metric space and proved some fixed point theorems for these mappings. Now, we use these notions in $b-$multiplicative metric spaces as follows.

Definition 2.9. *Suppose that $\mathcal{A}, \mathcal{B}$ are two self-mappings of a $b-$multiplicative metric space (U, d_b). Then, we say that $\mathcal{A}, \mathcal{B}$ are commuting mappings if $\mathcal{AB}u = \mathcal{BA}u,\ \forall u \in U$.*

Definition 2.10. *Let (U, d_b) be a $b-$multiplicative metric space. Then, we say that two self-mappings $\mathcal{A}$ and $\mathcal{B}$ on U are weakly commuting mappings if $d_b(\mathcal{AB}u, \mathcal{BA}u) \leq d_b(\mathcal{A}u, \mathcal{B}u)$, $\forall u \in U$.*

Remark 2.11. *Commuting mappings are weakly commuting mappings, but the converse is not true.*

Example 2.12. *Let $U = [0, 1]$. Define a mapping $d_b(u, v) : U \times U \to \mathbb{R}$ by $d_b(u, v) = e^{(u-v)^2}$ for all $u, v \in U$. Consider the self-mappings $f(u) = \frac{u}{2-u}$ and $g(u) = \frac{u}{2}$ for all $u \in U$.*
For any $u \in U$,

$$d_b(fgu, gfu) = e^{(\frac{u^2}{(4-u)(4-2u)})^2} \leq e^{(\frac{u^2}{4-2u})^2} = d_b(fu, gu).$$

Then, f and g are weakly commuting but f and g are not commuting since $fgu = \frac{u}{4-u} \neq \frac{u}{4-2u} = gfu$ for any non-zero $u \in U$.

The notion of compatible mapping in a metric space was introduced by Jungck [12] in 1986. In 2015, Kang et al. [13] introduced the concept of compatible mapping in multiplicative metric space.

Definition 2.13. We say that two self-mappings $\mathcal{A}$ and $\mathcal{B}$ of a $b-$multiplicative metric space (U, d_b) are compatible if $\lim_{n\to+\infty} d_b(\mathcal{AB}u_n, \mathcal{BA}u_n) = 1$, whenever $\{u_n\} \subset U$ such that $\lim_{n\to+\infty} \mathcal{A}u_n = \lim_{n\to+\infty} \mathcal{B}u_n$.

Remark 2.14. *Every weakly commuting mappings is compatible but the converse need not be true.*

In 1999, Pant [17] introduced the concept of reciprocally continuous mappings in metric space. In 2017, Jung et al. [10] introduced this concept in multiplicative metric space.

Definition 2.15. *Let (U, d_b) be a $b-$multiplicative metric space. We say that two self-mappings $\mathcal{A}$ and $\mathcal{B}$ on U are reciprocally continuous if $\lim_{n\to+\infty} \mathcal{AB}u_n = \mathcal{A}t$ and $\lim_{n\to+\infty} \mathcal{BA}u_n = \mathcal{B}t$, whenever $\{u_n\} \subset U$ such that $\lim_{n\to+\infty} \mathcal{A}u_n = \lim_{n\to+\infty} \mathcal{B}u_n = t$ for some $t \in U$.*

In 2009, Kumar et al. [15] introduced the concept of $R-$weakly commuting mappings of type (P) in metric spaces. Recently in 2016, Nagpal et al. [16] introduced this notion in multiplicative metric spaces.

Definition 2.16. *Let (U, d_b) be a $b-$multiplicative metric space and $\mathcal{A}, \mathcal{B}$ be two self-mappings on U. Then we say that $\mathcal{A}, \mathcal{B}$ are $R-$weakly commuting mappings of type (P) if there exist some $R > 0$ such that $d_b(\mathcal{AA}u, \mathcal{BB}u) \leq d_b(\mathcal{B}u, \mathcal{A}u)^R$ for every $u \in U$.*

In 2007, Singh and Singh [21] introduced the notion of compatible mappings of type (E) in metric space. Recently in 2020, Sharma et al. [19] used this notion in multiplicative metric space.

Definition 2.17. *Let (U, d_b) be a $b-$multiplicative metric space and $\mathcal{A}$ and $\mathcal{B}$ be two self-mappings on U. We say that $\mathcal{A}$ and $\mathcal{B}$ are compatible mappings of type (E) if $\lim_{n\to+\infty} \mathcal{AA}u_n = \lim_{n\to+\infty} \mathcal{AB}u_n = \mathcal{B}t$ and $\lim_{n\to+\infty} \mathcal{BB}u_n = \lim_{n\to+\infty} \mathcal{BA}u_n = \mathcal{A}t$, whenever $\{u_n\} \subset U$ such that $\lim_{n\to+\infty} \mathcal{A}u_n = \lim_{n\to+\infty} \mathcal{B}u_n = t$ for some $t \in U$.*

Example 2.18. *Let $U = [0, 1]$. Define the mapping $d_b : U \times U \to \mathbb{R}^+$ by $d_b(u, v) = e^{(u-v)^2}$ for all $u, v \in U$. Consider the self-mappings $\mathcal{A}$ and $\mathcal{B}$ defined as*

$$\mathcal{A}(u) = \begin{cases} 0, \text{ if } u \in [0, \frac{1}{2}] \setminus \{\frac{1}{4}\} \\ 1, \text{ if } u = \frac{1}{4}, \\ \frac{1-u}{2}, \text{ if } u \in (\frac{1}{2}, 1] \end{cases} \quad \text{and} \quad \mathcal{B}(u) = \begin{cases} 1, \text{ if } u \in [0, \frac{1}{2}] \setminus \{\frac{1}{4}\} \\ 0, \text{ if } u = \frac{1}{4} \\ \frac{u}{2}, \text{ if } u \in (\frac{1}{2}, 1] \end{cases}$$

Then, $\mathcal{A}$ and $\mathcal{B}$ are compatible mappings of type (E).

3. Main results

Here are our main results.

Theorem 3.1. *Let $\mathcal{P}, \mathcal{Q}, \mathcal{V}$ and $\mathcal{W}$ be self-mappings of a complete $b-$multiplicative metric space (U, d_b) satisfying the following conditions :*

(i) $\mathcal{P}(U) \subset \mathcal{W}(U)$ *and* $\mathcal{Q}(U) \subset \mathcal{V}(U)$,

(ii) $d_b^q(\mathcal{P}\rho, \mathcal{Q}\theta) \leq \Delta(\rho, \theta), \quad \forall \rho, \theta \in U$,
where

$$\begin{aligned} \Delta(\rho,\theta) &= \max\Big\{d_b(\mathcal{V}\rho, \mathcal{W}\theta), d_b(\mathcal{P}\rho, \mathcal{V}\rho), d_b(\mathcal{Q}\theta, \mathcal{W}\theta), d_b(\mathcal{P}\rho, \mathcal{W}\theta)^{\frac{1}{2s}}, d_b(\mathcal{Q}\theta, \mathcal{V}\rho)^{\frac{1}{2s}}\Big\}, \\ & \quad 1 \leq s < q < \infty; \end{aligned}$$

(iii) $\{\mathcal{P}, \mathcal{V}\}$ *and* $\{\mathcal{Q}, \mathcal{W}\}$ *are weakly commuting mappings. Furthermore, either* $\{\mathcal{P}, \mathcal{V}\}$ *or* $\{\mathcal{Q}, \mathcal{W}\}$ *is a compatible mapping of type* (E).

Then, $\mathcal{P}, \mathcal{Q}, \mathcal{V}$ and $\mathcal{W}$ have a unique common fixed point in U.

