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Best proximity point theorems by K, C and $\mathcal{MT}$ types in b-metric spaces with an application

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Abstract

In this paper, we introduce the concept of weak $\mathcal{MT}-K$ rational cyclic and weak $\mathcal{MT}-C$ rational cyclic conditions and a combination of both conditions in what we call weak $\mathcal{MT}-KC$ rational cyclic condition. We investigate some best proximity points theorems for a pair of mappings that satisfy these conditions that have been established in b-metric spaces. Our results include an application to the nonlinear integral equation as well.

Keywords: b-metric space, $\mathcal{MT}$-function, Rational cyclic condition, Cyclic map, Best proximity point.
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1. Introduction and preliminaries

In 1989, Bakhtin [2] introduced b-metric spaces as a generalization of metric spaces. Since then, several papers have been published on the fixed point theory and on the generalization of Banach contraction principle in such spaces.

The famous Banach's contraction principle states that every contraction self mapping on a complete metric space has a unique fixed point. This principle has been generalized and extended in several ways.

Let A and B be nonempty subsets of metric space (X, d) and let $T : A \to B$ be a non-self mapping. The equation $Tx = x$ does not necessarily have a solution. Best proximity point theorems analyze the existence of an approximate solution that is optimal. Best proximity point theorems are intended

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to furnish an approximate solution x that is optimal in the sense that the error $d(x, Tx)$ is minimum. Indeed, if A and B are nonempty subsets of X such that $d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$. then in light of the fact that $d(x, Tx)$ is at least $d(A, B)$. A best proximity point theorem guarantees the global minimization of $d(x, Tx)$ by the requirement that an approximate solution x satisfies the condition $d(x, Tx) = d(A, B)$. Such optimal approximate solutions are called best proximity points of the mapping T.

In 2003, Kirk et al. [12] introduced an interesting class of cyclic maps.

In 2005, the concepts of cyclic mapping and best proximity point were investigated by Eldreed, Kirk and Veeramani [7].

In 2006, Eldered and Veeramani [8] obtained some existence results about best proximity points of cyclic contraction mappings.

In [17], Sadiq Basha introduce the concept of K-cyclic and C-cyclic, for two mappings $T : A \to B$ and $S : B \to A$.

Motivated by the concepts of K-cyclic and C-cyclic mappings and the $\mathcal{MT}$-function, Lin et al. [15] introduced the concept of weak $\mathcal{MT} - K$ and $\mathcal{MT} - C$ conditions.

Best proximity points have been investigated for various types of maps on various spaces. For instance, in [16] and [10], best proximity points have been investigated in $\mathcal{S}$-metric space and $\mathcal{G}$-metric space.

In this paper, at the first we introduce the concept of b-metric space and introduce the concept of weak $\mathcal{MT} - K$ rational cyclic and weak $\mathcal{MT} - C$ rational cyclic conditions and a combination of both conditions in what we call weak $\mathcal{MT} - KC$ rational cyclic condition then investigate some best proximity points theorems for a pair of mappings satisfy these conditions have been established in b-metric spaces.

Definition 1.1. ([9]) *Let X be a (nonempty) set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is called a b-metric on X if the following conditions hold for all $x, y, z \in X$:*

(i) $d(x, y) = 0$ *if and only if* $x = y$,

(ii) $d(x, y) = d(y, x)$,

(iii) $d(x, y) \leq s[d(x, z) + d(z, y)]$ *(b-triangular inequality).*

Then, the pair (X, d) is called a b-metric space with parameter s.

Example 1.2. ([13]) *Let (X, d) be a metric space and let $\beta > 1, \lambda \geq 0$ and $\mu > 0$. For $x, y \in X$, set $\rho(x, y) = \lambda d(x, y) + \mu d(x, y)^{\beta}$. Then (X, ρ) is a b-metric space with the parameter $s = 2^{\beta-1}$ and not a metric space on X.*

Definition 1.3. ([3]) *Let (X, d) be a $b-$metric space. Then a sequence $\{x_n\}$ in X is called:*

(i) *$b-$convergent if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$. In this case, we write* $\lim\limits_{n\to\infty} x_n = x$.

(ii) *A $b-$Cauchy sequence if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.*

Lemma 1.4. ([1]) *Let (X, d) be a $b-$metric space with coefficient $s \geq 1$ and let $\{x_n\}$ and $\{y_n\}$ be $b-$convergent to points $x, y \in X$, respectively, Then we have*

$$\frac{1}{s^2} d(x, y) \leq \liminf_{n\to\infty} d(x_n, y_n) \leq \limsup_{n\to\infty} d(x_n, y_n) \leq s^2 d(x, y).$$

In particular if $x = y$, then we have $\lim_{n\to\infty} d(x_n, y_n) = 0$. Moreover, for each $z \in X$, we have

$$\frac{1}{s}d(x,z) \le \liminf_{n\to\infty} d(x_n,z) \le \limsup_{n\to\infty} d(x_n,z) \le sd(x,z).$$

Definition 1.5. ([11]) *Let A and B are nonempty subset of a nonempty set X, a map $T : A\cup B \to A\cup B$, is a cyclic map if $T(A) \subset B$ and $T(B) \subset A$.*

A point $x \in A \cup B$ is called a best proximity point for Tif

$$d(x,Tx) = d(A,B),$$

where $d(A,B) = \inf\{d(x,y) : x \in A, y \in B\}$.