Proof .
Since $\mathcal{P}(U) \subset \mathcal{W}(U)$ and $\mathcal{Q}(U) \subset \mathcal{V}(U)$, we can choose ρ_1, ρ_2 in U, starting with $\rho_0 \in U$ such that $\theta_0 = \mathcal{W}\rho_1 = \mathcal{P}\rho_0$ and $\theta_1 = \mathcal{V}\rho_2 = \mathcal{Q}\rho_1$. Continuing in this fashion, we construct sequences $\{\rho_n\}$ and $\{\theta_n\}$ in U such that

$$\begin{aligned} \theta_{2n} &= \mathcal{W}\rho_{2n+1} = \mathcal{P}\rho_{2n}, \\ \text{and } \theta_{2n+1} &= \mathcal{V}\rho_{2n+2} = \mathcal{Q}\rho_{2n+1}, \text{for each } n \in \mathbb{N} \cup \{0\}. \end{aligned}$$

From (ii), we obtain

$$\begin{aligned} d_b^q(\theta_{2n}, \theta_{2n+1}) &= d_b^q(\mathcal{P}\rho_{2n}, \mathcal{Q}\rho_{2n+1}) \leq \Delta(\rho_{2n}, \rho_{2n+1}) \\ &= \max\Big\{d_b(\mathcal{V}\rho_{2n}, \mathcal{W}\rho_{2n+1}), d_b(\mathcal{P}\rho_{2n}, \mathcal{V}\rho_{2n}), d_b(\mathcal{Q}\rho_{2n+1}, \mathcal{W}\rho_{2n+1}), \\ & \quad d_b(\mathcal{P}\rho_{2n}, \mathcal{W}\rho_{2n+1})^{\frac{1}{2s}}, d_b(\mathcal{Q}\rho_{2n+1}, \mathcal{V}\rho_{2n})^{\frac{1}{2s}}\Big\} \\ &= \max\Big\{d_b(\theta_{2n-1}, \theta_{2n}), d_b(\theta_{2n}, \theta_{2n-1}), d_b(\theta_{2n+1}, \theta_{2n}), \\ & \quad d_b(\theta_{2n}, \theta_{2n})^{\frac{1}{2s}}, d_b(\theta_{2n+1}, \theta_{2n-1})^{\frac{1}{2s}}\Big\} \\ &= \max\Big\{d_b(\theta_{2n-1}, \theta_{2n}), d_b(\theta_{2n+1}, \theta_{2n}), d_b(\theta_{2n+1}, \theta_{2n-1})^{\frac{1}{2s}}\Big\} \\ &\leq \max\Big\{d_b(\theta_{2n-1}, \theta_{2n}), d_b(\theta_{2n+1}, \theta_{2n}), \\ & \quad [d_b(\theta_{2n+1}, \theta_{2n})^s . d_b(\theta_{2n}, \theta_{2n-1})^s]^{\frac{1}{2s}}\Big\}. \end{aligned} \tag{3.1}$$

Suppose that $d_b(\theta_{2n-1}, \theta_{2n}) < d_b(\theta_{2n}, \theta_{2n+1})$, then from (3.1), we obtain

$$d_b^q(\theta_{2n}, \theta_{2n+1}) < d_b(\theta_{2n+1}, \theta_{2n})$$

a contradiction and hence $d_b(\theta_{2n}, \theta_{2n+1}) \leq d_b(\theta_{2n-1}, \theta_{2n})$, for all $n \geq 1$. Therefore from (3.1), we obtain

$$d_b(\theta_{2n}, \theta_{2n+1}) \leq d_b(\theta_{2n-1}, \theta_{2n})^{\frac{1}{q}}, \text{for all } n \geq 1. \tag{3.2}$$

Also from (ii), we obtain

$$\begin{aligned} d_b^q(\theta_{2n+2},\theta_{2n+1}) &= d_b^q(\mathcal{P}\rho_{2n+2},\mathcal{Q}\rho_{2n+1}) \le \Delta(\rho_{2n+2},\rho_{2n+1}) \\ &= \max\Big\{d_b(\mathcal{V}\rho_{2n+2},\mathcal{W}\rho_{2n+1}), d_b(\mathcal{P}\rho_{2n+2},\mathcal{V}\rho_{2n+2}), \\ &\quad d_b(\mathcal{Q}\rho_{2n+1},\mathcal{W}\rho_{2n+1}), d_b(\mathcal{P}\rho_{2n+2},\mathcal{W}\rho_{2n+1})^{\frac{1}{2s}}, \\ &\quad d_b(\mathcal{Q}\rho_{2n+1},\mathcal{V}\rho_{2n+2})^{\frac{1}{2s}}\Big\} \\ &= \max\Big\{d_b(\theta_{2n+1},\theta_{2n}), d_b(\theta_{2n+2},\theta_{2n+1}), d_b(\theta_{2n+1},\theta_{2n}), \\ &\quad d_b(\theta_{2n+2},\theta_{2n})^{\frac{1}{2s}}, d_b(\theta_{2n+1},\theta_{2n+1})^{\frac{1}{2s}}\Big\} \\ &\le \max\Big\{d_b(\theta_{2n+1},\theta_{2n}), d_b(\theta_{2n+2},\theta_{2n+1}), d_b(\theta_{2n+1},\theta_{2n}), \\ &\quad [d_b(\theta_{2n+2},\theta_{2n+1})^s.\ d_b(\theta_{2n+1},\theta_{2n})^s]^{\frac{1}{2s}}\Big\} \\ &\le d_b(\theta_{2n+1},\theta_{2n}) \end{aligned}$$

otherwise it leads to a contradiction that $d_b(\theta_{2n+2},\theta_{2n+1}) < (d_b(\theta_{2n+2},\theta_{2n+1}))^{\frac{1}{q}}$. Thus, we obtain

$$d_b(\theta_{2n+2},\theta_{2n+1}) \le (d_b(\theta_{2n+1},\theta_{2n}))^{\frac{1}{q}}, \text{ for all } n \in \mathbb{N}. \tag{3.3}$$

From (3.2) and (3.3), we write

$$\begin{aligned} d_b(\theta_n,\theta_{n+1}) &\le (d_b(\theta_{n-1},\theta_n))^{\frac{1}{q}} \\ &\le (d_b(\theta_{n-2},\theta_{n-1}))^{\frac{1}{q^2}} \\ &\le \dots \\ &\le (d_b(\theta_0,\theta_1))^{\frac{1}{q^n}}, \text{ for all } n \in \mathbb{N}. \end{aligned}$$

Let $n \in \mathbb{N}$ and $p \ge 1$, then by $b-$multiplicative triangular inequality, we have

$$\begin{aligned} d_b(\theta_n,\theta_{n+p}) &\le d_b(\theta_n,\theta_{n+1})^s.d_b(\theta_{n+1},\theta_{n+2})^{s^2} \dots d_b(\theta_{n+p-1},\theta_{n+p})^{s^{p-1}} \\ &\le d_b(\theta_0,\theta_1)^{\frac{s}{q^n}}.d_b(\theta_0,\theta_1)^{\frac{s^2}{q^{n+1}}} \dots d_b(\theta_0,\theta_1)^{\frac{s^{p-1}}{q^{n+p-1}}} \\ &= d_b(\theta_0,\theta_1)^{\frac{s}{q^n}(1+(\frac{s}{q})+\dots+(\frac{s}{q})^{p-2})} \\ &\le d_b(\theta_0,\theta_1)^{\frac{s}{q^n}(1+r+\dots+r^{p-2}+r^{p-1}+\dots)} \\ &= d_b(\theta_0,\theta_1)^{\frac{s}{q^n}(\frac{1}{1-r})}, \end{aligned}$$

where $1 \le s < q < \infty$ setting with $r = \frac{s}{q} < 1$. Letting $n \to +\infty$ in the above inequality, we get $d_b(\theta_n,\theta_{n+p}) \to 1$. Hence the sequence $\{\theta_n\}$ is a $b-$multiplicative Cauchy sequence. By the completeness of U, there exists $x^* \in U$ such that $\theta_n \to x^*$ as $n \to +\infty$. Consequently, the subsequences $\{\mathcal{P}\rho_{2n}\}$, $\{\mathcal{Q}\rho_{2n+1}\}$, $\{\mathcal{V}\rho_{2n+2}\}$, $\{\mathcal{W}\rho_{2n+1}\}$ converge to x^* as $n \to +\infty$.