Definition 1.6. ([7]) *Let A and B be nonempty subsets of a metric space (X,d). A cyclic map $T : A\cup B \to A\cup B$ is called a cyclic contradiction map if there exists $k \in [0,1)$ such that*

$$d(Tx,Ty) \le kd(x,y) + (k-1)d(A,B)$$

for all $x \in A$ and $y \in B$.

Definition 1.7. ([17]) *A pair of mappings $T : A \to B$ and $S : B \to A$ is said to form*

(*i*) *a K-cyclic mapping between A and B if there exists a nonnegative real number $k < \frac{1}{2}$ such that*

$$d(Tx,Sy) \le k[d(x,Tx) + d(y,Sy)] + (1-2k)d(A,B)$$

for all $x \in A$ and $y \in B$.

(*ii*) *A C-cyclic mapping between A and B if there exists a nonnegative real number $k < \frac{1}{2}$ such that*

$$d(Tx,Sy) \le k[d(x,Sy) + d(y,Tx)] + (1-2k)d(A,B)$$

for all $x \in A$ and $y \in B$.

Definition 1.8. ([5]) *A function $\varphi : [0,\infty) \to [0,1)$ is said to be an $\mathcal{MT}$-functions (or $\mathcal{R}$-functions). If*

$$\lim_{s\to t^+} \sup \varphi(s) < 1, \quad \text{for all } t \in [0,\infty).$$

Example 1.9. ([6]) *Let $\varphi : [0,\infty) \to [0,1)$ be defined by*

$$\varphi(t) = \begin{cases} \dfrac{\sin t}{t} & \text{if } t \in (0, \frac{\pi}{2}) \\ 0 & \text{otherwise.} \end{cases}$$

Since $\lim_{s\to 0^+} \sup \varphi(s) = 1$, φ is not a $\mathcal{MT}$-function.

Definition 1.10. *Let φ and ψ be functions from $[0,\infty)$ into $[0,1)$. The pair of functions φ and ψ is said to satisfy weak $\mathcal{MT}$-condition if $\varphi + \psi$ is an $\mathcal{MT}$- functions that mean*

$$\lim_{s\to t^+} \sup(\varphi(s) + \psi(s)) < 1, \quad \text{for all } t \in [0,\infty).$$

Motivated by the concepts of K-cyclic and C-cyclic mappings and the $\mathcal{MT}$ functions, Lin et. al. [14] introduced the concept of weak $\mathcal{MT}-K$ and $\mathcal{MT}-C$ conditions . we introduce this concepts in b-metric space.

Definition 1.11. ([14]) *Let A and B be nonempty subsets of a b-metric space (X, d) and $T : A \to B$ and $S : B \to A$ be maps. The pair of maps T and S is said to satisfy*

(*i*) *weak $b - \mathcal{MT} - \mathcal{K}$ cyclic condition if there exists an $\mathcal{MT}$- function such that*

$$d(Tx, Sy) \leq \frac{1}{2s}\varphi(d(x, y))[d(x, Tx) + d(y, Sy)] + \left(1 - \frac{1}{s}\varphi(d(x, y))\right) d(A, B)$$

for all $x \in A$ and $y \in B$.

(*ii*) *weak $b - \mathcal{MT} - \mathcal{C}$ cyclic condition if there exists an $\mathcal{MT}$ function such that*

$$d(Tx, Sy) \leq \frac{1}{2s}\varphi(d(x, y))[d(x, Sy) + d(y, Tx)] + \left(1 - \frac{1}{s}\varphi(d(x, y))\right) d(A, B)$$

for all $x \in A$ and $y \in B$.

Theorem 1.12. ([5]) *Let $\varphi : [0, \infty) \to [0, 1)$ be a function. Then the following statements are equivalent.*

(*i*) *φ is an $\mathcal{MT}$-function.*

(*ii*) *For any non increasing sequence $\{x_n\}_{n\in\mathbb{N}}$ in $[0, 1), 0 \leq \sup_{n\in\mathbb{N}} \varphi(x_n) < 1$.*

(*iii*) *φ is a function of contractive factor, for any strictly decreasing sequence $\{x_n\}_{n\in\mathbb{N}}$ in $[0, 1)$, we have $0 \leq \sup_{n\in\mathbb{N}} \varphi(x_n) < 1$.*

2. Main Results

We introduce the concept of weak $b - \mathcal{MT} - K$ rational cyclic and weak $b - \mathcal{MT} - C$ rational cyclic conditions and a combination of both conditions in what we call weak $b - \mathcal{MT} - KC$rational cyclic condition. Some best proximity point theorems for a pair of mappings satisfy these conditions have been established in b-metric space. In the next definition , we generalize Definition [4, Definition 3.1] to b-metric spaces.

Definition 2.1. *Let A and B be nonempty subsets of a b-metric space (X, d) and $T : A \longrightarrow B$ and $S : B \longrightarrow A$ be maps. The pair of maps T and S is said to satisfy*

(*i*) *weak $b - \mathcal{MT} - K$ rational cyclic condition if there exists an $\mathcal{MT}$ function φ such that*

$$d(Tx, Sy) \leq \frac{1}{2s}\varphi(d(x, y))\frac{d(x, Tx)d(x, Sy)}{d(x, Tx) + d(y, Sy)} + \frac{1}{s}(1 - \varphi(d(x, y)))d(A, B) \tag{2.1}$$

for all $x \in A, y \in B$ and $x \neq Sy$.

(*ii*) *weak $b - \mathcal{MT} - C$ rational cyclic condition if there exists an $\mathcal{MT}$ function φ such that*

$$d(Tx, Sy) \leq \frac{1}{2s}\varphi(d(x, y))\frac{d(x, Tx)d(x, Sy)}{d(x, Sy) + d(y, Tx)} + \frac{1}{s}(1 - \varphi(d(x, y)))d(A, B) \tag{2.2}$$

for all $x \in A, y \in B$ and $x \neq Sy$.

(iii) *weak $b-\mathcal{MT}-KC$ rational cyclic condition if there exists a pair of $\mathcal{MT}$ function φ,ψ such that*

$$\begin{aligned} d(Tx,Sy) \leq& \frac{k_1}{s}\varphi(d(x,y))\frac{d(x,Tx)d(x,Sy)}{d(x,Tx)+d(y,Sy)}+\frac{k_2}{s}\psi(d(x,y))\frac{d(x,Tx)d(x,Sy)}{d(x,Sy)+d(y,Tx)} \\ &+\frac{1}{s}(1-2[k_1\varphi(d(x,y))+k_2\psi(d(x,y))])d(A,B) \end{aligned} \tag{2.3}$$

for all $x\in A, y\in B$ and $x\neq Sy$ and $k_1,k_2\leq \frac{1}{2}$.

Theorem 2.2. *Let A and B be nonempty subsets of a b-metric space (X,d) and $T:A\longrightarrow B$ and $S:B\longrightarrow A$ be maps. If the pair of maps T and S satisfy the weak $b-\mathcal{MT}-KC$ rational cyclic condition then there exists a sequence $\{x_n\}$ in X such that*

$$\lim_{s\to\infty} d(x_n,x_{n+1}) = d(A,B).$$

Moreover, the following statements hold.

(i) *If $\{x_{2n}\}_{n\in\mathbb{N}}$ has a convergent subsequence in A, then there exists a point $x\in A$ such that $d(x,Tx)=d(A,B)$.*

(ii) *If $\{x_{2n-1}\}_{n\in\mathbb{N}}$ has a convergent subsequence in B, then there exists a point $y\in B$ such that $d(y,Sy)=d(A,B)$.*

Proof . Suppose $x_0\in A$. set $x_{2n+1}=Tx_{2n}$ and $x_{2n}=Sx_{2n-1}$ for all $n\in\mathbb{N}\cup\{0\}$. since $T(A)\subseteq B$ and $S(B)\subseteq A$, we have $\{x_{2n}\}_{n\in\mathbb{N}}\subset A$ and $\{x_{2n-1}\}_{n\in\mathbb{N}}\subset B$. define $\varphi_0:=\varphi\big(d(x_0,x_1)\big)$ and $\psi_0:=\psi\big(d(x,x_1)\big)$, by Definition2.1 we have

$$\begin{aligned} d(x_1,x_2) &= d(Tx_0,Sx_1) \\ &\leq \frac{k_1}{s}\varphi_0\frac{d(x_0,x_1)d(x_0,x_2)}{d(x_0,x_1)+d(x_1,x_2)}+\frac{k_2}{s}\psi_0\frac{d(x_0,x_1)d(x_0,x_2)}{d(x_0,x_2)+d(x_1,x_1)} \\ &+ \left(\frac{1}{s}\big(1-2(k_1\varphi_0+k_2\psi_0)\big)\right)d(A,B) \\ &\leq k_1\varphi_0\frac{d(x_0,x_1)d(x_0,x_2)}{\frac{1}{s}d(x_0x_2)}+k_2\psi_0\frac{d(x_0,x_1)d(x_0,x_2)}{d(x_0,x_2)} \\ &\quad +\Big(1-2\big(k_1\varphi_0+k_2\psi_0\big)\Big)d(A,B) \\ &\leq (k_1\varphi_0+k_2\psi_0)d(x_0,x_1)+\Big(1-2\big(k_1\varphi_0+k_2\psi_0\big)\Big)d(A,B). \end{aligned}$$

therefore,

$$\Big(1-\big(k_1\varphi_0+k_2\psi_0\big)\Big)d(x_1,x_2)\leq \big(k_1\varphi_0+k_2\psi_0\big)d(x_0,x_1)+\Big(1-2(k_1\varphi_0+k_2\psi_0)\Big)d(A,B).$$

it follows that

$$d(x_1,x_2)\leq \frac{(k_1\varphi_0+k_2\psi_0)}{1-(k_1\varphi_0+k_2\psi_0)}d(x_0,x_1)+\frac{1-2(k_1\varphi_0+k_2\psi_0)}{1-(k_1\varphi_0+k_2\psi_0)}d(A,B).$$