Since $\mathcal{P}$ and $\mathcal{V}$ are compatible mappings of type (E), we have, $\lim_{n\to+\infty} \mathcal{V}\mathcal{V}\rho_{2n} = \lim_{n\to+\infty} \mathcal{V}\mathcal{P}\rho_{2n} = \mathcal{P}x^*$ and $\lim_{n\to+\infty} \mathcal{P}\mathcal{P}\rho_{2n} = \lim_{n\to+\infty} \mathcal{P}\mathcal{V}\rho_{2n} = \mathcal{V}x^*$
Also, $\mathcal{P}$ and $\mathcal{V}$ are weakly commuting mappings. By Remark (2.14), we have,

$$\lim_{n\to+\infty} d_b(\mathcal{P}\mathcal{V}\rho_{2n},\mathcal{V}\mathcal{P}\rho_{2n}) = 1$$

that is, $\mathcal{P}x^* = \mathcal{V}x^*$.
Since $\mathcal{P}(U) \subset \mathcal{W}(U)$, there exists a point y^* in U such that $\mathcal{P}x^* = \mathcal{W}y^*$. Using (ii), we have,

$$\begin{aligned}
d_b^q(\mathcal{P}x^*, \mathcal{Q}y^*) &\leq \Delta(x^*, y^*)\\
&= \max\Big\{d_b(\mathcal{V}x^*, \mathcal{W}y^*), d_b(\mathcal{P}x^*, \mathcal{V}x^*), d_b(\mathcal{Q}y^*, \mathcal{W}y^*),\\
&\quad d_b(\mathcal{P}x^*, \mathcal{W}y^*)^{\frac{1}{2s}}, d_b(\mathcal{Q}y^*, \mathcal{V}x^*)^{\frac{1}{2s}}\Big\}\\
&= \max\Big\{d_b(\mathcal{P}x^*, \mathcal{P}x^*), d_b(\mathcal{P}x^*, \mathcal{P}x^*), d_b(\mathcal{Q}y^*, \mathcal{P}x^*),\\
&\quad d_b(\mathcal{P}x^*, \mathcal{P}x^*)^{\frac{1}{2s}}, d_b(\mathcal{Q}y^*, \mathcal{P}x^*)^{\frac{1}{2s}}\Big\}\\
&= \max\Big\{1, d_b(\mathcal{Q}y^*, \mathcal{P}x^*)\Big\}\\
&= d_b(\mathcal{P}x^*, \mathcal{Q}y^*)\\
\Longrightarrow d_b(\mathcal{P}x^*, \mathcal{Q}y^*) &= 1.
\end{aligned}$$

Therefore, $\mathcal{P}x^* = \mathcal{Q}y^*$. Thus, $\mathcal{P}x^* = \mathcal{V}x^* = \mathcal{Q}y^* = \mathcal{W}y^*$. Since weakly commutativity of $\mathcal{P}$ and $\mathcal{V}$ implies that $d_b(\mathcal{P}\mathcal{V}x^*, \mathcal{V}\mathcal{P}x^*) \leq d_b(\mathcal{P}x^*, \mathcal{V}x^*)$ implies $\mathcal{P}\mathcal{V}x^* = \mathcal{V}\mathcal{P}x^*$ and $\mathcal{P}\mathcal{P}x^* = \mathcal{P}\mathcal{V}x^* = \mathcal{V}\mathcal{P}x^* = \mathcal{V}\mathcal{V}x^*$.
Similarly, $\mathcal{Q}\mathcal{Q}y^* = \mathcal{W}\mathcal{W}y^*$.

Again, from (ii), we have

$$\begin{aligned}
d_b^q(\mathcal{P}x^*, \mathcal{P}\mathcal{P}x^*) &= d_b^q(\mathcal{Q}y^*, \mathcal{P}\mathcal{P}x^*)\\
&= d_b^q(\mathcal{P}\mathcal{P}x^*, \mathcal{Q}y^*)\\
&\leq \Delta(\mathcal{P}x^*, y^*)\\
&= \max\Big\{d_b(\mathcal{V}\mathcal{P}x^*, \mathcal{W}y^*), d_b(\mathcal{P}\mathcal{P}x^*, \mathcal{V}\mathcal{P}x^*),\\
&\quad d_b(\mathcal{Q}y^*, \mathcal{W}y^*), d_b(\mathcal{P}\mathcal{P}x^*, \mathcal{W}y^*)^{\frac{1}{2s}}, d_b(\mathcal{Q}y^*, \mathcal{V}\mathcal{P}x^*)^{\frac{1}{2s}}\Big\}\\
&= \max\Big\{d_b(\mathcal{P}\mathcal{P}x^*, \mathcal{P}x^*), d_b(\mathcal{P}\mathcal{P}x^*, \mathcal{P}\mathcal{P}x^*), d_b(\mathcal{Q}y^*, \mathcal{Q}y^*),\\
&\quad d_b(\mathcal{P}\mathcal{P}x^*, \mathcal{P}x^*)^{\frac{1}{2s}}, d_b(\mathcal{P}x^*, \mathcal{P}\mathcal{P}x^*)^{\frac{1}{2s}}\Big\}\\
&= \max\Big\{1, d_b(\mathcal{P}\mathcal{P}x^*, \mathcal{P}x^*)\Big\}\\
&= d_b(\mathcal{P}x^*, \mathcal{P}\mathcal{P}x^*).
\end{aligned}$$

Therefore, $\mathcal{P}\mathcal{P}x^* = \mathcal{P}x^*$. Thus, $\mathcal{P}x^* = \mathcal{P}\mathcal{P}x^* = \mathcal{V}\mathcal{P}x^*$.

Thus, $\mathcal{P}x^*$ is a common fixed point of $\mathcal{P}$ and $\mathcal{V}$. Again, from (ii), we have

$$\begin{aligned} d_b^q(\mathcal{Q}y^*, \mathcal{Q}\mathcal{Q}y^*) &= d_b^q(\mathcal{P}x^*, \mathcal{Q}\mathcal{Q}y^*) \\ &\leq \Delta(x^*, \mathcal{Q}y^*) \\ &= \max\Big\{d_b(\mathcal{V}x^*, \mathcal{W}\mathcal{Q}y^*), d_b(\mathcal{P}x^*, \mathcal{V}x^*), d_b(\mathcal{Q}\mathcal{Q}y^*, \mathcal{W}\mathcal{Q}y^*), \\ &\quad d_b(\mathcal{P}x^*, \mathcal{W}\mathcal{Q}y^*)^{\frac{1}{2s}}, d_b(\mathcal{Q}\mathcal{Q}y^*, \mathcal{V}x^*)^{\frac{1}{2s}}\Big\} \\ &= \max\Big\{d_b(\mathcal{Q}\mathcal{Q}y^*, \mathcal{Q}y^*), d_b(\mathcal{Q}y^*, \mathcal{Q}y^*), d_b(\mathcal{Q}\mathcal{Q}y^*, \mathcal{Q}\mathcal{Q}y^*), \\ &\quad d_b(\mathcal{Q}y^*, \mathcal{Q}\mathcal{Q}y^*)^{\frac{1}{2s}}, d_b(\mathcal{Q}\mathcal{Q}y^*, \mathcal{Q}y^*)^{\frac{1}{2s}}\Big\} \\ &= \max\Big\{1, d_b(\mathcal{Q}\mathcal{Q}y^*, \mathcal{Q}y^*)\Big\} \\ &= d_b(QQx^*, Qy^*). \end{aligned}$$

Therefore, $\mathcal{Q}\mathcal{Q}y^* = \mathcal{Q}y^*$. Thus, $\mathcal{Q}y^* = \mathcal{Q}\mathcal{Q}y^* = \mathcal{W}\mathcal{Q}y^*$. Therefore, $\mathcal{Q}y^*$ is common fixed point of $\mathcal{Q}$ and $\mathcal{W}$. If $\mathcal{Q}y^* = \mathcal{P}x^* = z$, then $\mathcal{P}z = \mathcal{V}z = \mathcal{Q}z = \mathcal{W}z = z$. Thus, the common fixed point of $\mathcal{P}, \mathcal{Q}, \mathcal{V}$ and $\mathcal{W}$ is z. Let, if possible, w be another common fixed point of $\mathcal{P}, \mathcal{Q}, \mathcal{V}$ and $\mathcal{W}$. Then, from (ii), we have

$$\begin{aligned} d_b^q(z, w) &= d_b^q(\mathcal{P}z, \mathcal{Q}w) \leq \Delta(z, w) \\ &= \max\Big\{d_b(\mathcal{V}z, \mathcal{W}w), d_b(\mathcal{P}z, \mathcal{V}z), d_b(\mathcal{Q}w, \mathcal{W}w), d_b(\mathcal{P}z, \mathcal{W}z)^{\frac{1}{2s}}, d_b(\mathcal{Q}w, \mathcal{V}z)^{\frac{1}{2s}}\Big\} \\ &= \max\Big\{d_b(z, w), d_b(z, z), d_b(w, w), d_b(z, z)^{\frac{1}{2s}}, d_b(w, z)^{\frac{1}{2s}}\Big\} \\ &= \max\{1, d_b(z, w)\} \\ &= d_b(z, w). \end{aligned}$$