We give the proof only for the case $k_1 \geq k_2$; The same reasoning applies to the case $k_1 \leq k_2$. Now ,if we suppose $k_1 \geq k_2$, we would have

$$d(x_1,x_2) \leq \frac{\varphi_0 + \frac{k_2}{k_1}\psi_0}{\frac{1}{1_1} - \left(\varphi_0 + \frac{k_2}{k_1}\psi_0\right)} d(x_0,x_1) + \left(1 - \frac{\varphi_0 + \frac{k_2}{k_1}\psi_0}{\frac{1}{k_1} - \left(\varphi_0 + \dfrac{k_2}{k_1}\psi_0\right)}\right) d(A,B).$$

Accordingly

$$d(x_1,x_2) - d(A,B) \leq \lambda_0 \left(d(x_0,x_1) - d(A,B)\right)$$

where $\lambda_0 := \dfrac{\varphi_0 + \frac{k_2}{k_1}\psi_0}{\frac{1}{k_1} - \left(\varphi_0 + \frac{k_2}{k_1}\psi_0\right)}$.

We now apply Definition2.1 again, we have

$$d(x_2,x_3) - d(A,B) \leq \lambda_1 \left(d(x_0,x_1) - d(A,B)\right).$$

By induction, we have

$$d(x_n,x_{n+1}) - d(A,B) \leq \lambda_{n-1} \left(d(x_{n-1},x_n) - d(A,B)\right), \tag{2.4}$$

where

$$\lambda_{n-1} := \frac{\varphi_{n-1} + \frac{k_2}{k_1}\psi_{n-1}}{\frac{1}{k_1} - \left(\varphi_{n-1} + \frac{k_2}{k_1}\psi_{n-1}\right)}.$$

Since $\varphi(t) + \psi(t) < 1$, $k_1, k_2 \leq \frac{1}{2}$ and $k_1 \geq k_2$, for all $t \in [0,\infty)$ we see that

$$\frac{1}{k_1} - \left(\varphi(t) + \frac{k_2}{k_1}\psi(t)\right) > 1.$$

Consequently

$$\frac{\varphi(t) + \frac{k_2}{k_1}\psi(t)}{\dfrac{1}{k_1} - \left(\varphi(t) + \frac{k_2}{k_1}\psi(t)\right)} < 1$$

for all $t \in [0,\infty)$. By (2.4) we thus get $\lambda_{n-1} < 1$ therefore

$$d(x_n,x_{n+1}) - d(A,B) < d(x_{n-1},x_n) - d(A,B)$$

This gives $d(x_n,x_{n+1}) < d(x_{n-1},x_n)$ for all n. So $\{d(x_n,x_{n+1})\}$ is a strictly decreasing sequence. Since φ and ψ satisfy the weak $\mathcal{MT}-$ condition, by Theorem 1.1, we have

$$0 \leq \sup_{n\in\mathbb{N}}(\varphi_n + \psi_n) < 1.$$

Choose $\gamma = \sup\limits_{n\in\mathbb{N}}(\varphi_n + \psi_n)$, so $0 \leq \gamma < 1$. Since

$$\varphi_n + \frac{k_2}{k_1}\psi_n \leq \varphi_n + \psi_n \leq \gamma$$

we have

$$\frac{1}{k_1} - \varphi_n - \frac{k_2}{k_1}\psi_n \geq \frac{1}{k_1} - \gamma$$

and

$$\frac{\varphi_n + \frac{k_2}{k_1}\psi_n}{\frac{1}{k_1} - \varphi_n - \frac{k_2}{k_1}\psi_n} \leq \frac{\gamma}{\frac{1}{k_1} - \gamma}$$

for all $n \in \mathbb{N}$. So

$$0 \leq \sup_{n\in\mathbb{N}} \frac{\varphi_n + \frac{k_2}{k_1}\psi_n}{\frac{1}{k_1} - \varphi_n - \frac{k_2}{k_1}\psi_n} \leq \frac{\gamma}{\frac{1}{k_1} - \gamma} < 1.$$

Let

$$\lambda := \sup_{n\in\mathbb{N}} \frac{\varphi_n + \frac{k_2}{k_1}\psi_n}{\frac{1}{k_1} - \varphi_n - \frac{k_2}{k_1}\psi_n} \leq \frac{\gamma}{\frac{1}{k_1} - \gamma}$$

then $\lambda \in [0, 1)$. By (2.4) we see that

$$\begin{aligned} d(x_n, x_{n+1}) - d(A,B) &\leq \frac{\varphi_{n-1} + \frac{k_2}{k_1}\psi_{n-1}}{\frac{1}{k_1} - (\varphi_{n-1} + \frac{k_2}{k_1}\psi_{n-1})} \\ &\leq \lambda\big(d(x_{n-1}, x_n) - d(A,B)\big) \\ &\leq \lambda^2\big(d(x_{n-2}, x_{n-1}) - d(A,B)\big) \\ &\vdots \\ &\leq \lambda^n\big(d(x_0, x_1) - d(A,B)\big). \end{aligned}$$

Since $\lambda \in [0, 1)$, it follows that $\lim_{n\to\infty} \lambda^n = 0$ and

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = d(A,B). \tag{2.5}$$

Now, let $\{x_{2n_j}\}$ be a convergent subsequence of $\{x_{2n}\}$ and $x_{2n_j} \to x$ as $j \to \infty$ for some $x \in A$. By Definition2.1we get

$$\begin{aligned} d(Tx, x_{2n_j}) &= d(Tx, Sx_{2n_{j-1}}) \\ &\leq \big(\frac{k_1}{s}\varphi_{2n_{j-1}} + \frac{k_2}{s}\psi_{2n_{j-1}}\big)d(x_{2n_{j-1}}, x_{2n_j}) \\ &+ \frac{1}{s}\big(1 - 2(k_1\varphi_{2n_{j-1}} + k_2\psi_{2n_{j-1}})\big)d(A,B). \end{aligned}$$

for all $j \in \mathbb{N}$. Since $x_{2n_j} \to x$ as $j \to \infty$, by taking the lim sup as $j \to \infty$ in above inequality and using (2.5), and by Lemma1.1 we have $\frac{1}{s}d(Tx, x) \leq \frac{1}{s}d(A,B)$. Then $d(Tx, x) \leq d(A,B)$ On the other hand, since $d(A,B) = \inf\{d(x,y) : x \in A, y \in B\}$ we have $d(Tx, x) \geq d(A,B)$, therefore $d(Tx, x) = d(A,B)$, and (i) is proved. The conclusion (ii) can be verified by using a similar argument as the proof of (i). The same reasoning applies to the case $k_1 \leq k_2$. The proof is completed. □

Our main Theorem 2.2 improves Theorem 3.2 in [4] to metric space by $s = 1$. Also Theorem 3.4, Corollary 3.5 and Corollary 3.6 from [4] are special case of the following Corollaries in b-metric spaces.

Example 2.3. *Let $X = \{1, 2, 3\}$ and $d : X \times X \to [0, \infty)$ be defined as follows :*

1. $d(1, 2) = d(2, 1) = 1$

2. $d(1,3) = d(3,1) = \dfrac{1}{9}$
3. $d(2,3) = d(3,2) = \dfrac{6}{9}$
4. $d(1,1) = d(2,2) = d(3,3) = 0$

It is easy to check that $(\mathbb{X}, d)$ *is a b-metric space with constant* $s = \dfrac{3}{2}$. *Consider* $A = \{1,2\}, B = \{2,3\}$, *define the mappings* $T : A \to B$ *and* $S : B \to A$ *as follows:* $Tx = x+1$ *and* $Sy = y-1$ *Then* $d(A,B) = 0, A_0 = A, B_0 = B, S(A_0) \subseteq B_0$ *and* $T(B_0) \subseteq A_0$.
Define $\varphi(t) = \frac{t+0.3}{t+0.4}$ *and* $\psi(t) = \frac{t+0.1}{t+0.2}$; *for all* $t \in [0,\infty)$ *and setting* $k_1 = k_2 = \frac{1}{2}$. *It is easily seen that* T *and* S *satisfies the weak* $b - \mathcal{MT} - KC$ *rational cyclic condition for all* $x \in A, y \in B$ *and* $x \neq Sy$ *and* $k_1, k_2 \leq \dfrac{1}{2}$.

Consider sequence $a_n = 1$ *in* X, *by Lemma 1.4 we will have*

$$\lim_{n\to\infty} d(a_n, a_{n+1}) = d(A,B) = 0.$$

Here 0 is a best proximity point of T *and* S.

Corollary 2.4. *Let* A *and* B *be nonempty subsets of a b-metric space* (X,d) *and* $T : A \longrightarrow B$ *and* $S : B \longrightarrow A$ *be maps. If the pair of maps* T *and* S *satisfy the weak* $b - \mathcal{MT} - K$ *rational cyclic condition then there exists a sequence* $\{x_n\}$ *in* X *such that*

$$\lim_{S\to\infty} d(x_n, x_{n+1}) = d(A,B).$$

Moreover, the following statements hold.

(*i*) *If* $\{x_{2n}\}_{n\in\mathbb{N}}$ *has a convergent subsequence in* A, *then there exists a point* $x \in A$ *such that* $d(x,Tx) = d(A,B)$.