Therefore, $d_b(z, w) = 1$. Thus, $z = w$, which is a contradiction. Thus, $\mathcal{P}, \mathcal{Q}, \mathcal{V}$ and $\mathcal{W}$ have a unique common fixed point in U. □

Corollary 3.2. *Let $\mathcal{P}, \mathcal{Q}$ be self-mappings of a complete $b-$multiplicative metric space (U, d_b) satisfying*

$$d_b^q(\mathcal{P}\rho, \mathcal{Q}\theta) \leq \max\Big\{d_b(\rho, \theta), d_b(\mathcal{P}\rho, \rho), d_b(\mathcal{Q}\theta, \theta), d_b(\mathcal{P}\rho, \theta)^{\frac{1}{2s}}, d_b(\mathcal{Q}\theta, \rho)^{\frac{1}{2s}}\Big\},\ 1 \leq s < q < \infty.$$

Then, $\mathcal{P}$ and $\mathcal{Q}$ have a unique common fixed point in U.

Proof . By considering $\mathcal{V} = \mathcal{W} = \mathcal{I}_U$ (identity mapping on U), then by Theorem 3.1 gives that $\mathcal{P}$ and $\mathcal{Q}$ have a unique common fixed point. □

Corollary 3.3. *Theorem 3.1 remains true if the condition, that is, $\{\mathcal{P}, \mathcal{V}\}$ is a compatible mapping of type (E) is replaced by the condition that $\{\mathcal{P}, \mathcal{V}\}$ is a compatible pair of reciprocally continuous mappings.*

Corollary 3.4. *Let $\mathcal{P}$ be a self-mapping of a complete $b-$multiplicative metric space (U, d_b) satisfying*

$$d_b^q(\mathcal{P}\rho, \mathcal{P}\theta) \leq \max\Big\{d_b(\rho, \theta), d_b(\mathcal{P}\rho, \rho), d_b(\mathcal{P}\theta, \theta), d_b(\mathcal{P}\rho, \theta)^{\frac{1}{2s}}, d_b(\mathcal{P}\theta, \rho)^{\frac{1}{2s}}\Big\},\ 1 \leq s < q < \infty.$$

Then, $\mathcal{P}$ has a unique fixed point in U.

Proof . Take $\mathcal{V}$ and $\mathcal{W}$ as identity mappings on U and $\mathcal{P} = \mathcal{Q}$ and then apply Theorem 3.1. □

Theorem 3.5. *Let $\mathcal{P}, \mathcal{Q}, \mathcal{V}$ and $\mathcal{W}$ be self-mappings of a complete $b-$multiplicative metric space (U, d_b) satisfying the following conditions :*

(i) $\mathcal{P}(U) \subset \mathcal{W}(U)$ *and* $\mathcal{Q}(U) \subset \mathcal{V}(U)$,

(ii) $d_b^q(\mathcal{P}\rho, \mathcal{Q}\theta) \leq \Delta(\rho, \theta), \quad \forall \rho, \theta \in U,$

where

$$\begin{aligned}\Delta(\rho,\theta) &= \max\left\{d_b(\mathcal{V}\rho,\mathcal{W}\theta), d_b(\mathcal{P}\rho,\mathcal{V}\rho), d_b(\mathcal{Q}\theta,\mathcal{W}\theta), d_b(\mathcal{P}\rho,\mathcal{W}\theta)^{\frac{1}{2s}}, d_b(\mathcal{Q}\theta,\mathcal{V}\rho)^{\frac{1}{2s}}\right\},\\ &\quad 1 \leq s < q < \infty;\end{aligned}$$

(iii) $\{\mathcal{P}, \mathcal{V}\}$ *and* $\{\mathcal{Q}, \mathcal{W}\}$ *are pairs of $R-$weakly commuting mappings of type* (P);

(iv) $\{\mathcal{P}, \mathcal{V}\}$ *is a compatible mapping of type* (E).

Then, $\mathcal{P}, \mathcal{Q}, \mathcal{V}$ and $\mathcal{W}$ have a unique common fixed point in U.

Proof . Following the proof of Theorem 3.1, we know that the sequence $\{\theta_n\}$ in U defined by

$$\begin{aligned}\theta_{2n+1} &= \mathcal{W}\rho_{2n+1} = \mathcal{P}\rho_{2n}\\ \text{and } \theta_{2n+2} &= \mathcal{V}\rho_{2n+2} = \mathcal{Q}\rho_{2n+1} \text{ for } n = 0, 1, 2, \ldots\end{aligned}$$

is a $b-$multiplicative Cauchy sequence. From the completeness of U, there exists $x^* \in$ U such that $\theta_n \to x^*$ as $n \to +\infty$. As a result, $\{\mathcal{P}\rho_{2n}\}$, $\{\mathcal{Q}\rho_{2n+1}\}$, $\{\mathcal{V}\rho_{2n+2}\}$, $\{\mathcal{W}\rho_{2n+1}\}$ converge to x^* as $n \to +\infty$.

Since $\{\mathcal{P}, \mathcal{V}\}$ is a compatible mapping of type (E),

$$\lim_{n\to+\infty} \mathcal{V}\mathcal{V}\rho_{2n} = \lim_{n\to+\infty} \mathcal{V}\mathcal{P}\rho_{2n} = \mathcal{P}x^*.$$

$$\lim_{n\to+\infty} \mathcal{P}\mathcal{P}\rho_{2n} = \lim_{n\to+\infty} \mathcal{P}\mathcal{V}\rho_{2n} = \mathcal{V}x^*.$$

Since the pair $\{\mathcal{P}, \mathcal{V}\}$ is R-weakly commuting mapping of type (P), we have

$$d_b(\mathcal{V}\mathcal{V}\rho_{2n}, \mathcal{P}\mathcal{P}\rho_{2n}) \leq d_b(\mathcal{P}\rho_{2n}, \mathcal{V}\rho_{2n})^R.$$

Taking $n \to +\infty$, we have

$$\begin{aligned}\lim_{n\to+\infty} d_b(\mathcal{V}\mathcal{V}\rho_{2n}, \mathcal{P}\mathcal{P}\rho_{2n}) &\leq \lim_{n\to+\infty} d_b(\mathcal{P}\rho_{2n}, \mathcal{V}\rho_{2n})^R\\ \Longrightarrow \lim_{n\to+\infty} d_b(\mathcal{P}\mathcal{P}\rho_{2n}, \mathcal{P}x^*) &\leq 1\end{aligned}$$

Therefore, $\lim_{n\to+\infty} \mathcal{P}\mathcal{P}\rho_{2n} = \mathcal{P}x^*$. Using (ii), we have

$$\begin{aligned}d_b^q(\mathcal{P}\mathcal{P}\rho_{2n}, \mathcal{Q}\rho_{2n+1}) &\leq \Delta(\mathcal{P}\rho_{2n}, \rho_{2n+1})\\ &= \max\Big\{d_b(\mathcal{V}\mathcal{P}\rho_{2n}, \mathcal{W}\rho_{2n+1}), d_b(\mathcal{P}\mathcal{P}\rho_{2n}, \mathcal{V}\mathcal{P}\rho_{2n}),\\ &\qquad d_b(\mathcal{Q}\rho_{2n+1}, \mathcal{W}\rho_{2n+1}), d_b(\mathcal{P}\mathcal{P}\rho_{2n}, \mathcal{W}\rho_{2n+1})^{\frac{1}{2s}}, d_b(\mathcal{Q}\rho_{2n+1}, \mathcal{V}\mathcal{P}\rho_{2n})^{\frac{1}{2s}}\Big\}\end{aligned}$$