(*ii*) *If* $\{x_{2n-1}\}_{n\in\mathbb{N}}$ *has a convergent subsequence in* B, *then there exists a point* $y \in B$ *such that* $d(y,Sy) = d(A,B)$.

Proof . The proof follows by taking $k_1 = \dfrac{1}{2}$ and $k_2 = 0$ in Theorem 2.2. □

Corollary 2.5. *Let* A *and* B *be nonempty subsets of a b-metric space* (X,d) *and* $T : A \longrightarrow B$ *and* $S : B \longrightarrow A$ *be maps. If the pair of maps* T *and* S *satisfy the weak* $b - \mathcal{MT} - C$ *rational cyclic condition then there exists a sequence* $\{x_n\}$ *in* X *such that*

$$\lim_{S\to\infty} d(x_n, x_{n+1}) = d(A,B).$$

Moreover, the following statements hold.

(*i*) *If* $\{x_{2n}\}_{n\in\mathbb{N}}$ *has a convergent subsequence in* A, *then there exists a point* $x \in A$ *such that* $d(x,Tx) = d(A,B)$.

(*ii*) *If* $\{x_{2n-1}\}_{n\in\mathbb{N}}$ *has a convergent subsequence in* B, *then there exists a point* $y \in B$ *such that* $d(y,Sy) = d(A,B)$.

Proof . The proof follows by taking $k_1 = 0$ and $k_2 = \dfrac{1}{2}$ in Theorem 2.2. □ In the next definition, we generalize Definition [4, Definition 3.5] to b-metric spaces.

Definition 2.6. *Let A and B be nonempty subsets of a b-metric space (X, d) and $T : A \cup B \longrightarrow A \cup B$ is said to satisfy*

(i) *weak $b - \mathcal{MT} - K$ rational cyclic condition if there exists an $\mathcal{MT}$ function φ such that*

$$d(Tx, Sy) \leq \frac{1}{2s}\varphi(d(x, y))\frac{d(x, Tx)d(x, Ty)}{d(x, Tx) + d(y, Ty)} + \frac{1}{s}(1 - \varphi(d(x, y)))d(A, B) \tag{2.6}$$

for all $x \in A, y \in B$ and $x \neq Ty$.

(ii) *weak $b - \mathcal{MT} - C$ rational cyclic condition if there exists an $\mathcal{MT}$ function φ such that*

$$d(Tx, Ty) \leq \frac{1}{2s}\varphi(d(x, y))\frac{d(x, Tx)d(x, Ty)}{d(x, Ty) + d(y, Tx)} + \frac{1}{s}(1 - \varphi(d(x, y)))d(A, B) \tag{2.7}$$

for all $x \in A, y \in B$ and $x \neq Ty$.

(iii) *weak $b - \mathcal{MT} - KC$ rational cyclic condition if there exists an $\mathcal{MT}$ function φ, ψ such that*

$$\begin{aligned} d(Tx, Ty) \leq &\frac{k_1}{s}\varphi(d(x, y))\frac{d(x, Tx)d(x, Ty)}{d(x, Tx) + d(y, Ty)} + \frac{k_2}{s}\psi(d(x, y))\frac{d(x, Tx)d(x, Ty)}{d(x, Ty) + d(y, Tx)} \\ &+ \frac{1}{s}(1 - 2[k_1\varphi(d(x, y))) + k_2\psi(d(x, y))])d(A, B) \end{aligned} \tag{2.8}$$

for all $x \in A, y \in B, x \neq Ty$ and $k_1, k_2 \leq \frac{1}{2}$.

The next conclusion follows easily from Theorem 2.2.

Corollary 2.7. *Let A and B be nonempty subsets of a b-metric space (X, d) and $T : A \cup B \longrightarrow A \cup B$ be maps. If map T satisfy the weak $b-\mathcal{MT}-KC$ rational cyclic condition then there exists a sequence $\{x_n\}$ in $A \cup B$ such that*

$$\lim_{S\to\infty} d(x_n, x_{n+1}) = d(A, B).$$

Moreover, the following statements hold.

(i) *If $\{x_{2n}\}_{n\in\mathbb{N}}$ has a convergent subsequence in A, then there exists a point $x \in A$ such that $d(x, Tx) = d(A, B)$.*

(ii) *If $\{x_{2n-1}\}_{n\in\mathbb{N}}$ has a convergent subsequence in B, then there exists a point $y \in B$ such that $d(y, Ty) = d(A, B)$.*

Corollary 2.8. *Let A and B be nonempty subsets of a b-metric space (X, d) and $T : A \cup B \longrightarrow A \cup B$ be maps. If map T satisfy the weak $b-\mathcal{MT}-K$ rational cyclic condition then there exists a sequence $\{x_n\}$ in $A \cup B$ such that*

$$\lim_{S\to\infty} d(x_n, x_{n+1}) = d(A, B).$$

Moreover, the following statements hold.

(i) *If $\{x_{2n}\}_{n\in\mathbb{N}}$ has a convergent subsequence in A, then there exists a point $x \in A$ such that $d(x, Tx) = d(A, B)$.*

(ii) *If $\{x_{2n-1}\}_{n\in\mathbb{N}}$ has a convergent subsequence in B, then there exists a point $y \in B$ such that $d(y, Ty) = d(A, B)$.*

Corollary 2.9. *Let A and B be nonempty subsets of a b-metric space (X,d) and $T : A\cup B \longrightarrow A\cup B$ be maps. If map T satisfy the weak $b-\mathcal{MT}-C$ rational cyclic condition then there exists a sequence $\{x_n\}$ in $A\cup B$ such that*

$$\lim_{S\to\infty} d(x_n, x_{n+1}) = d(A,B).$$

Moreover, the following statements hold.

- *(i) If $\{x_{2n}\}_{n\in\mathbb{N}}$ has a convergent subsequence in A, then there exists a point $x \in A$ such that $d(x,Tx) = d(A,B)$.*
- *(ii) If $\{x_{2n-1}\}_{n\in\mathbb{N}}$ has a convergent subsequence in B, then there exists a point $y \in B$ such that $d(y,Ty) = d(A,B)$.*

Theorem 2.10. *Let A and B be nonempty subsets of a b-metric space (X,d) and $T : A \longrightarrow B$ and $S : B \to A$ be maps. If the pair of map T and S satisfy the weak $b-\mathcal{MT}-KC$ rational cyclic condition then for $k_2 < \dfrac{1}{2}$, the sequence $\{x_n\}$ is bounded.*

Proof . By Theorem 2.10, we have $\lim_{n\to\infty} d(x_n, x_{n+1}) = d(A,B)$. Since $\{d(x_{2n-1}, x_{2n})\}$ is a subsequence of $\{d(x_n, x_{n+1})\}$, we have

$$\lim_{n\to\infty} d(x_{2n-1}, x_{2n}) = d(A,B),$$

Thus, $\{d(x_{2n-1}, x_{2n})\}$ is bounded. Therefore there exists $L > 0$ such that

$$d(x_{2n-1}, x_{2n}) \leq \frac{1}{s}L, \qquad \text{for all } \ n \in \mathbb{N}.$$

For each $n \in \mathbb{N}$, Since φ and ψ satisfies the weak $b-\mathcal{MT}-KC$ rational cyclic condition,it follows that

$$\begin{aligned}
d(Tx_0, x_{2n}) &= d(Tx_0, Sx_{2n-1}) \\
&\leq \frac{k_1}{s}\varphi(d(x_0, x_{2n-1}))\frac{d(x_0,Tx_0)d(x_0,x_{2n})}{d(x_0,Tx_0)+d(x_{2n-1},x_{2n})} \\
&+ \frac{k_2}{s}\psi(d(x_0, x_{2n-1}))\frac{d(x_0,Tx_0)d(x_0,x_{2n})}{d(x_0,x_{2n})+d(x_{2n-1},Tx_0)} \\
&+ \frac{1}{s}(1-2[k_1\varphi(d(x_0,x_{2n-1}))+k_2\psi(d(x_0,x_{2n-1}))])d(A,B) \\
d(Tx_0, x_{2n}) &= d(Tx_0, Sx_{2n_1}) \\
&\leq \big(\frac{k_1}{s}\varphi_{2n_1}+\frac{k_2}{s}\psi_{2n_1}\big)d(x_{2n_1}, x_{2n}) \\
&+ \frac{1}{s}\big(1-2(k_1\varphi_{2n_1}+k_2\psi_{2n_1})\big)d(A,B). \\
&\leq \frac{1}{s}((k_1+k_2)L + d(A,B))
\end{aligned}$$

Choose $K := (k_1+k_2)L + d(A,B)$ then

$$d(Tx_0, x_{2n}) \leq \frac{1}{s}K. \tag{2.9}$$

Furthermore,

$$\begin{aligned} d(Tx_0, x_{2n+1}) &\leq s(d(Tx_0, x_{2n}) + d(x_{2n}, x_{2n+1})) \\ &\leq s\frac{1}{s}(L+K). \end{aligned} \tag{2.10}$$

Hence, from (2.9) and (2.10) for all $n \in \mathbb{N}$, we have $d(Tx_0, x_n) \leq L + K$. which means that $\{x_n\}$ is bounded. □

Corollary 2.11. *Let A and B be nonempty subsets of a b-metric space (X,d) and $T : A \longrightarrow B$ and $S : B \longrightarrow A$ be maps. If the pair of maps T and S satisfy the weak $b - \mathcal{MT} - K$ rational cyclic condition then there the sequence $\{x_n\}$ i bounded.*

Proof . The proof follows by taking $k_1 = \dfrac{1}{2}$ and $k_2 = 0$ in Theorem 2.10. □