Taking $n \to +\infty$ in the above inequality, we obtain

$$\begin{aligned}d_b^q(\mathcal{P}x^*, x^*) &\leq \max\left\{d_b(\mathcal{P}x^*, x^*), d_b(\mathcal{P}x^*, \mathcal{P}x^*), d_b(x^*, x^*), d_b(\mathcal{P}x^*, x^*)^{\frac{1}{2s}}, d_b(x^*, \mathcal{P}x^*)^{\frac{1}{2s}}\right\}\\ &= \max\{d_b(\mathcal{P}x^*, x^*), 1\}\\ &= d_b(\mathcal{P}x^*, x^*)\end{aligned}$$

which implies that $d_b(\mathcal{P}x^*, x^*) = 1$, i.e., $\mathcal{P}x^* = x^*$. Thus, $\mathcal{P}x^* = \mathcal{V}x^* = x^*$. Since $x^* = \mathcal{P}x^* \in \mathcal{P}(U) \subset \mathcal{W}(U)$, so there exists $y^* \in U$ such that $x^* = \mathcal{W}y^*$. Then,

$$\begin{aligned} d_b^q(x^*, \mathcal{Q}y^*) &= d_b^q(\mathcal{P}x^*, \mathcal{Q}y^*) \leq \Delta(x^*, y^*) \\ &= \max\left\{d_b(\mathcal{V}x^*, \mathcal{W}y^*), d_b(\mathcal{P}x^*, \mathcal{V}x^*), d_b(\mathcal{Q}y^*, \mathcal{W}y^*), d_b(\mathcal{P}x^*, \mathcal{W}y^*)^{\frac{1}{2s}}, d_b(\mathcal{Q}y^*, \mathcal{V}x^*)^{\frac{1}{2s}}\right\} \\ &= \max\left\{d_b(x^*, x^*), d_b(x^*, x^*), d_b(\mathcal{Q}y^*, x^*), d_b(x^*, x^*)^{\frac{1}{2s}}, d_b(\mathcal{Q}y^*, x^*)^{\frac{1}{2s}}\right\} \\ &= \max\{1, d_b(\mathcal{Q}y^*, x^*)\} \\ &= d_b(x^*, \mathcal{Q}y^*) \end{aligned}$$

which implies that $d_b(x^*, \mathcal{Q}y^*) = 1$, that is, $\mathcal{Q}y^* = x^*$. Since $\mathcal{Q}$ and $\mathcal{W}$ are $R-$weakly commuting mappings of type (P), we have

$$d_b(\mathcal{Q}x^*, \mathcal{W}x^*) = d_b(\mathcal{Q}\mathcal{Q}y^*, \mathcal{W}\mathcal{W}y^*) \leq d_b(\mathcal{W}y^*, \mathcal{Q}y^*)^R = d_b(x^*, x^*)^R = 1.$$

Therefore, $d_b(\mathcal{Q}x^*, \mathcal{W}x^*) = 1$, so $\mathcal{Q}x^* = \mathcal{W}x^*$. Lastly, we have

$$\begin{aligned} d_b^q(x^*, \mathcal{Q}x^*) &= d_b^q(\mathcal{P}x^*, \mathcal{Q}x^*) \leq \Delta(x^*, x^*) \\ &= \max\left\{d_b(\mathcal{V}x^*, \mathcal{W}x^*), d_b(\mathcal{P}x^*, \mathcal{V}x^*), d_b(\mathcal{Q}x^*, \mathcal{W}x^*), d_b(\mathcal{P}x^*, \mathcal{W}x^*)^{\frac{1}{2s}}, d_b(\mathcal{Q}x^*, \mathcal{V}x^*)^{\frac{1}{2s}}\right\} \\ &= \max\left\{d_b(x^*, \mathcal{Q}x^*), d_b(x^*, x^*), d_b(\mathcal{Q}x^*, \mathcal{Q}x^*), d_b(\mathcal{Q}x^*, x^*)^{\frac{1}{2s}}, d_b(\mathcal{Q}x^*, x^*)^{\frac{1}{2s}}\right\} \\ &= d_b(x^*, \mathcal{Q}x^*) \end{aligned}$$

which implies that $d_b(\mathcal{Q}x^*, x^*) = 1$, that is, $\mathcal{Q}x^* = x^* = \mathcal{W}x^*$. In addition, we prove that $\mathcal{P}, \mathcal{Q}, \mathcal{V}$ and $\mathcal{W}$ have a unique common fixed point. Suppose that $z \in U$ is another common fixed point of $\mathcal{P}, \mathcal{Q}, \mathcal{V}$ and $\mathcal{W}$. Then,

$$\begin{aligned} d_b^q(x^*, z) &= d_b^q(\mathcal{P}x^*, \mathcal{Q}z) \leq \Delta(x^*, z) \\ &= \max\left\{d_b(\mathcal{V}x^*, \mathcal{W}z), d_b(\mathcal{P}x^*, \mathcal{V}x^*), d_b(\mathcal{Q}z, \mathcal{W}z), d_b(\mathcal{P}x^*, \mathcal{W}z)^{\frac{1}{2s}}, d_b(\mathcal{Q}z, \mathcal{V}x^*)^{\frac{1}{2s}}\right\} \\ &= \max\{d_b(x^*, z), d_b(x^*, x^*), d_b(z, z), d_b(x^*, z)^{\frac{1}{2s}}, d_b(z, x^*)^{\frac{1}{2s}}\} \\ &= d_b(x^*, z) \end{aligned}$$

which implies that $d_b(x^*, z) = 1$, that is, $x^* = z$, which is a contradiction. Hence, $\mathcal{P}, \mathcal{Q}, \mathcal{V}$ and $\mathcal{W}$ have a unique common fixed point. □

Theorem 3.6. Let $\mathcal{P}, \mathcal{Q}, \mathcal{V}$ and $\mathcal{W}$ be self mappings of a complete $b-$multiplicative metric space (U, d_b) such that $\mathcal{P}(U) \subseteq \mathcal{W}(U)$, $\mathcal{Q}(U) \subseteq \mathcal{V}(U)$ and satisfying

$$d_b(\mathcal{P}\rho, \mathcal{Q}\theta) \leq \Delta_1(\rho, \theta), \ \forall \rho, \theta \in U, \tag{3.4}$$

where

$$\Delta_1(\rho, \theta) = \max\left\{d_b(\mathcal{V}\rho, \mathcal{W}\theta), d_b(\mathcal{P}\rho, \mathcal{V}\rho), d_b(\mathcal{Q}\theta, \mathcal{W}\theta), \frac{1}{2}\big(d_b(\mathcal{V}\rho, \mathcal{Q}\theta) + d_b(\mathcal{P}\rho, \mathcal{W}\theta)\big)\right\}^{\lambda}$$

where $\lambda \in (0, 1)$, then $\mathcal{P}, \mathcal{Q}, \mathcal{V}$ and $\mathcal{W}$ have a unique common fixed point in U provided that $\mathcal{V}$ and $\mathcal{W}$ are continuous and pairs $\{\mathcal{P}, \mathcal{V}\}$ and $\{\mathcal{Q}, \mathcal{W}\}$ are compatible.

Proof . Following the proof of Theorem 3.1, we know that the sequence $\{\theta_n\}$ in U defined by

$$\begin{aligned} \theta_{2n+1} &= \mathcal{W}\rho_{2n+1} = \mathcal{P}\rho_{2n} \\ \text{and } \theta_{2n+2} &= \mathcal{V}\rho_{2n+2} = \mathcal{Q}\rho_{2n+1} \text{ for } n = 0, 1, 2, \ldots . \end{aligned}$$

is a $b-$multiplicative Cauchy sequence. By the completeness of U, there exists $x^* \in U$ such that $\theta_n \to x^*$ as $n \to +\infty$. As a result, $\{\mathcal{P}\rho_{2n}\}$, $\{\mathcal{Q}\rho_{2n+1}\}$, $\{\mathcal{V}\rho_{2n+2}\}$, $\{\mathcal{W}\rho_{2n+1}\}$ converge to x^* as $n \to +\infty$. Since $\mathcal{V}$ is continuous, therefore,

$$\lim_{n\to+\infty} \mathcal{V}^2\rho_{2n+2} = \mathcal{V}x^* \text{ and } \lim_{n\to+\infty} \mathcal{V}\mathcal{P}\rho_{2n} = \mathcal{V}x^*.$$