3. Application

Let $X = C[0,1]$ be the set of all real continuous functions on $[0,1]$ and X equipped with the b-metric below,

$$d(f,g) = \max\{(|f(t) - g(t)|)^2 : 0 \leq t \leq 1\}, \quad f, g \in X.$$

Then (X,d) is a complete b-metric space with parameter $s = 2$. Now, consider integral equations:

$$f(t) = \int_0^1 G(t,s)k_1(t,s,f(s))\mathrm{d}s$$

and

$$g(t) = \int_0^1 G(t,s)k_2(t,s,g(s))\mathrm{d}s$$

where $G : [0,1] \times [0,1] \to \mathbb{R}$ and $k_1, k_2 = [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ are continuous functions.
Suppose that:

1. For all $t, s \in [0,1]$ and $f, g \in X$, we have:

   $$0 \leq \int_0^1 G(t,s)k_1(t,s,f(s))\mathrm{d}s \leq \frac{1}{3}$$

   and

   $$0 \leq \int_0^1 G(t,s)k_2(t,s,g(s))\mathrm{d}s \leq \frac{1}{2}.$$

   Define:

   $$A = \{f \in X | 0 \leq f(t) \leq \frac{1}{2}\}$$

   and

   $$B = \{g \in X | 0 \leq g(t) \leq \frac{1}{3}\},$$

   then $d(A,B) = \dfrac{1}{4}$.
2. For all $t, s \in [0,1]$, we have:

   $$\max_{0 \leq t \leq 1} \int_0^1 |G(t,s)|^2 \mathrm{d}s \leq \frac{1}{2} d(A,B)$$

3. For all $t, s \in [0,1]$ and $f, g \in X$, we have:

$$\begin{aligned}|k_1(t,s,f(s)) - k_2(t,s,g(s))| &\leq \frac{1}{\sqrt{2}}|f(s)-g(s)| \\ &\leq |f(s) - Tf(s)| \\ &\leq |f(s) - Sg(s)|\end{aligned}$$

Let $T : A \to B$, $S : B \to A$ be mappings defined by:

$$Tf(t) = \int_0^1 G(t,s)k_1(t,s,f(s))\mathrm{d}s.$$

and

$$Sg(t) = \int_0^1 G(t,s)k_2(t,s,g(s))\mathrm{d}s.$$

Consider $\varphi(t) = \psi(t) = \dfrac{t}{2}$ and fix $k_1 = k_2 = \dfrac{1}{4}$.

We have to show that the operator T and S satisfies all conditions of Theorem 2.2 for any $f \in A$ and $g \in B$, $f \neq sg$, we have

$$\begin{aligned}d(Tf, Sg) &= \max_{0\leq t\leq 1}\{(|Tf(t) - Sg(t)|)^2\} \\ &= \max_{0\leq t\leq 1}\{(|\int_0^1 G(t,s)k_1(t,s,f(s))\mathrm{d}s - \int_0^1 G(t,s)k_2(t,s,g(s))\mathrm{d}s|)^2\} \\ &\leq \max_{0\leq t\leq 1}\{(\int_0^1 |G(t,s)|(|k_1(t,s,f(s)) - k_2(t,s,g(s))|)\mathrm{d}s)^2\} \\ &\leq \max_{0\leq t\leq 1}\{\int_0^1 |G(t,s)^2\mathrm{d}s \int_0^1 |k_1(t,s,f(s)) - k_2(t,s,g(s))|^2\mathrm{d}s\} \\ &\leq \frac{1}{2}d(A,B)\int_0^1 \frac{1}{2}|f(s)-g(s)|^2|f(s)-Tf(s)|^2|f(s)-Sg(s)|^2\mathrm{d}s \\ &\leq \frac{1}{8}(\frac{1}{2}d(f,g)d(f,Tf)d(f,Sg)) \\ &\leq \frac{1}{8}[(\frac{1}{2}d(f,g)d(f,Tf)d(f,Sg))(\frac{1}{d(f,Tf)+d(g,Sg)} + \frac{1}{d(f,Sg)+d(g,Tf)})] \\ &\leq \frac{1}{8}[\varphi(d(f,g))\frac{d(f,Tf)d(f,Sg)}{d(f,Tf)+d(g,Sg)} + \psi(d(f,g))\frac{d(f,Tf)d(f,Sg)}{d(f,Sg)+d(g,Tf)} \\ &+ (1 - \frac{1}{2}(\varphi(d(f,g)) + \psi(d(f,g)))]\end{aligned}$$

Hence, all of the hypotheses of Theorem 2.2 for $s = 2, k_1 = k_2 = \dfrac{1}{4}$ and $\psi(t) = \varphi(t) = \frac{t}{2}$ are satisfied.

Conclusion

In this paper, we present the concept of b-metric space and weak $\mathcal{MT} - KC$ rational cyclic condition. Also we have achieved some best proximity points theorems for a pair of mappings satisfy these condition in b-metric spaces. Theorem 2.2 improves Theorem 3.2 in [4] to metric space. Also other results from [4] are special case of the our Corollaries in b-metric spaces. We give an example to show the validity of our result and an application to nonlinear integral in clusion for the applicability purpose.

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