Since the pair $\{\mathcal{P}, \mathcal{V}\}$ is compatible, $\lim_{n\to+\infty} d_b(\mathcal{P}\mathcal{V}\rho_{2n}, \mathcal{V}\mathcal{P}\rho_{2n}) = 1$. So, we have $\lim_{n\to+\infty} \mathcal{P}\mathcal{V}\rho_{2n} = \mathcal{V}x^*$. Putting $\rho = \mathcal{V}\rho_{2n}$ and $\theta = \rho_{2n+1}$ in (3.4), we obtain

$$\begin{aligned} d_b(\mathcal{P}\mathcal{V}\rho_{2n}, \mathcal{Q}\rho_{2n+1}) \leq \max\Big\{& d_b(\mathcal{V}^2\rho_{2n}, \mathcal{W}\rho_{2n+1}), d_b(\mathcal{P}\mathcal{V}\rho_{2n}, \mathcal{V}^2\rho_{2n}), d_b(\mathcal{Q}\rho_{2n+1}, \mathcal{W}\rho_{2n+1}), \\ & \frac{1}{2}\big(d_b(\mathcal{V}^2\rho_{2n}, \mathcal{Q}\rho_{2n+1}) + d_b(\mathcal{P}\mathcal{V}\rho_{2n}, \mathcal{W}\rho_{2n+1})\big)\Big\}^\lambda \end{aligned}$$

Letting $n \to +\infty$ in the above inequality, we get

$$\begin{aligned} d_b(\mathcal{V}x^*, x^*) &\leq \max\Big\{d_b(\mathcal{V}x^*, x^*), d_b(x^*, x^*), d_b(x^*, x^*), \frac{1}{2}\big(d_b(\mathcal{V}x^*, x^*) + d_b(\mathcal{V}x^*, x^*)\big)\Big\}^\lambda \\ &= \max\{d_b(\mathcal{V}x^*, x^*), 1\}^\lambda \\ &= d_b^\lambda(\mathcal{V}x^*, x^*) \end{aligned}$$

which implies that $\mathcal{V}x^* = x^*$. Using the continuity of $\mathcal{W}$, we obtain,

$$\lim_{n\to+\infty} \mathcal{W}^2\rho_{2n+1} = \mathcal{W}x^* \text{ and } \lim_{n\to+\infty} \mathcal{W}\mathcal{Q}\rho_{2n+1} = \mathcal{W}x^*.$$

Since $\mathcal{Q}$ and $\mathcal{W}$ are compatible,

$$\lim_{n\to+\infty} d(\mathcal{Q}\mathcal{W}\rho_{2n+1}, \mathcal{W}\mathcal{Q}\rho_{2n+1}) = 1.$$

So, we have, $\lim_{n\to+\infty} \mathcal{Q}\mathcal{W}\rho_{2n+1} = \mathcal{Q}x^*$. Putting $\rho = \rho_{2n}$ and $\theta = \mathcal{W}\rho_{2n+1}$ in (3.4), we obtain,

$$\begin{aligned} d_b(\mathcal{P}\rho_{2n}, \mathcal{Q}\mathcal{W}\rho_{2n+1}) \leq \max\Big\{& d_b(\mathcal{V}\rho_{2n}, \mathcal{W}^2\rho_{2n+1}), d_b(\mathcal{P}\rho_{2n}, \mathcal{V}\rho_{2n}), d_b(\mathcal{Q}\mathcal{W}\rho_{2n+1}, \mathcal{W}^2\rho_{2n+1}), \\ & \frac{1}{2}\big(d_b(\mathcal{V}\rho_{2n}, \mathcal{Q}\mathcal{W}\rho_{2n+1}) + d_b(\mathcal{P}\rho_{2n}, \mathcal{W}^2\rho_{2n+1})\big)\Big\}^\lambda \end{aligned}$$

Letting $n \to +\infty$ in the above inequality, we have

$$\begin{aligned} d_b(x^*, \mathcal{W}x^*) &\leq \max\big\{d_b(x^*, \mathcal{W}x^*), 1, 1, \frac{1}{2}\big(d_b(x^*, \mathcal{W}x^*) + d_b(x^*, \mathcal{W}x^*)\big)\big\}^\lambda \\ &= d_b^\lambda(x^*, \mathcal{W}x^*) \end{aligned}$$

This shows that $\mathcal{W}x^* = x^*$ and

$$\begin{aligned} d_b(\mathcal{P}x^*, \mathcal{Q}\rho_{2n+1}) \leq \max\Big\{& d_b(\mathcal{V}x^*, \mathcal{W}\rho_{2n+1}), d_b(\mathcal{P}x^*, \mathcal{V}x^*), d_b(\mathcal{Q}\rho_{2n+1}, \mathcal{W}\rho_{2n+1}), \\ & \frac{1}{2}\big(d_b(\mathcal{V}x^*, \mathcal{Q}\rho_{2n+1}) + d_b(\mathcal{P}x^*, \mathcal{W}\rho_{2n+1})\big)\Big\}^\lambda. \end{aligned}$$

Taking the upper limit as $n \to +\infty$ and using $\mathcal{V}x^* = \mathcal{W}x^* = x^*$, we have

$$\begin{aligned} d_b(\mathcal{P}x^*, x^*) &\leq \max\left\{d_b(\mathcal{V}x^*, x^*), d_b(x^*, x^*), d_b(x^*, x^*), \frac{1}{2}\big(d_b(\mathcal{V}x^*, x^*) + d_b(\mathcal{P}x^*, x^*)\big)\right\}^\lambda \\ &\leq \max\{d_b(x^*, x^*), d_b(\mathcal{P}x^*, x^*)\}^\lambda \\ &= d_b^\lambda(\mathcal{P}x^*, x^*) \end{aligned}$$

which implies that $d_b(\mathcal{P}x^*, x^*) = 1$ and hence $\mathcal{P}x^* = x^*$. Finally, from (3.4) and the fact that $\mathcal{P}x^* = \mathcal{V}x^* = \mathcal{W}x^* = x^*$, we have

$$\begin{aligned} d_b(x^*, \mathcal{Q}x^*) &= d_b(\mathcal{P}x^*, \mathcal{Q}x^*) \\ &\leq \max\big\{d_b(\mathcal{V}x^*, \mathcal{W}x^*), d_b(\mathcal{P}x^*, \mathcal{V}x^*), d_b(\mathcal{Q}x^*, \mathcal{W}x^*), \\ &\qquad \frac{1}{2}\big(d_b(Sx^*, Qx^*) + d_b(Px^*, Tx^*)\big)\big\}^\lambda \\ &= \max\{1, 1, d_b(\mathcal{Q}x^*, x^*), \frac{1}{2}(d_b(x^*, \mathcal{Q}x^*) + 1)\} \\ &= d_b^\lambda(x^*, \mathcal{Q}x^*) \end{aligned}$$

this shows that $\mathcal{Q}x^* = x^*$. Hence, there exists a common fixed point for the mappings $\mathcal{P}, \mathcal{Q}, \mathcal{V}$ and $\mathcal{W}$. Let, if possible, $z \in U$ be another common fixed point of $\mathcal{P}, \mathcal{Q}, \mathcal{V}$ and $\mathcal{W}$, then

$$\begin{aligned} d_b(x^*, z) &= d_b(\mathcal{P}x^*, \mathcal{Q}z) \\ &\leq \max\left\{d_b(\mathcal{V}x^*, \mathcal{W}z), d_b(\mathcal{P}x^*, \mathcal{V}x^*), d_b(\mathcal{Q}z, \mathcal{W}z), \frac{1}{2}(d_b(\mathcal{V}x^*, \mathcal{Q}z) + d_b(\mathcal{P}x^*, \mathcal{W}z))\right\}^\lambda \\ &= \max\{d_b(x^*, z), 1\}^\lambda \\ &= d_b^\lambda(x^*, z). \end{aligned}$$

which implies that $x^* = z$. Hence, $\mathcal{P}, \mathcal{Q}, \mathcal{V}$ and $\mathcal{W}$ have a unique common fixed point in U. □

Example 3.7. *Let $U = [0, 1]$. We define a mapping $d_b : U \times U \to \mathbb{R}^+$ by $d_b(\rho, \theta) = e^{(\rho-\theta)^2}$, for all $\rho, \theta \in U$. Obviously, (U, d_b) is a complete $b-$multiplicative metric space. Consider the self-mappings;*

$$\mathcal{P}(\rho) = \mathcal{Q}(\rho) = \begin{cases} \frac{\rho}{2}, & \text{if } \rho \in \big[0, \frac{1}{2}\big) \\ \frac{1}{4}, & \text{if } \rho \in \big[\frac{1}{2}, 1\big] \end{cases}$$

and $\mathcal{V}(\rho) = \mathcal{W}(\rho) = \begin{cases} \frac{\rho}{3}, & \text{if } \rho \in \big[0, \frac{1}{2}\big) \\ \frac{1}{4}, & \text{if } \rho \in \big[\frac{1}{2}, 1\big] \end{cases}$

Since $\mathcal{P}(U) = \mathcal{Q}(U) = \big[0, \frac{1}{4}\big]$, $\mathcal{V}(U) = \mathcal{W}(U) = \big[0, \frac{1}{6}\big) \cup \big\{\frac{1}{4}\big\}$, so, we have $\mathcal{V}(U) \subset \mathcal{Q}(U)$ and $\mathcal{W}(U) \subset \mathcal{P}(U)$.

Now we show that $\{\mathcal{P}, \mathcal{V}\}$ is a compatible mapping of type (E). For this, we define a sequence $\{\rho_n\}$ where $\rho_n = \frac{1}{n}$, for $n \geq 1$. Then, $\lim_{n\to+\infty} \mathcal{P}\rho_n = \lim_{n\to+\infty} \frac{\rho_n}{2} = \lim_{n\to+\infty} \frac{1}{2n} = 0 = t$. Also we have, $\lim_{n\to+\infty} \mathcal{P}\mathcal{P}\rho_n = \lim_{n\to+\infty} \mathcal{P}(\frac{\rho_n}{2}) = 0$, $\lim_{n\to+\infty} \mathcal{P}\mathcal{V}\rho_n = \lim_{n\to+\infty} \mathcal{P}(\frac{\rho_n}{3}) = 0, \mathcal{P}(0) = 0$ and $\lim_{n\to+\infty} \mathcal{V}\mathcal{V}\rho_n =$

$\lim_{n\to+\infty}\mathcal{V}(\frac{\rho_n}{3})=0,\ \ \lim_{n\to+\infty}\mathcal{V}\mathcal{P}\rho_n=\lim_{n\to+\infty}\mathcal{V}(\frac{\rho_n}{2})=0, \mathcal{V}(0)=0.$

Therefore, $\{\mathcal{P},\mathcal{V}\}$ is a compatible mapping of type (E) and also we have $\mathcal{P}(0)=\mathcal{V}(0)$, so $\{\mathcal{P},\mathcal{V}\}$ is a compatible mapping. Also, we have $d_b(\mathcal{P}\mathcal{V}\rho,\mathcal{V}\mathcal{P}\rho)\le d_b(\mathcal{P}\rho,\mathcal{V}\rho)$ for all $\rho\in U$ and hence $\mathcal{P}$ and $\mathcal{V}$ are weakly commuting mappings. Similarly, $\mathcal{Q}$ and $\mathcal{W}$ are also weakly commuting mappings. For $\rho,\theta\in[0,\frac{1}{2})$,

$$\begin{aligned} d_b^q(\mathcal{P}\rho,\mathcal{Q}\theta) &\le \max\left\{d_b(\mathcal{V}\rho,\mathcal{W}\theta), d_b(\mathcal{P}\rho,\mathcal{V}\rho), d_b(\mathcal{Q}\theta,\mathcal{W}\theta), d_b(\mathcal{P}\rho,\mathcal{W}\theta)^{\frac{1}{2s}}, d_b(\mathcal{Q}\theta,\mathcal{V}\rho)^{\frac{1}{2s}}\right\} \\ \Longrightarrow e^{(\frac{\rho}{2}-\frac{\theta}{3})^2 q} &\le \max\left\{1,\ e^{(\frac{\rho}{6})^2},\ e^{(\frac{\theta}{6})^2},\ e^{(\frac{\rho}{2}-\frac{\theta}{3})^2\frac{1}{2s}},\ e^{(\frac{\theta}{2}-\frac{\rho}{3})^2\frac{1}{2s}}\right\}. \end{aligned}$$

Because $\theta=\ln\rho$ is an increasing function, so

$$(\frac{\rho}{2}-\frac{\theta}{3})^2 q\le\max\left\{0,\ (\frac{\rho}{6})^2,\ (\frac{\theta}{6})^2,\ (\frac{\rho}{2}-\frac{\theta}{3})^2\frac{1}{2s},\ (\frac{\theta}{2}-\frac{\rho}{3})^2\frac{1}{2s}\right\}$$

which is true for all $\rho,\theta\in U$. All the conditions of Theorem 3.1 are satisfied, so by Theorem 3.1, we obtain a unique common fixed point. Here, the unique common fixed point is 0.

Example 3.8. *Let $U=[0,1]$. Define a mapping $d_b: U\times U\to\mathbb{R}^+$ by $d_b(\rho,\theta)=e^{(\rho-\theta)^2}$ for all $\rho,\theta\in U$. Obviously, (U,d_b) is a complete $b-$multiplicative metric space. Consider the self-mappings*

$$\mathcal{P}(\rho)=(\tfrac{\rho}{2})^4,\quad \mathcal{Q}(\rho)=(\tfrac{\rho}{3})^2,\quad \mathcal{V}(\rho)=(\tfrac{\rho}{2})^2,\quad \mathcal{W}(\rho)=(\tfrac{\rho}{3}).$$

One can easily see that $\mathcal{V}(U)$ and $\mathcal{W}(U)$ are continuous mappings also $(\mathcal{P},\mathcal{V})$ and $(\mathcal{Q},\mathcal{W})$ are compatible. For each $\rho,\theta\in U$, we have

$$\begin{aligned} d_b(\mathcal{P}\rho,\mathcal{Q}\theta) &= e^{(\mathcal{P}\rho-\mathcal{Q}\theta)^2} \\ &= e^{\left((\frac{\rho}{2})^4-(\frac{\theta}{3})^2\right)^2} \\ &= e^{\left((\frac{\rho}{2})^2+\frac{\theta}{3}\right)^2\left((\frac{\rho}{2})^2-\frac{\theta}{3}\right)^2} \\ &\le e^{\left((\frac{\rho}{2})^2-\frac{\theta}{3}\right)^2} \\ &= d_b(\mathcal{V}\rho,\mathcal{W}\theta) \\ &\le \Delta_1(\rho,\theta). \end{aligned}$$

Thus, $\mathcal{P},\mathcal{Q},\mathcal{V}$ and $\mathcal{W}$ satisfy all the conditions of Theorem 3.6. By theorem 3.6, we will have a unique common fixed point. Here, 0 is the unique common fixed point of $\mathcal{P},\mathcal{Q},\mathcal{V}$ and $\mathcal{W}$.

4. Application

In this section, we study the existence and uniqueness of the solution of a system of multiplicative integral equations. Consider the integral equation

$$\rho(t)=\int_a^b K_i\big(t,s,\rho(s)\big)^{ds},\quad \text{for}\quad i=1,2\quad\text{and}\quad s,t\in[a,b] \tag{4.1}$$

where $a,b\in\mathbb{R}$ and $K_i:[a,b]\times[a,b]\times\mathbb{R}$. The purpose of this section is to give an existence theorem for a solution of (4.1) using Theorem 3.1.

Consider the space $U=C[a,b]$ of real continuous functions defined on $[a,b]$. Obviously this space, with the $b-$multiplicative metric given by

$$d_b(\rho,\theta)=\begin{cases}\sup_{t\in[a,b]}\left|\frac{\rho(t)}{\theta(t)}\right|^2, & \text{if } \frac{\rho(t)}{\theta(t)}>1\\ \sup_{t\in[a,b]}\left|\frac{\theta(t)}{\rho(t)}\right|^2, & \text{if } \frac{\rho(t)}{\theta(t)}<1\end{cases}$$

is a complete $b-$multiplicative metric space.

Theorem 4.1. *Assume that*

(i) for each $t, s \in [a, b]$ *and* $\rho, \theta \in U$*, there exists a constant* $\eta > 0$ *such that*

$$\Big|\frac{K_1(t,s,\rho(s))}{K_2(t,s,\theta(s))}\Big|^2 \leq \Big(\Big|\frac{\rho(s)}{\theta(s)}\Big|\Big)^{\eta},$$

(ii) the constant η *is such that* $\eta < \frac{1}{q(b-a)}$, $1 < q < \infty$,

(iii) $K_i : [a,b] \times [a,b] \times \mathbb{R} \to \mathbb{R}$ *is continuous.*

Then the system (4.1) have a unique common solution in U*.*

Proof . Consider the two mappings $\mathcal{P}, \mathcal{Q} : U \to U$ defined as

$$\begin{aligned}
\mathcal{P}\rho(t) &= \int_a^b K_1(t,s,\rho(s))^{ds}, \\
\mathcal{Q}\rho(t) &= \int_a^b K_2(t,s,\rho(s))^{ds}, \quad s, t \in [a,b],
\end{aligned}$$

The system (4.1) has a common solution if and only if the self-mappings $\mathcal{P}$ and $\mathcal{Q}$ have a common fixed point in U. Since K_i are continuous, so $\mathcal{P}$ and $\mathcal{Q}$ are continuous, then $\{\mathcal{I}_U, \mathcal{P}\}$ and $\{\mathcal{I}_U, \mathcal{Q}\}$ are compatible mappings of type (E). Further we have

$$\begin{aligned}
\Big|\frac{\mathcal{P}\rho(t)}{\mathcal{Q}\theta(t)}\Big|^2 &= \Big(\int_a^b \Big|\frac{K_1(t,s,\rho(s))}{K_2(t,s,\theta(s))}\Big|^{ds}\Big)^2 \\
&\leq \Big(\int_a^b \Big(\Big|\frac{\rho(s)}{\theta(s)}\Big|^{\eta}\Big)^{ds}\Big)^2 \\
&\leq \Big(\int_a^b \big(d_b(\rho,\theta)^{\frac{\eta}{2}}\big)^{ds}\Big)^2 \\
&= \Big(\big(d_b(\rho,\theta)^{b-a}\big)^{\frac{\eta}{2}}\Big)^2 \\
&= d_b(\rho,\theta)^{\eta(b-a)} \\
&< d_b(\rho,\theta)^{\frac{1}{q}} \\
&\leq \max\Big\{d_b(\rho,\theta), d_b(\mathcal{P}\rho,\rho), d_b(\mathcal{Q}\theta,\theta), d_b(\mathcal{P}\rho,\theta)^{\frac{1}{2s}}, d_b(\mathcal{Q}\theta,\rho)^{\frac{1}{2s}}\Big\}^{\frac{1}{q}}, \\
&\qquad \text{where } 1 \leq s < \infty.
\end{aligned}$$

Therefore

$$\begin{aligned}
d_b^q(\mathcal{P}\rho, \mathcal{Q}\theta) &\leq \max\Big\{d_b(\rho,\theta), d_b(\mathcal{P}\rho,\rho), d_b(\mathcal{Q}\theta,\theta), d_b(\mathcal{P}\rho,\theta)^{\frac{1}{2s}}, d_b(\mathcal{Q}\theta,\rho)^{\frac{1}{2s}}\Big\}, \\
&\qquad \text{where } 1 \leq s < q < \infty.
\end{aligned}$$

for all $\rho, \theta \in U$. Consequently, all the hypotheses of Theorem 3.1 (with $\mathcal{V} = \mathcal{W} = \mathcal{I}_U$) hold. Then $\mathcal{P}$ and $\mathcal{Q}$ have a unique common fixed point and so the system (4.1) have a unique common solution. □

5. $R-$multiplicative metric space

In this section, we introduce the notion of $R-$multiplicative metric space with some examples. At the end of this section, we have an open question.

Recently, Eshagi Gordji et al. [6] introduced the concept of orthogonal set (for more details, we refer [2],[5]) and Khalehogli et al. [14] introduced the notion of $R-$metric spaces and gave a real generalization of the Banach fixed point theorem. Inspired by the works of Khalehogli et al. [14] , we introduce the concept of $R-$multiplicative metric space as follows.

Definition 5.1. *Suppose (U, d) is a multiplicative metric space and R is a relation on U. Then, the triple (U, d, R) is called $R-$multiplicative metric space.*

Definition 5.2. *[6] Let $U \neq \emptyset$ and $\perp \subseteq U \times U$ be a binary relation. If $\perp$ satisfies the following conditions:*

$$\exists x_0 : (\forall y, y \perp x_0) \quad or \quad (\forall y, x_0 \perp y),$$

then it is called an orthogonal set (briefly O-set) and it is denoted by $(U, \perp)$.

If $R = \perp \subseteq U \times U$, then we say that the triplet $(U, d, \perp)$ is orthogonal multiplicative metric space.

Example 5.3. *Let $U = (0, +\infty)$ and $d : U^n \times U^n \to \mathbb{R}$ be defined by*

$$d(x, y) = |\frac{x_1}{y_1}|\ |\frac{x_2}{y_2}| \dots |\frac{x_n}{y_n}|,$$

where $x = (x_1, x_2, \dots, x_n),\ y = (y_1, y_2, \dots, y_n) \in U^n$ and $\Big|\ .\ \Big| : \mathbb{R}^+ \to \mathbb{R}^+$ is defined by

$$|a| = \begin{cases} a, \text{if } a \geq 1 \\ \frac{1}{a}, \text{if } a < 1 \end{cases}$$

Obviously, d is a multiplicative metric. Define $xRy, \quad x, y \in U^n$ if $|xy| \leq \max\{|x|, |y|\}$. Then (U^n, d, R) is a $R-$multiplicative metric space.

Example 5.4. *Let $U = [0, +\infty)$ and $d : U \times U \to \mathbb{R}$ be defined by $d(x, y) = e^{|x-y|}, \forall x, y \in U$. Also, define a relation R on U as $\{(x, y) : x \leq y,\ x, y \in U\}$. Then, (U, d, R) is an $R-$multiplicative metric space.*

Definition 5.5. *Let $\{x_n\}$ be a sequence in an $R-$multiplicative metric space (U, d, R). Then*

(i) $\{x_n\}$ is called an $R-$multiplicative sequence if $x_n R x_{n+k}$ for each $n, k \in \mathbb{N}$.

(ii) $\{x_n\}$ is said to converge to $x \in U$ if for every $\varepsilon > 1$, there is an integer N such that $d(x_n, x) < \varepsilon$ if $n \geq N$. In this case, we write $x_n \xrightarrow{R} x$.

(iii) $\{x_n\}$ in U is said to be an $R-$multiplicative Cauchy sequence if for every $\varepsilon > 1$, there exists an integer N such that $d(x_n, x_m) < \varepsilon$ if $n, m \geq N$. It is clear that $x_n R x_m$ or $x_m R x_n$.

(iv) U is said to be $R-$multiplicative complete if every $R-$multiplicative Cauchy sequence in U converges to a point in U.

Definition 5.6. *Let $\mathcal{A} : U \to U$ be a mapping. Then, $\mathcal{A}$ is called $R-$multiplicative preserving if xRy, then $\mathcal{A}xR\mathcal{A}y$ for all $x, y \in U$.*

Definition 5.7. *Let $\mathcal{A} : U \to U$ be a mapping. Then, $\mathcal{A}$ is said to be $R-$multiplicative continuous at $x \in U$ if for every $R-$multiplicative sequence $\{x_n\}$ in U with $x_n \overset{R}{\to} x$, we have $\mathcal{A}x_n \to \mathcal{A}x$. Also, $\mathcal{A}$ is said to be $R-$multiplicative continuous on U if $\mathcal{A}$ is $R-$multiplicative continuous in each $x \in U$.*

Definition 5.8. *A mapping $\mathcal{A} : U \to U$ is said to be an $R-$multiplicative contraction with Lipschitz constant $0 < \lambda < 1$ if for all $x, y \in U$ such that xRy, we have*

$$d\big(\mathcal{A}x, \mathcal{A}y\big) \leq d(x, y)^{\lambda}$$

Open question:

Let U be an $R-$multiplicative metric space. Let $\mathcal{A} : U \to U$ be $R-$multiplicative continuous, $R-$multiplicative contraction with Lipschitz constant $\lambda \in (0, 1)$ and $R-$multiplicative preserving. Does $\mathcal{A}$ has a unique fixed point if $\exists x_0 \in U$ such that $x_0 R y$ for all $y \in \mathcal{A}(U)$?

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